On (a,d)-antimagic labelings of $H_n$, $FL_n$ and $mC_n$

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Abstract

In this paper, we derive the necessary condition for an (a,d)-antimagic labeling of some new classes of graphs such as $H_n$, $FL_n$ and $mC_n$. We prove that $H_n$ is $(7n + 2, 1)$-antimagic and $mC_n$ is $(\frac{mn+3}{2}, 1)$-antimagic. Also we prove that $FL_n$ has no $(\frac{n+1}{2}, 4)$-antimagic labeling.

Keywords: antimagic labeling, (a, d)-antimagic labeling.

Mathematics Subject Classification : 05C78

DOI: 10.19184/ijc.2020.4.2.3

1. Introduction

The graphs considered here are finite, undirected and simple. The vertex[edge] set of a graph $G$ is denoted by $V(G)$[$E(G)$] respectively. The weight $w(v)$ of a vertex $v \in V(G)$ under an edge labeling $f$ is the sum of the values $f(e)$ assigned to all edges incident to the vertex $v$.

A connected graph $G(V, E)$ is said to be (a,d)-antimagic if there exists positive integers (a,d) and a bijection $f : E(G) \rightarrow \{1, 2, ..., |E(G)|\}$ such that the induced mapping $g_f : V(G) \rightarrow W$ is also a bijection where $W = \{w(v) | v \in V(G)\} = \{a, a + d, ..., a + (|V(G)| - 1)d\}$.

If $G(V, E)$ is $(a, d)$-antimagic and $f : E(G) \rightarrow \{1, 2, ..., |E(G)|\}$ is a corresponding bijective map of $G$ then $f$ is said to be an $(a, d)$-antimagic labeling of $G$.

Hartsfield and Ringel [6] introduced the concepts of an antimagic graph. An antimagic graph $G$ is a graph whose edges can be labeled with the integers $1, 2, ..., |E(G)|$ so that sum of the labels...
at any given vertex is different from the sum of the labels at any other vertex, that is no two vertices receives the same weight. Some results about antimagic graphs can be seen in [1, 2, 3, 5].

We introduce a the new class of graph $H_n$. The graph $H_n$, $n \geq 3$ is defined as follows: Let $I = \{1, 2, \ldots, n\}$ be an index set. We denote the vertex set of $H_n$ by

$$V(H_n) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_{2n}, z_1, z_2, \ldots, z_n\}$$

and the edge set of $H_n$ by

$$E(H_n) = \{x_ix_{i+1}/i \in I\} \cup \{x_iy_{2i}/i \in I\} \cup \{y_iy_{i+1}/1 \leq i \leq 2n\} \cup \{y_{2i}z_i/i \in I\} \cup \{z_iz_{i+1}/i \in I\}.$$ 

We make convention that $x_{n+1} = x_1$, $y_{2n+1} = y_1$ and $z_{n+1} = z_1$. In $H_n$, $|V(H_n)| = 4n$ and $|E(H_n)| = 6n$.

In [7], The flower graph $FL_n$, $n \geq 2$ is defined as follows:

Let $I = \{1, 2, \ldots, n\}$ be an index set. We denote the vertex set of $FL_n$ by

$$V(FL_n) = \{u, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_{n-1}\}$$

and the edge set of $FL_n$ by

$$E(FL_n) = \{uu_i/i \in I\} \cup \{uv_i/i \in I - \{n\}\} \cup \{u_iu_{i-1}/i \in I - \{1\}\} \cup \{u_1u_n\}.$$ 

In $FL_n$, $|V(FL_n)| = 2n$, $|E(FL_n)| = 3n - 1$.

In [4], The graph $mC_n$, $m \geq 1, n \geq 3$ is defined as follows: Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$ be two index sets. We denote the vertex set of $mC_n$ by

$$V(mC_n) = \{x_{i,j}/i \in I \text{ and } j \in J\}$$

and the edge set of $mC_n$ by

$$E(mC_n) = \{x_{i,j}x_{i,j+1}/i \in I \text{ and } j \in J\}.$$ 

We make the convention that for $i \in I, j \in J, x_{i,n+1} = x_{i,1}$. In $mC_n$, $|V(mC_n)| = |E(mC_n)| = mn$.

In this paper we prove that $H_n$ is $(7n + 2, 1)$-antimagic and $mC_n$ is $(\frac{mn+3}{2}, 1)$-antimagic. Also we prove that $FL_n$ has no $(\frac{n+1}{2}, 4)$-antimagic labeling.

2. Main Results

The following theorem gives the necessary condition for an $(a, d)$-antimagic labeling of the graph $H_n$.

**Theorem 2.1.** If $H_n$, $n \geq 3$ is $(a, d)$-antimagic, then $d$ is odd and $a = 9n - \frac{d}{2}(4n - 1) + \frac{3}{2}$. 


Proof. Let \( H_n, n \geq 3 \) be \((a, d)\)- antimagic. Then

\[
W = \{ w(v) | v \in V(G) \} = \{ a, a + d, \ldots, a + (4n - 1)d \}.
\]

Therefore, \( \sum_{v \in V(H_n)} w(v) = 2n[2a + (4n - 1)d] \). Since \( H_n \) is \((a, d)\)- antimagic, the edges of \( H_n \) are labeled by the set of integers \( \{1, 2, \ldots, 6n\} \). Also each of the edge labels is used twice in the computation of the weights of the vertices. Therefore \( \sum_{v \in V(H_n)} w(v) = 2(1 + 2 + \ldots + 6n) = 6n(6n + 1) \). Therefore \( 2n[2a + (4n - 1)d] = 6n(6n + 1) \) implies \( 2a + (4n - 1)d = 3(6n + 1) \). Thus \( d = \frac{3(6n + 1) - 2n}{4n - 1} \). Here \( a = 6 \) is the minimal value of weight which can be assigned to a vertex of degree three. For \( a \geq 6, 0 < d < \frac{9}{2} \). \( d \) is odd for \( n \geq 3 \). Therefore \( d = 1, 3 \). For \( d = 1, a = 7n + 2 \) and for \( d = 3, a = 3n + 3 \). Also \( 2a + (4n - 1)d = 18n + 3 \) implies \( a = 9n - \frac{d}{2}(4n - 1) + \frac{3}{2} \). Therefore (1) has exactly two different solutions \((a, d) = (7n + 2, 1)\) and \((a, d) = (3n + 3, 3)\) respectively.

The following theorem shows the existence of \((7n + 2, 1)\)- antimagic labeling in \( H_n \).

**Theorem 2.2.** For \( n \geq 3 \), \( H_n \) has \((7n + 2, 1)\)- antimagic labeling.

Proof. Define a mapping \( f : E(H_n) \to \{1, 2, \ldots, 6n\} \) as follows:

\[
\begin{align*}
    f(x_iy_{2i}) &= n - i + 1 \quad \text{for } 1 \leq i \leq n, \\
    f(y_{2i-1}z_i) &= 2n - i + 1 \quad \text{for } 1 \leq i \leq n, \\
    f(y_iy_{i+1}) &= \begin{cases}
        \frac{4n+i+1}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq 2n - 1 \\
        \frac{10n+2}{2} & \text{if } i \text{ is even and } 1 \leq i \leq 2n \\
    \end{cases} \\
    f(x_iy_{i+1}) &= 3n + i \quad \text{for } 1 \leq i \leq n, \\
    f(y_{2i-1}z_i) &= 4n + i \quad \text{for } 1 \leq i \leq n.
\end{align*}
\]

Here the labeling \( f \) uses each integer \( 1, 2, \ldots, 6n \) exactly once.

The weight of the corresponding vertices are

\[
\begin{align*}
    w(x_i) &= f(x_ix_{i+1}) + f(x_ix_{i-1}) + f(x_iy_{2i}) = 7n + i \text{ for } 2 \leq i \leq n, \\
    w(x_1) &= f(x_1x_2) + f(x_1x_n) + f(x_1y_2) = 8n + 1, \\
    w(y_{2i}) &= f(y_{2i-1}y_{2i}) + f(y_{2i}y_{2i+1}) + f(x_1y_{2i}) = 8n + i + 1 \text{ for } 1 \leq i \leq n, \\
    w(y_{2i+1}) &= f(y_{2i+1}y_{2i}) + f(y_{2i+1}y_{2i+2}) + f(z_{i+1}y_{2i+1}) \\
    &= 9n + i + 1 \text{ for } 1 \leq i \leq n - 1, \\
    w(y_1) &= f(y_1y_2) + f(y_1y_n) + f(y_1z_1) = 10n + 1, \\
    w(z_i) &= f(z_{i+1}z_{i-1}) + f(z_iz_{i+1}) + f(z_{i}y_{2i-1}) = 10n + i \text{ for } 2 \leq i \leq n, \\
    w(z_1) &= f(z_1z_2) + f(z_1z_n) + f(z_1y_1) = 11n + 1.
\end{align*}
\]

Each vertex of \( H_n \) receives exactly one label of weight from \( \{7n + 2, 7n + 3, \ldots, 11n + 1\} \). Therefore, \( H_n \) has \((7n + 2, 1)\)- antimagic labeling.

The following theorem gives the necessary condition for an \((a, d)\)- antimagic labeling of the graph \( FL_n \).

**Theorem 2.3.** Let \( FL_n, n \geq 2 \) be \((a, d)\)- antimagic.

If \( n \) is even, then either \( d = 1 \) and \( a = \frac{7n-2}{2} \) or \( d = 3 \) and \( a = \frac{3n}{2} \).

If \( n \) is odd, then either \( d = 2 \) and \( a = \frac{5n-1}{2} \) or \( d = 4 \) and \( a = \frac{n+1}{2} \).
Proof. Let $FL_n$, $n \geq 2$ be $(a,d)$-antimagic. Then

$$W = \{w(v)/v \in V(FL_n)\} = \{a, a + d, ..., a + (2n - 1)d\}$$

Therefore, $\sum_{v \in V(FL_n)} w(v) = n[2a + (2n - 1)d]$. Since $FL_n$ is $(a,d)$-antimagic, edges of $FL_n$ are labeled by the set of integers $\{1, 2, ..., 3n - 1\}$. Also each of the edge labeling is used twice in the computation of weights of the vertices. Therefore $\sum_{v \in V(FL_n)} w(v) = 2(1 + 2 + ... + 3n - 1) = (3n - 1)3n$. Therefore $n[2a + (2n - 1)d] = 3n(3n - 1)$ implies $2a + (2n - 1)d = 3(3n - 1)$. Thus $d = \frac{3(3n - 1) - 2a}{2n - 1}$. Here $a = 3$ is the minimal value of weight which can be assigned to a vertex of weight two.

If $a \geq 3$, $0 < d < \frac{9}{2}$.

When $n$ is even, $d$ is odd. Therefore $d = 1, 3$.

- For $d = 1$, $a = \frac{7n - 2}{2}$.
- For $d = 3$, $a = \frac{3n}{2}$.

When $n$ is odd, $d$ is even. Therefore $d = 2, 4$.

- For $d = 2$, $a = \frac{5n - 1}{2}$.
- For $d = 4$, $a = \frac{n + 1}{2}$.

\[\square\]

Observation 2.1. By direct verification, we have

1) $FL_2$ has no $(3,3)$-antimagic labeling.
2) $FL_2$ has no $(6,1)$-antimagic labeling.
3) $FL_3$ has no $(2,4)$-antimagic labeling.
4) $FL_3$ has $(7,2)$-antimagic labeling.
5) $FL_4$ has $(13,1)$-antimagic labeling.
6) $FL_5$ has no $(3,4)$-antimagic labeling.

The following theorem shows that $FL_n$ has no $(\frac{n+1}{2},4)$-antimagic labeling for $n \geq 9$ and $n$ is odd.

Theorem 2.4. $FL_n$ is not $(\frac{n+1}{2},4)$-antimagic for $n \geq 7$ and $n$ is odd.

Proof. The vertex and edge set of $FL_n$ are $V(FL_n) = \{u, u_1, u_2, ..., u_n, v_1, v_2, ..., v_{n-1}\}$ and $E(FL_n) = \{uu_i/i \in I\} \cup \{v_iv_i/i \in I - \{n\}\} \cup \{u_iv_{i-1}/i \in I - \{1\}\} \cup \{u_1u_n\}$. Then $|V(FL_n)| = 2n$ and $|E(FL_n)| = 3n - 1$. Also $d(u) = n$, $d(u_i) = 3$ for all $1 \leq i \leq n$ and $d(v_i) = 2$ for all $1 \leq i \leq n - 1$.

Suppose $FL_n$ is $(\frac{n+1}{2},4)$-antimagic. Then there exists a labeling

$$f : E(FL_n) \rightarrow \{1, 2, ..., 3n - 1\}$$

and $\{w(v)/v \in V(FL_n)\} = \{\frac{n+1}{2}, \frac{n+1}{2} + 4, ..., \frac{n+1}{2} + (2n - 1)4\}$.

Suppose $w(v_i) = \frac{n+1}{2} + (2n - 1)4$ for some $i = 1, 2, ..., n - 1$. Since $d(v_i) = 2$ for all $i = 1, 2, ..., n - 1$, the maximum possible labels which are assigned to the edges incident with $v_i$ are $3n - 1$ and $3n - 2$. But sum of these labels less than $\frac{n+1}{2} + (2n - 1)4$. Hence $w(v_i) \neq \frac{n+1}{2} + (2n - 1)4$ for all $1 \leq i \leq n - 1$. Also we can’t assign the weights $\frac{n+1}{2} + (2n - \frac{n+3}{2})4$, $\frac{n+1}{2} + (2n - \frac{n+5}{2})4$, $\frac{n+1}{2} + (2n - \frac{n+7}{2})4$, $\frac{n+1}{2} + (2n - \frac{n+9}{2})4$, $\frac{n+1}{2} + (2n - \frac{n+11}{2})4$.
Therefore, \( FL_n \) is odd. Thus both \( d \) and \( w \) are labeled by the set of integers \( \{1, 2, ..., mn\} \). Also each of the edge labeling is used twice in the computation of weights of the vertices. Therefore \( \sum_{v \in V(mC_n)} w(v) = 2(2a + (mn - 1)d) = mn(mn + 1) \). Therefore \( d = \frac{2(mn + 1) - 2a}{mn - 1} \). Here \( a = 3 \) is the minimal value of weight which can be assigned to a vertex of degree two. For \( a \geq 3, 0 < d < 2 \). Therefore \( d = 1 \). When \( d = 1, a = \frac{mn+3}{2} \). Since \( a \) is an integer, \( mn + 3 \equiv 0(\text{mod}2) \). Therefore \( mn \) is odd. Thus both \( m \) and \( n \) are odd.

The following theorem shows the existence of \((a,d)\)-antimagic labeling in \( mC_n \).

**Theorem 2.6.** For \( m \geq 1 \) odd and \( n \geq 3 \) odd, \( mC_n \) is \((\frac{mn+3}{2}, 1)\)-antimagic.

**Proof.** Define the edge labeling \( f : E(mC_n) \to \{1, 2, ..., mn\} \) by

\[
f(x_{i,1}x_{i,n}) = i \quad \text{for} \quad 1 \leq i \leq n,
\]

\[
f(x_{m-i,j}x_{m-i,j+1}) = \begin{cases} \frac{mj+i+1}{2(j+1)m+1} & \text{for} \quad i = 1, 3, ..., m - 2 \quad \text{and} \quad j = 2, 4, ..., n - 1, \\ \frac{mn+1}{2} & \text{for} \quad i = 0, 2, ..., m - 1 \quad \text{and} \quad j = 2, 4, ..., n - 1, \\ 
\end{cases}
\]

\[
f(x_{m-i,n-k}x_{m-i,n-k}+1) = \begin{cases} \frac{(mn+1)+i-2m}{2} & \text{for} \quad i = 0, 2, ..., m - 1 \quad \text{and} \quad k = 2, 4, ..., n - 1, \\ \frac{(mn+1)+i-(k-1)m}{2} & \text{for} \quad i = 1, 3, ..., m - 2 \quad \text{and} \quad k = 2, 4, ..., n - 1. \\ \end{cases}
\]

Here the labeling \( f \) uses each integer 1, 2, ..., \( mn \) exactly once.

The weights of the vertices of \( mC_n \) under the labeling of \( f \) are given below.

\[
w(x_{i,j}) = \begin{cases} f(x_{i,j}x_{i,j+1}) + f(x_{i,j-1}x_{i,j}) & \text{for} \quad i = 1, 2, ..., m \quad \text{and} \quad j = 1, 2, ..., n - 1, \\ f(x_{i-1,n}x_{i,n}) + f(x_{i,n}x_{i,1}) & \text{for} \quad i = 1, 2, ..., m \quad \text{and} \quad j = n, \\ \end{cases}
\]

\[
w(x_{i,1}) = \begin{cases} \frac{mn+2m+i+2}{2} & \text{for} \quad i = 1, 3, ..., m, \\ \frac{mn+3m+i+2}{2} & \text{for} \quad i = 2, 4, ..., m - 1, \\ \end{cases}
\]

\[
\{w(x_{i,1})\}/i = \begin{cases} \frac{mn+3}{2} + m, \frac{mn+3}{2} + m + 1, ..., \frac{mn+3}{2} + 2m - 1 \end{cases}
\]

\[
w(x_{i,n}) = \begin{cases} \frac{mn+m+i+2}{2} & \text{for} \quad i = 1, 3, ..., m, \\ \frac{mn+i+1}{2} & \text{for} \quad i = 2, 4, ..., m - 1, \\ \end{cases}
\]
On \((a,d)\)-antimagic labelings of \(H_n\), \(FL_n\) and \(mC_n\)  

\[ \{w(x_{i,n})/i = 1, 2, ..., m\} = \left\{ \frac{mn+3}{2}, \frac{mn+3}{2} + 1, ..., \frac{mn+3}{2} + m - 1, \right\} \] and  

\[ \{w(x_{i,j})/i = 1, 2, ..., m \text{ and } j = 2, 3, ..., n-1\} = \left\{ \frac{mn+3}{2} + 2m, \frac{mn+3}{2} + 2m + 1, ..., \frac{mn+3}{2} + mn - 1 \right\}. \]  

Each vertex of \(mC_n\) receives exactly one weight from \(\left\{ \frac{mn+3}{2}, \frac{mn+3}{2} + 1, ..., \frac{mn+3}{2} + mn - 1 \right\}\). Hence for both \(m \geq 1\) and \(n \geq 3\) odd, \(mC_n\) is \((\frac{mn+3}{2}, 1)\)-antimagic.

![Figure 1. \(H_4\) is \((30, 1)\)-antimagic](image)

3. Open Problems and Conjectures

Problems:
- Determine whether \(FL_n\) is \((\frac{5n-1}{2}, 2)\)-antimagic or not for \(n\) is odd and \(n \geq 5\).
- Determine whether \(H_n\) is \((3n + 3, 3)\)-antimagic or not for \(n \geq 3\).

From our experience on this labeling, we propose the following conjectures.

Conjectures:
- For \(n\) is even and \(n \geq 6\), \(FL_n\) is not \((\frac{7n-2}{2}, 1)\)-antimagic.
- For \(n\) is even and \(n \geq 4\), \(FL_n\) is not \((\frac{3n}{2}, 3)\)-antimagic.
Figure 2. $F_3$ is $(7, 2)$-antimagic and $F_4$ is $(13, 1)$-antimagic

Figure 3. $3C_5$ is $(9, 1)$-antimagic
Acknowledgement

We thank anonymous reviewer for the valuable comments on an earlier version of the manuscript.

References


