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On (a,d)-antimagic labelings of H_n, FL_n and mC_n

R.Ramalakshmi[a], KM. Kathiresan[b]

[a]*Department of Mathematics, Rajapalayam Rajus' College, Rajapalayam -626 117 , India.*
[b]*Centre for Graph Theory, Ayya Nadar Janaki Ammal College, Sivakasi-626 124, India.*

ramimani20@gmail.com, kathir2esan@yahoo.com

Abstract

In this paper, we derive the necessary condition for an (a,d)-antimagic labeling of some new classes of graphs such as H_n, FL_n and mC_n. We prove that H_n is $(7n+2,1)$-antimagic and mC_n is $(\frac{mn+3}{2},1)$-antimagic. Also we prove that FL_n has no $(\frac{n+1}{2},4)$-antimagic labeling.

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1. Introduction

The graphs considered here are finite, undirected and simple. The vertex[edge] set of a graph G is denoted by $V(G)[E(G)]$ respectively. The weight $w(v)$ of a vertex $v \in V(G)$ under an edge labeling f is the sum of the values $f(e)$ assigned to all edges incident to the vertex v.

A connected graph $G(V,E)$ is said to be (a,d)-antimagic if there exists positive integers (a,d) and a bijection $f : E(G) \to \{1,2,...,E(G)\}$ such that the induced mapping $g_f : V(G) \to W$ is also a bijection where $W = \{w(v)|v \in V(G)\} = \{a, a+d, ..., a+(|V(G)|-1)d\}$.

If $G(V,E)$ is (a,d)-antimagic and $f : E(G) \to \{1,2,...,|E(G)|\}$ is a corresponding bijective map of G then f is said to be an (a,d)-antimagic labeling of G.

Hartsfield and Ringel [6] introduced the concepts of an antimagic graph. An antimagic graph G is a graph whose edges can be labeled with the integers $1,2,...,|E(G)|$ so that sum of the labels

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at any given vertex is different from the sum of the labels at any other vertex, that is no two vertices receives the same weight. Some results about antimagic graphs can be seen in [1, 2, 3, 5].

We introduce a the new class of graph H_n. The graph H_n, $n \geq 3$ is defined as follows: Let $I = \{1, 2, ..., n\}$ be an index set. We denote the vertex set of H_n by

$$V(H_n) = \{x_1, x_2, ..., x_n, y_1, y_2, ..., y_{2n}, z_1, z_2, ..., z_n\}$$

and the edge set of H_n by

$$E(H_n) = \{x_i x_{i+1}/i \in I\} \cup \{x_i y_{2i}/i \in I\} \cup \{y_i y_{i+1}/1 \leq i \leq 2n\} \cup \{y_{2i-1} z_i/i \in I\} \cup \{z_i z_{i+1}/i \in I\}.$$

We make convention that $x_{n+1} = x_1$, $y_{2n+1} = y_1$ and $z_{n+1} = z_1$. In H_n, $|V(H_n)| = 4n$ and $|E(H_n)| = 6n$.

In [7], The flower graph FL_n, $n \geq 2$ is defined as follows:
Let $I = \{1, 2, ..., n\}$ be an index set. We denote the vertex set of FL_n by

$$V(FL_n) = \{u, u_1, u_2, ..., u_n, v_1, v_2, ..., v_{n-1}\}$$

and the edge set of FL_n by

$$E(FL_n) = \{uu_i/i \in I\} \cup \{u_i v_i/i \in I - \{n\}\} \cup \{u_i v_{i-1}/i \in I - \{1\}\} \cup \{u_1 u_n\}.$$

In FL_n, $|V(FL_n)| = 2n$, $|E(FL_n)| = 3n - 1$.

In [4], The graph mC_n, $m \geq 1, n \geq 3$ is defined as follows: Let $I = \{1, 2, ..., m\}$ and $J = \{1, 2, ..., n\}$ be two index sets. We denote the vertex set of mC_n by

$$V(mC_n) = \{x_{i,j}/i \in I \text{ and } j \in J\}$$

and the edge set of mC_n by

$$E(mC_n) = \{x_{i,j} x_{i,j+1}/i \in I \text{ and } j \in J\}.$$

We make the convention that for $i \in I, j \in J$, $x_{i,n+1} = x_{i,1}$. In mC_n, $|V(mC_n)| = |E(mC_n)| = mn$.

In this paper we prove that H_n is $(7n + 2, 1)$-antimagic and mC_n is $(\frac{mn+3}{2}, 1)$-antimagic. Also we prove that FL_n has no $(\frac{n+1}{2}, 4)$-antimagic labeling.

2. Main Results

The following theorem gives the necessary condition for an (a, d)-antimagic labeling of the graph H_n.

Theorem 2.1. *If $H_n, n \geq 3$ is (a, d)-antimagic, then d is odd and $a = 9n - \frac{d}{2}(4n - 1) + \frac{3}{2}$.*

Proof. Let $H_n, n \geq 3$ be (a, d)-antimagic. Then

$$W = \{w(v)|v \in V(G)\} = \{a, a+d, ..., a+(4n-1)d.\}$$

Therefore, $\sum_{v \in V(H_n)} w(v) = 2n[2a + (4n-1)d]$. Since H_n is (a, d)-antimagic, the edges of H_n are labeled by the set of integers $\{1, 2, ..., 6n\}$. Also each of the edge labels is used twice in the computation of the weights of the vertices. Therefore $\sum_{v \in V(H_n)} w(v) = 2(1+2+...+6n) = 6n(6n+1)$. Therefore $2n[2a+(4n-1)d] = 6n(6n+1)$ implies $2a+(4n-1)d = 3(6n+1)$. Thus $d = \frac{3(6n+1)-2a}{4n-1}$. Here $a = 6$ is the minimal value of weight which can be assigned to a vertex of degree three. For $a \geq 6, 0 < d < \frac{9}{2}$. d is odd for $n \geq 3$. Therefore $d = 1, 3$.For $d = 1, a = 7n+2$ and for $d = 3, a = 3n+3$. Also $2a + (4n-1)d = 18n+3$ implies $a = 9n - \frac{d}{2}(4n-1) + \frac{3}{2}$. Therefore (1) has exactly two different solutions $(a, d) = (7n+2, 1)$ and $(a, d) = (3n+3, 3)$ respectively. □

The following theorem shows the existence of $(7n+2, 1)$-antimagic labeling in H_n.

Theorem 2.2. *For $n \geq 3$, H_n has $(7n+2, 1)$-antimagic labeling.*

Proof. Define a mapping $f : E(H_n) \to \{1, 2, ..., 6n\}$ as follows:
$f(x_i y_{2i}) = n - i + 1$ for $1 \leq i \leq n$,
$f(y_{2i-1} z_i) = 2n - i + 1$ for $1 \leq i \leq n$,
$$f(y_i y_{i+1}) = \begin{cases} \frac{4n+i+1}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq 2n-1 \\ \frac{10n+i}{2} & \text{if } i \text{ is even and } 1 \leq i \leq 2n, \end{cases}$$
$f(x_i x_{i+1}) = 3n + i$ for $1 \leq i \leq n$,
$f(y_{2i-1} z_i) = 4n + i$ for $1 \leq i \leq n$.
Here the labeling f uses each integer $1, 2, ..., 6n$ exactly once.

The weight of the corresponding vertices are
$w(x_i) = f(x_i x_{i+1}) + f(x_i x_{i-1}) + f(x_i y_{2i}) = 7n + i$ for $2 \leq i \leq n$.
$w(x_1) = f(x_1 x_2) + f(x_1 x_n) + f(x_1 y_2) = 8n + 1$.
$w(y_{2i}) = f(y_{2i-1} y_{2i}) + f(y_{2i} y_{2i+1}) + f(x_i y_{2i}) = 8n + i + 1$ for $1 \leq i \leq n$.
$w(y_{2i+1}) = f(y_{2i+1} y_{2i}) + f(y_{2i+1} y_{2i+2}) + f(z_{i+1} y_{2i+1})$
$= 9n + i + 1$ for $1 \leq i \leq n-1$.
$w(y_1) = f(y_1 y_2) + f(y_1 y_2) + f(y_1 z_1) = 10n + 1$.
$w(z_i) = f(z_i z_{i-1}) + f(z_i z_{i+1}) + f(z_i y_{2i-1}) = 10n + i$ for $2 \leq i \leq n$.
$w(z_1) = f(z_1 z_2) + f(z_1 z_n) + f(z_1 y_1) = 11n + 1$.
Each vertex of H_n receives exactly one label of weight from $\{7n+2, 7n+3, ..., 11n+1\}$
Therefore, H_n has $(7n+2, 1)$-antimagic labeling. □

The following theorem gives the necessary condition for an (a, d)-antimagic labeling of the graph FL_n.

Theorem 2.3. *Let $FL_n, n \geq 2$ be (a, d)-antimagic.*
If n is even, then either $d = 1$ and $a = \frac{7n-2}{2}$ or $d = 3$ and $a = \frac{3n}{2}$.
If n is odd, then either $d = 2$ and $a = \frac{5n-1}{2}$ or $d = 4$ and $a = \frac{n+1}{2}$.

Proof. Let $FL_n, n \geq 2$ be (a,d)-antimagic. Then

$$W = \{w(v)/v \in V(FL_n)\} = \{a, a+d, ..., a+(2n-1)d.\}$$

Therefore, $\sum_{v \in V(FL_n)} w(v) = n[2a + (2n-1)d]$. Since FL_n is (a,d)-antimagic, edges of FL_n are labeled by the set of integers $\{1, 2, \ldots, 3n-1\}$. Also each of the edge labeling is used twice in the computation of weights of the vertices. Therefore $\sum_{v \in V(FL_n)} w(v) = 2(1+2+...+3n-1) = (3n-1)3n$.Therefore $n[2a+(2n-1)d] = 3n(3n-1)$ implies $2a+(2n-1)d = 3(3n-1)$. Thus $d = \frac{3(3n-1)-2a}{2n-1}$. Here $a = 3$ is the minimal value of weight which can be assigned to a vertex of degree two.
If $a \geq 3$, $0 < d < \frac{9}{2}$.
When n is even, d is odd. Therefore $d = 1, 3$.
For $d = 1, a = \frac{7n-2}{2}$.
For $d = 3, a = \frac{3n}{2}$.
When n is odd, d is even. Therefore $d = 2, 4$.
For $d = 2, a = \frac{5n-1}{2}$.
For $d = 4, a = \frac{n+1}{2}$.

□

Observation 2.1. *By direct verification, we have*
1) FL_2 *has no* $(3,3)$*-antimagic labeling.*
2) FL_2 *has no* $(6,1)$*-antimagic labeling.*
3) FL_3 *has no* $(2,4)$*-antimagic labeling.*
4) FL_3 *has* $(7,2)$*-antimagic labeling.*
5) FL_4 *has* $(13,1)$*-antimagic labeling.*
6) FL_5 *has no* $(3,4)$*-antimagic labeling.*

The following theorem shows that FL_n has no $(\frac{n+1}{2}, 4)$-antimagic labeling for $n \geq 9$ and n is odd.

Theorem 2.4. FL_n *is not* $(\frac{n+1}{2}, 4)$*-antimagic for* $n \geq 7$ *and* n *is odd.*

Proof. The vertex and edge set of FL_n are $V(FL_n) = \{u, u_1, u_2, ..., u_n, v_1, v_2, ..., v_{n-1}\}$ and $E(FL_n) = \{uu_i/i \in I\} \cup \{u_iv_i/i \in I - \{n\}\} \cup \{u_iv_{i-1}/i \in I - \{1\}\} \cup \{u_1u_n\}$. Then $|V(FL_n)| = 2n$ and $|E(FL_n)| = 3n-1$. Also $d(u) = n$, $d(u_i) = 3$ for all $1 \leq i \leq n$ and $d(v_i) = 2$ for all $1 \leq i \leq n-1$.

Suppose FL_n is $(\frac{n+1}{2}, 4)$-antimagic. Then there exists a labeling

$$f : E(FL_n) \rightarrow \{1, 2, ..., 3n-1\}$$

and $\{w(v)/v \in V(FL_n)\} = \left\{\frac{n+1}{2}, \frac{n+1}{2}+4, ..., \frac{n+1}{2}+(2n-1)4\right\}$.

Suppose $w(v_i) = \frac{n+1}{2} + (2n-1)4$ for some $i = 1, 2, ..., n-1$. Since $d(v_i) = 2$ for all $i = 1, 2, ..., n-1$, the maximum possible labels which are assigned to the edges incident with v_i are $3n-1$ and $3n-2$. But sum of these labels less than $\frac{n+1}{2}+(2n-1)4$. Hence $w(v_i) \neq \frac{n+1}{2}+(2n-1)4$ for all $1 \leq i \leq n-1$. Also we can't assign the weights $\frac{n+1}{2} + (2n - \frac{n+3}{2})4$, $\frac{n+1}{2} + (2n -$

$\frac{n+1}{2})4, ..., \frac{n+1}{2} + (2n-2)4$ in any v_i for $1 \leq i \leq n-1$. These weights can be assigned in u or u_i for $1 \leq i \leq n$.

Let $K = \{\frac{n+1}{2} + (2n - \frac{n+3}{2})4, \frac{n+1}{2} + (2n - \frac{n+1}{2})4, ..., \frac{n+1}{2} + (2n-2)4\}$. Then $|K| = \frac{n+1}{2}$. In FL_n, each u_i is of degree three and there is no common edge to two different u_i except u_1 and u_n. If $w(u_i) = k$ where $k \in K$, then the sum of any $3(\frac{n+1}{2})$ numbers from the list $\{1, 2, ..., 3n-1\}$ is always less than sum of elements in K for all $n \geq 7$, which is not possible.
Therefore FL_n is not $(\frac{n+1}{2}, 4)$-antimagic labeling for $n \geq 7$ and n is odd. □

The following theorem gives the necessary condition for (a, d)-antimagic labeling of the graph mC_n.

Theorem 2.5. *If $mC_n, m \geq 1, n \geq 3$ is (a, d)-antimagic, then $(a, d) = (\frac{mn+3}{2}, 1)$ and both m and n are odd.*

Proof. Let $mC_n, m \geq 1, n \geq 3$ be (a, d)-antimagic. Then $W = \{w(v)/v \in V(mC_n)\} = \{a, a+d, ..., a+(mn-1)d\}$. Therefore, $\sum_{v \in V(mC_n)} w(v) = \frac{mn}{2}[2a + (mn-1)d]$. Since mC_n is (a, d)-antimagic, edges of mC_n are labeled by the set of integers $\{1, 2, ..., mn\}$. Also each of the edge labeling is used twice in the computation of weights of the vertices. Therefore $\sum_{v \in V(mC_n)} w(v) = 2(1 + 2 + ... + mn) = mn(mn+1)$. Therefore $\frac{mn}{2}[2a + (mn-1)d] = mn(mn+1)$. Implies $2a + (mn-1)d = 2(mn+1)$. Thus $d = \frac{2(mn+1)-2a}{mn-1}$. Here $a = 3$ is the minimal value of weight which can be assigned to a vertex of degree two. For $a \geq 3, 0 < d < 2$. Therefore $d = 1$. When $d = 1$, $a = \frac{mn+3}{2}$. Since a is an integer, $mn + 3 \equiv 0 (mod 2)$. Therefore mn is odd. Thus both m and n are odd. □

The following theorem shows the existence of (a, d)-antimagic labeling in mC_n.

Theorem 2.6. *For $m \geq 1$ odd and $n \geq 3$ odd, mC_n is $(\frac{mn+3}{2}, 1)$-antimagic.*

Proof. Define the edge labeling $f : E(mC_n) \to \{1, 2, ..., mn\}$ by

$f(x_{i,1}x_{i,n}) = i$ for $1 \leq i \leq n$,

$$f(x_{m-i,j}x_{m-i,j+1}) = \begin{cases} \frac{mj+i+1}{2} & \text{for } i = 1, 3, ..., m-2 \text{ and } j = 2, 4, ..., n-1 \\ \frac{(j+1)m+i+1}{2} & \text{for } i = 0, 2, ..., m-1 \text{ and } j = 2, 4, ..., n-1, \end{cases}$$

$$f(x_{m-i,n-k}x_{m-i,n-k+1}) = \begin{cases} (mn+1) + \frac{i-km}{2} & \text{for } i = 0, 2, ..., m-1 \text{ and } k = 2, 4, ..., n-1, \\ (mn+1) + \frac{i-(k-1)m}{2} & \text{for } i = 1, 3, ..., m-2 \text{ and } k = 2, 4, ..., n-1. \end{cases}$$

Here the labeling f uses each integer $1, 2, ..., mn$ exactly once.
The weights of the vertices of mC_n under the labeling of f are given below.

$$w(x_{i,j}) = \begin{cases} f(x_{i,j}x_{i,j+1}) + f(x_{i,j-1}x_{i,j}) & \text{for } i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n-1, \\ f(x_{i,n-1}x_{i,n}) + f(x_{i,n}x_{i,1}) & \text{for } i = 1, 2, ..., m \text{ and } j = n, \end{cases}$$

$$w(x_{i,1}) = \begin{cases} \frac{mn+2m+i+2}{2} & \text{for } i = 1, 3, ..., m, \\ \frac{mn+3m+i+2}{2} & \text{for } i = 2, 4, ..., m-1, \end{cases}$$

$$\{w(x_{i,1})/i = 1, 2, ..., m\} = \{\frac{mn+3}{2} + m, \frac{mn+3}{2} + m + 1, ..., \frac{mn+3}{2} + 2m - 1\},$$

$$w(x_{i,n}) = \begin{cases} \frac{mn+m+i+2}{2} & \text{for } i = 1, 3, ..., m, \\ \frac{mn+i+1}{2} & \text{for } i = 2, 4, ..., m-1, \end{cases}$$

$\{w(x_{i,n})/i = 1, 2, ..., m\} = \left\{\frac{mn+3}{2}, \frac{mn+3}{2} + 1, ..., \frac{mn+3}{2} + m - 1, \right\}$ and

$\{w(x_{i,j})/i = 1, 2, ..., m$ and $j = 2, 3, ..., n-1\}$
$= \left\{\frac{mn+3}{2} + 2m, \frac{mn+3}{2} + 2m + 1, ..., \frac{mn+3}{2} + mn - 1\right\}.$

Each vertex of mC_n receives exactly one weight from $\left\{\frac{mn+3}{2}, \frac{mn+3}{2} + 1, ..., \frac{mn+3}{2} + mn - 1\right\}$. Hence for both $m \geq 1$ and $n \geq 3$ odd, mC_n is $(\frac{mn+3}{2}, 1)$-antimagic. □

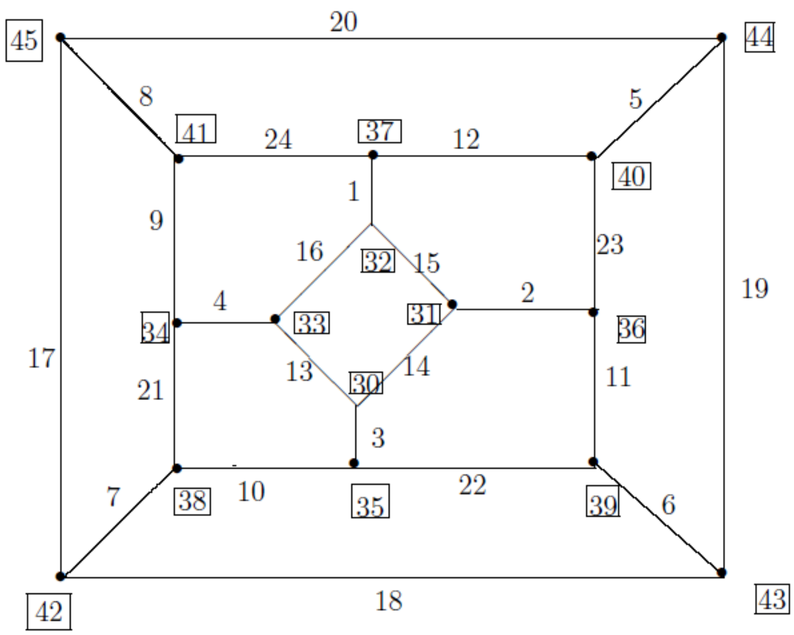

Figure 1. H_4 is $(30, 1)$-antimagic

3. Open Problems and Conjectures

Problems:

• Determine whether FL_n is $(\frac{5n-1}{2}, 2)$-antimagic or not for n is odd and $n \geq 5$.
• Determine whether H_n is $(3n + 3, 3)$-antimagic or not for $n \geq 3$.

From our experience on this labeling, we propose the following conjectures.

Conjectures:

• For n is even and $n \geq 6$, FL_n is not $(\frac{7n-2}{2}, 1)$-antimagic.
• For n is even and $n \geq 4$, FL_n is not $(\frac{3n}{2}, 3)$-antimagic.

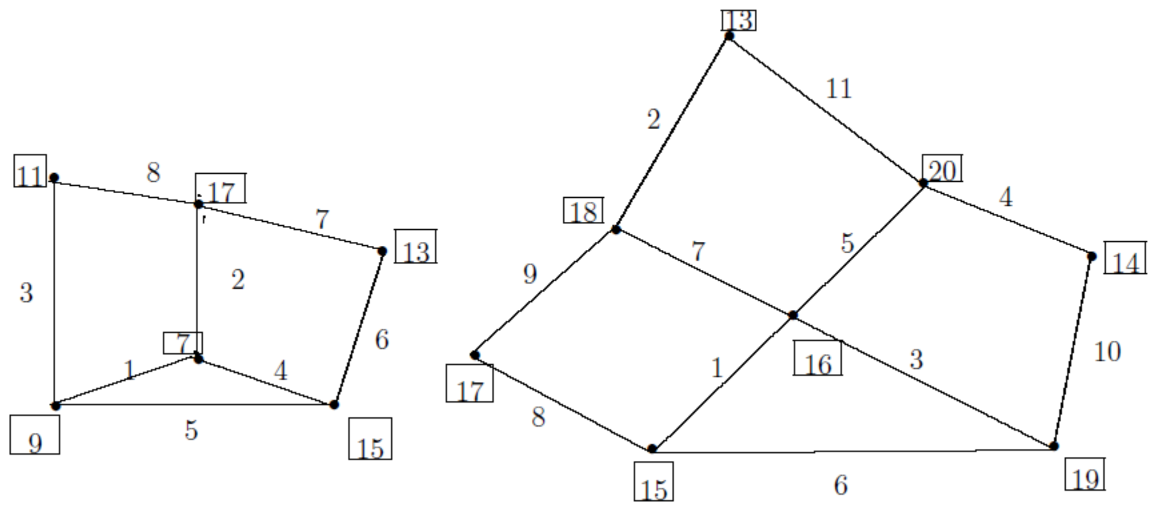

Figure 2. F_3 is $(7, 2)$-antimagic and F_4 is $(13, 1)$-antimagic

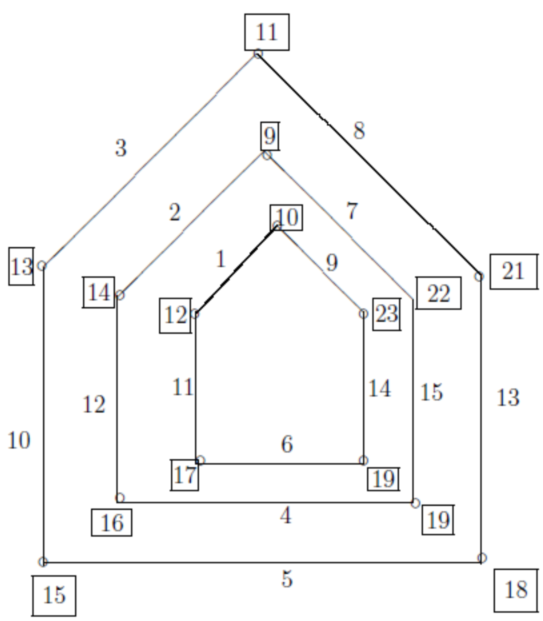

Figure 3. $3C_5$ is $(9, 1)$-antimagic

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