Indonesian Journal of Combinatorics **4** (1) (2020), 10–20

INDONESIAN JOURNAL OF COMBINATORICS

A note on edge irregularity strength of some graphs

I Nengah Suparta, I Gusti Putu Suharta

Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Pendidikan Ganesha
Jalan Udayana 11 Singaraja 81116 Bali, Indonesia

nengah.suparta@undiksha.ac.id, igp.suharta@undiksha.ac.id

Abstract

Let $G(V, E)$ be a finite simple graph and k be some positive integer. A vertex k-labeling of graph $G(V, E)$, $\phi : V \to \{1, 2, \ldots, k\}$, is called *edge irregular k-labeling* if the edge weights of any two different edges in G are distinct, where the edge weight of $e = xy \in E(G)$, $w_\phi(e)$, is defined as $w_\phi(e) = \phi(x) + \phi(y)$. The *edge irregularity strength* for graph G is the minimum value of k such that ϕ is irregular edge k-labeling for G. In this note we derive the edge irregularity strength of chain graphs $mK_3 - path$ for $m \not\equiv 3 (\text{mod } 4)$ and $C[C_n^{(m)}]$ for all positive integers $n \equiv 0 (\text{mod } 4)$ and m. We also propose bounds for the edge irregularity strength of join graph $P_m + \overline{K_n}$ for all integers $m, n \geq 3$.

Keywords: k-labeling, edge irregular k-labeling, edge irregularity strength for graph
Mathematics Subject Classification: 05C12, 05C78
DOI: 10.19184/ijc.2020.4.1.2

1. Introduction

Let $G(V, E)$ be a finite simple graph and k be some positive integer. A vertex k-labeling of graph $G(V, E)$, $\phi : V \to \{1, 2, \ldots, k\}$, is called *edge irregular k-labeling* if the edge weights of any two different edges in G are distinct, where the edge weight of $e = xy \in E(G)$, $w_\phi(e)$, is defined as $w_\phi(e) = \phi(x) + \phi(y)$. The *edge irregularity strength* for graph G is the minimum value of k such that ϕ is irregular edge k-labeling for G.

Received: 12 Nov 2019, Revised: 17 Feb 2020, Accepted: 13 April 2020.

Let $G(V, E)$ be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. We denote by xy the edge having end vertices x and y, with $x, y \in V(G)$. Let k be some positive integer. A vertex k-labeling of graph $G(V, E)$, $\phi : V \to \{1, 2, \ldots, k\}$, is called *edge irregular k-labeling* if the edge weights of any two different edges in G are distinct, where the edge weight of $e = xy \in E(G)$, $w_\phi(e)$, is defined as $w_\phi(e) = \phi(x) + \phi(y)$. The *edge irregularity strength* for graph G, $es(G)$, is the minimum value of k such that ϕ is irregular edge k-labeling for G.

Following [3], we mean a *block* of a graph is a maximal subgraph with no cut vertex. A graph H is called a *block-cut-vertex* graph of graph G if the vertices of H are blocks and cut-vertices of G, and two vertices in H are adjacent whenever one vertex is a block in G and the other one is a cut-vertex in G belonging to the block. A *chain graph* is a graph with blocks $B_1, B_2, \ldots, B_m$ such that for every i, B_i and B_{i+1} have a common vertex in such a way that the block-cut-vertex graph is a path. If $B_i = B_j = B$ for all $i, j \in \{1, 2, \ldots, m\}$, then the chain graph is denoted by $C[B^{(m)}]$. Furthermore, if B is identical to the complete graph of n vertices, $C[K_n^{(m)}]$ is frequently denoted by $mK_n - path$.

Edge irregularity strength for some graphs have been established. (See eg. [1], [4], [5]). In this note we derive the edge irregularity strength of chain graphs $mK_3 - path$ for $m \not\equiv 3 (\text{mod } 4)$ and $C[C_n^{(m)}]$ for all positive integers $n \equiv 0 (\text{mod } 4)$ and m. We also propose bounds for the edge irregularity strength of join graph $P_m + \overline{K_n}$ for all integers $m, n \geq 3$.

2. Main Results

Through out this paper, we restrict our discussion only for finite simple graph. Let G be a graph. For some vertex $v \in G$, $d(v)$ stands for the degree of the vertex v. The maximum degree of G, $\Delta(G)$, is defined as the maximum value of $d(v)$, $v \in G$. The label of vertex v will be frequently denoted by $l(v)$. Moreover, if x is a real number and s is the smallest integer such that $s \geq x$, then we write $s = \lceil x \rceil$.

The following lemma is for facilitating the proof of Theorem 2.1.

Lemma 2.1. *For any positive integer r, we have*

a. $\lceil \frac{3(2r+3)}{2} \rceil = 5 + 3r$.

b. $\lceil \frac{3(4r+6)}{2} \rceil = 9 + 6r$.

Proof. We will only prove the *a* part of the lemma. The *b* part is omitted.
The proof is carried out using mathematical induction principles for r.
For $r = 1$, by inspection we can see that the relation is true. Now assume the lemma is true for any positive integer $r = s$. Thus we have $\lceil \frac{3(2s+3)}{2} \rceil = 5 + 3s$. Consider $r = s + 1$. We have

$$\lceil \frac{3(2(s+1)+3)}{2} \rceil = \lceil \frac{3((2s+3)+2)}{2} \rceil = \lceil \frac{3(2s+3)}{2} + 3 \rceil = \lceil \frac{3(2s+3)}{2} \rceil + 3.$$

Based on the induction assumption, $\lceil \frac{3(2s+3)}{2} \rceil = 5 + 3s$, we obtain $\lceil \frac{3(2(s+1)+3)}{2} \rceil = 5 + 3(s + 1)$. Therefore, we may conclude that $\lceil \frac{3(2r+3)}{2} \rceil = 5 + 3r$ for all positive integer r. $\square$

We will start the main discussion with a fundamental theorem on edge irregularity strength of simple graphs.

Theorem 2.1 (Ahmad, Al-Mushayt, Băca [2]). *If $G(V, E)$ is a simple graph with maximum degree $\Delta(G)$, then $es(G) \geq max\{\lceil \frac{|E(G)|+1}{2} \rceil, \Delta(G)\}$.*

Regarding the irregularity strength of $mK_3 - path$, we present the following theorem which is due to Ahmad, Gupta, and Simanjuntak [1].

Theorem 2.2. *For any positive integer m, $\lceil \frac{3(m+1)}{2} \rceil \leq es(mK_3\text{-}path) \leq 2m + 1$.*

Following this theorem, they then propose an open problem below.

Open Problem 1. *For any positive integer m, determine the irregularity strength of mK_3-path.*

With respect to this problem, in this discussion we derive a partial solution for it, that is for $m \not\equiv 3$(mod 4). The following Subsection 2.1 will give explanation on how we derive the partial answer.

2.1. Irregularity Strength of $mK_3 - path$

As a motivation we will show some examples of some labeling of $mK_3 - path$ with their irregularity strengths: chain graph $4K_3 - path$ with $es(4K_3\text{-}path) = 8 = \lceil \frac{3(m+1)}{2} \rceil$(see Figure 1); chain graph $5K_3 - path$ with $es(5K_3 - path) = 9 = \lceil \frac{3(m+1)}{2} \rceil$(see Figure 2).

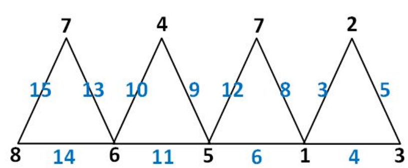

Figure 1. es($4K_3$-$path$) = 8

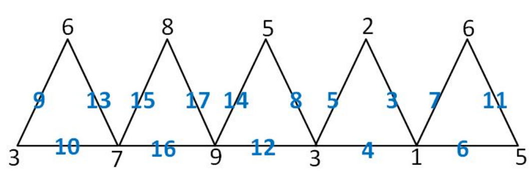

Figure 2. es($5K_3 - path$) = 9

In the following, we will show how we define an irregular labeling for $mK_3 - path$, $m \not\equiv 3$(mod 4), and show that $es(mK_3 - path) = \lceil \frac{3(m+1)}{2} \rceil$.

First, we denote the vertices of chain graph $mK_3 - path$ as we see in Figure 3. Thus, the chain graph $mK_3 - path$ has the following elements:

- $V(mK_3\text{-}path) = \{x_i : 1 \leq i \leq m + 1\} \cup \{y_j : 1 \leq j \leq m\}$, and

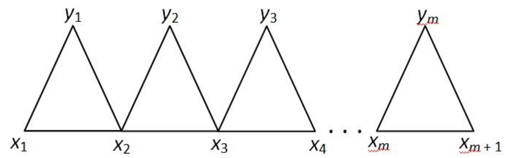

Figure 3. Vertex names for mK_3-*path*

- $E(mK_3\text{-}path) = \{x_i x_{i+1}, x_i y_i, x_{i+1} y_i : 1 \le i \le m\}$.

We will proceed using mathematical induction principles for m. We consider two cases on m: m even and $m \equiv 1 (\text{mod } 4)$.
Case m even.
First we introduce two $2K_3 - paths$ having different irregular labeling. The one in Figure 4 we call as *adder A* and the other in Figure 5 we call as *adder B*.

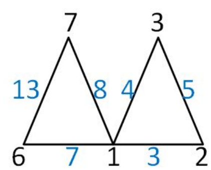

Figure 4. Adder A

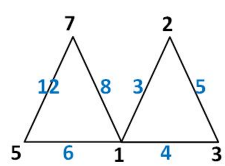

Figure 5. Adder B

Remark From these two adders we have the following important observations with respect to inductive process: If we add by 3 all vertex labels of adder A (resp. adder B), then from the resulting labeling we get that $l(x_3)$ of adder A(resp. adder B) is the same as $l(x_1)$ of adder B(resp. adder A). Then we identify these two vertices x_3 of adder A(adder B) and x_1 of adder B(adder A), to have $4K_3 - path$ with an irregular labeling. This remark is indeed needed for concluding the labeling irregularity of the resulting graph through mathematical induction process. We will call a *derivation graph* for the resulting graph which is obtained by adding all vertex labels of graph with the same constant (in this instance the constant is 3).

Furthermore, we create a seed graph $2K_3 - path$ as is shown in Figure 6, for the commencement of inductive process. Here $m = 2$, and we can immediately see that this graph has edge irregular labeling with $es(2K_3 - path) = 5 = \lceil \frac{3(m+1)}{2} \rceil$. The next process of induction is conducted as follows. All labels of this seed graph $2K_3 - path$ are added up by constant 3. The resulting graph $2K_3 - path$ will have $l(x_3) = 2 + 3 = 5$. It is easy to see that the irregularity of induced edge labels

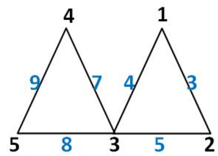

Figure 6. $2K_3$-$path$ with $es(2K_3 - path) = 5$

are maintained for the derivation graph, since all vertex labels increase to the same constant 3. This irregularity property always holds any time we produce derivation graphs. Then we identify vertex x_{m+1} from the derivation of $mK_3 - path$ with vertex x_1 of adder A if $l(x_{m+1}) = 6$ or of adder B if $l(x_{m+1}) = 5$. For example, on the derivation of $2K_3 - path$ we have that $l(x_{2+1}) = l(x_3) = 2 + 3 = 5$. This label is the same as $l(x_1)$ of adder B. Thus, we identify vertex x_3 of the derivation of $2K_3-path$ with x_1 of adder B. The resulting chain graph $4K_3-path$ has edge irregular labeling with $es(4K_3 - path) = 8 = \lceil\frac{3(4+1)}{2}\rceil$. This chain graph is shown in Figure 1.

Now observe the derivation of the resulting $4K_3 - path$. Since the rightmost two blocks of the resulting chain graph $4K_3 - path$ are adder B, as described in the remark, we can identify vertex x_5 from the derivation of $4K_3 - path$ with vertex x_1 of adder A. The resulting chain graph $6K_3 - path$ is shown in Figure 7.

Let $m = 2l$ for some positive integer l. Continue this identifying process to produce $(m + 2)K_3-path$ from the derivation of mK_3-path and adder A or from the derivation of mK_3-path and adder B as follows: If l is even, we identify vertex x_{m+1} from the derivation of mK_3-path with vertex x_1 of adder A, and if l is odd, we identify vertex x_{m+1} from the derivation of $mK_3 - path$ with vertex x_1 of adder B.

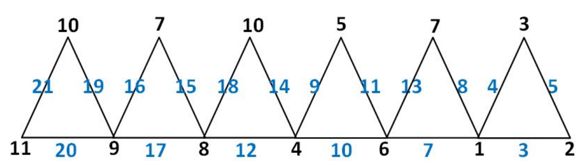

Figure 7. $6K_3 - path$ with $es(6K_3$-$path) = 11$

Let r be the number of times we repeat identification process for producing chain graph $(2 + 2r)K_3 - path$. We see that each identification process results in the increase of irregularity strength by 3. Since the seed chain graph $2K_3 - path$ has $es(2K_3 - path) = 5$, then we get that $es((2 + 2r)K_3 - path) = 5 + 3r$ which is by Lemma 2.1, equal to $\lceil\frac{3(2r+3)}{2}\rceil = \lceil\frac{3((2r+2)+1)}{2}\rceil = \lceil\frac{3(m+1)}{2}\rceil$. Therefore, we can conclude that for case m even, we obtain that $es(mK_3 - path) = \lceil\frac{3(m+1)}{2}\rceil$.

Now we discuss for case $m \equiv 1(\text{mod } 4)$.
It is clear that $es(K_3 - path) = 3$. So, for $m = 1$ we have $es(mK_3 - path) = 3 = \lceil\frac{3(m+1)}{2}\rceil$. For $m = 5$, an irregular labeled chain graph $5K_3 - path$ is shown in Figure 2. Observe that all induced edge labels of this chain graph take all integers from 3 up until 17. This indicates that the largest

vertex label is the irregularity strength of the graph. Thus, $es(5K_3 - path) = 9$.

We will use this chain graph as the seed graph to construct an irregular labeling for chain graph $mK_3 - path$ with $m \geq 5$. Note that $l(x_6) = 5$.

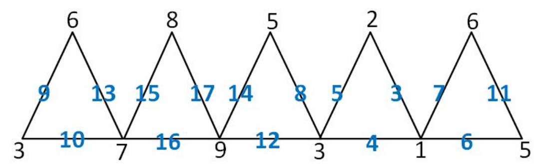

Figure 8. $es(5K_3$-$path)$ = 9

Furthermore, we construct a chain graph $4K_3 - path$, with irregular labeling as is shown in Figure 9. Observe that the induced edge labels of this adder C also run from 3 up to 14, and that $l(x_1) = 11$ and $l(x_5) = 5$.

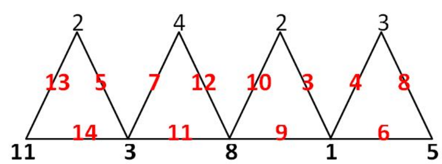

Figure 9. Adder C

The technique we use to produce bigger $mK_3 - path$ is the same as the technique we applied for m even. First add by 6 to all vertex labels of $mK_3 - path$, $m \geq 5$. The resulting derivation graph will also have irregular labeling with the smallest edge label equals $3+6+6 = 15$. This edge label is the successor of the largest edge label of adder C which is equal to $l(x_1x_2) = 14$. Moreover, we have $l(x_{m+1}) = 5 + 6 = 11$ in the derivation graph which is the same as $l(x_1)$ in adder C. We identify these two vertices to produce $(m+4)K_3 - path$ having irregular labeling. For $m = 5$, we have $es((m+4)K_3 - path) = 9 + 6 = 15$ as is shown in Figure 10. Continuing this identification process, we see that the resulting chain graph $(m + 4)K_3 - path$ has $l(x_{m+1}) = l(x_{10}) = 5$ because this comes from $l(x_5)$ of adder C. Continuing this process we will have irregular labeling for $mK_3 - path$, with $m \geq 5$.

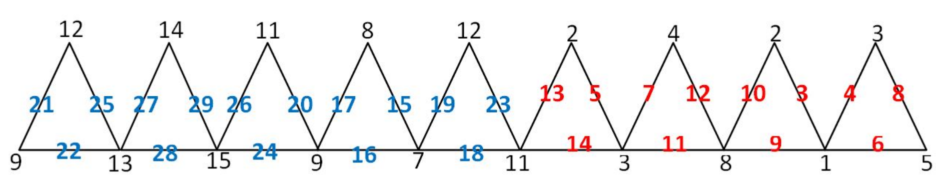

Figure 10. Chain graph $9K_3$-$path$ with $es(9K_3$-$path) = 15$.

The irregularity strength of the chain graph mK_3-path is derived as follows. We know already that $es(5K_3-path) = 9$. Let r stand for the number of times we do identification processes. Thus, after conducting r times identification processes, we have $m = 5 + 4r$ and $es(mK_3 - path) = 9 + 6r$. Using Lemma 2.1 we can conclude that $es(mK_3 - path) = 9 + 6r = \lceil\frac{3(4r+6)}{2}\rceil = \lceil\frac{3(m+1)}{2}\rceil$. Therefore, for case $m \equiv 1(\text{mod } 4)$ we also have that $es(mK_3 - path) = \lceil\frac{3(m+1)}{2}\rceil$.

Based on this above observation we proved already the following theorem.

Theorem 2.3. *For all positive integers $m \not\equiv 3$ (mod 4), $es(mK_3\text{-}path) = \lceil\frac{3(m+1)}{2}\rceil$.*

Based on this result, with respect to irregularity strength of $mK_3 - path$, we have the following reduced open problem instead of the one proposed by Ahmad, Gupta, and Simanjuntak in [1].

Open Problem 2. *For any positive ingeter $m \equiv 3(\text{mod } 4)$, determine the $es(mK_3 - path)$.*

2.2. Irregularity strength of $C[C_n^{(m)}]$ for $n \equiv 0$(mod 4)

In [1] it is shown that for cycle graph of 4 vertices, C_4, the irregularity strength of chain graph $C[C_4^{(m)}]$ is $2m + 1$. Then, from this fact they proposed the following conjecture.

Conjecture 1. *For $m \geq 2, n \geq 5$, the edge irregularity strength of $C[C_n^{(m)}]$ is $\lceil\frac{mn+1}{2}\rceil$.*

Here we also address this formulated conjecture, and introduce a solution for $n \equiv 0(\text{mod } 4)$ as we describe below. First we name vertices of $C[C_n^{(m)}]$ as shown in Figure 11. Therefore, the graph $C[C_n^{(m)}]$ has elements:

$$\begin{aligned} V(C[C_n^{(m)}]) &= \{x_0, y_0\} \cup \{x_1^i, x_2^i, \ldots, x_{n-2}^i : 1 \leq i \leq m\} \\ &\cup \{c_1, c_2, \ldots, c_{m-1}\}, \text{ and} \\ E(C[C_n^{(m)}]) &= \{c_i x_{n-3}^i, c_i x_{n-2}^i, c_i x_1^{i+1}, c_i x_2^{i+1} : 1 \leq i \leq m-1\} \\ &\cup \{x_j^i x_{j+2}^i : 1 \leq i \leq m; 1 \leq j \leq n-4\} \\ &\cup \{x_0 x_i^1, x_0 x_2^1, y_0 x_{n-3}^m, y_0 x_{n-2}^m\}. \end{aligned}$$

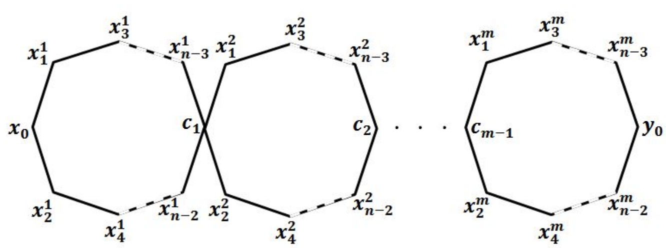

Figure 11. Chain graph $C[C_n^{(m)}]$ with its vertex labels

Furthermore, we will show that for all $n \equiv 0(\text{mod } 4)$ and all positive integer m, $es(C[C_n^{(m)}]) = \lceil\frac{mn+1}{2}\rceil$. Since n is even, we have that $\lceil\frac{mn+1}{2}\rceil = \frac{mn}{2} + \lceil\frac{1}{2}\rceil = \frac{mn}{2} + 1$. This is formulated as the following theorem.

Theorem 2.4. *For all $n \equiv 0(\text{mod } 4)$ and all positive integer m, we have $es(C[C_n^{(m)}]) = \frac{mn}{2} + 1$.*

Proof. To this, we label graph $C[C_n^{(m)}]$ using the following function

$$\begin{aligned} f(x_0) &= 1, \\ f(y_0) &= \tfrac{mn}{2} + 1, \\ f(c_i) &= \tfrac{ni}{2} + 1 \text{ for } 1 \le i \le m-1, \end{aligned}$$

and for $1 \le i \le m$,

$$f(x_j^i) = \begin{cases} \frac{n}{2}(i-1) + \frac{j+1}{2} & j = 1, 5, \ldots, N^1, \\ \frac{n}{2}(i-1) + \frac{j+2}{2} & j = 2, 4, \ldots, n-2, \\ \frac{n}{2}(i-1) + \frac{j+3}{2} & j = 3, 7, \ldots, N^3. \end{cases}$$

where $N^1 := max\{s \in \mathbb{Z}^+ : s \le n-3, s \equiv 1(\text{mod } 4)\}$ and $N^3 := max\{s \in \mathbb{Z}^+ : s \le n-3, s \equiv 3(\text{mod } 4)\}$.

Note that the largest vertex label is equal to $f(y_0) = \frac{mn}{2} + 1$. So, we need only to show that the labeling function f gives irregular labeling for $C[C_n^{(m)}]$. This will be completed as the following.

Let us see the case $m = 1$.
Using the above labeling function f, we can see that:

1) $\{f(x_0) = 1\}$,
2) $\{f(x_j^1) : j = 2, 4, \ldots, n-2\} = \{2, 3, \ldots, \frac{n}{2}\}$,
3) $\{f(x_j^1) : j = 1, 5, \ldots, N^1\} = \{1, 3, 5, \ldots, \frac{N^1+1}{2}\}$,
4) $\{f(x_j^1) : j = 3, 7, \ldots, N^3\} = \{3, 5, 7, \ldots, \frac{N^3+3}{2}\}$, and
5) $f(y_0) = \frac{n}{2} + 1$.

Recall that

$$\begin{aligned} E(C[C_n^{(m)}]) = \; & \{c_i x_{n-3}^i, c_i x_{n-2}^i, c_i x_1^{i+1}, c_i x_2^{i+1} : 1 \le i \le m-1\} \\ & \cup \{x_j^i x_{j+2}^i : 1 \le i \le m; 1 \le j \le n-4\} \\ & \cup \{x_0 x_i^1, x_0 x_2^1, y_0 x_{n-3}^m, y_0 x_{n-2}^m\}. \end{aligned}$$

So, the edge labels of $C[C_n^{(1)}]$, $L(C[C_n^{(1)}])$, is equal to

$$\begin{aligned} L(C[C_n^{(1)}]) &= \emptyset \cup \{4, 5, 6, \ldots, n-2, n-1\} \cup \{2, 3, n, n+1\} \\ &= \{2, 3, \ldots, n-2, n-1, n, n+1\}. \end{aligned}$$

From this last equation, we conclude that the function f is irregular labeling for the case $m = 1$.

Now let us see for case $m \ge 2$.
If we rename x_0 and y_0 as c_0 and c_m, respectively, then we see that $f(c_i) = f(c_{i-1}) + \frac{n}{2}$, with $1 \le i \le m$; and $f(x_j^i) = f(x_j^{i-1}) + \frac{n}{2}$, with $2 \le i \le m$. Since the function f for $C[C_n^{(1)}]$ is irregular, we may conclude that f is irregular labeling for all positive integer m and all positive integer $n \equiv 0(\text{mod } 4)$. Thus we may conclude that the theorem is proved. □

We mention again that this result confirms a partial portion of above Conjecture 1. Therefore, the remaining problem now is as the following conjecture.

Conjecture 2. *For potive integers $m \geq 2, n \geq 5, n \not\equiv 0$(mod 4), the edge irregularity strength of $C[C_n^{(m)}]$ is $\lceil \frac{mn+1}{2} \rceil$.*

2.3. Irregularity strength for $P_m + \overline{K_n}$

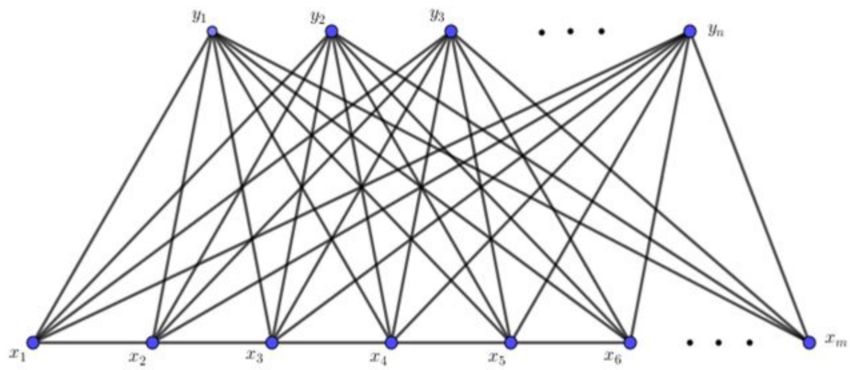

Figure 12. Vertex names for $P_m + \overline{K_n}$

Now we will proceed to address irregularity strength of join graph $P_m + \overline{K_n}$ which is also discussed in [1] for certain cases. Some bounds for the irregularity strength of this graph for all integers $n \geq 3$ and $3 \leq m \leq 6$ were proposed in [1].

In this paper, we derive bounds for the irregularity strength of the graph for all integers $m \geq 3$ and $n \geq 3$. We will see later that for $3 \leq m \leq 6$ and $n \geq 3$, our bounds are the same as in [1]. Therefore, our bounds can be considered as some extension of those in [1].

Before we formulate the bounds, we will start the process by firstly naming vertices of graph $P_m + \overline{K_n}$ as in Figure 12. So, the graph has elements as follows.

$$\begin{aligned} V(P_m + \overline{K_n}) &= \{x_1, x_2, \ldots, x_m\} \cup \{y_1, y_2, \ldots, y_n\}, \text{ and} \\ E(P_m + \overline{K_n}) &= \{x_i x_{i+1} : i \in \{1, 2, \ldots, m-1\}\} \\ &\quad \cup \{x_i y_j : i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}\} \end{aligned}$$

Theorem 2.5. *For positive integers $m, n \geq 3$, we have*

$$\lceil \tfrac{m(n+1)}{2} \rceil \leq es(P_m + \overline{K_n}) \leq \begin{cases} \frac{(2n+1)m+1}{2} - n, \text{for } m \text{ odd} \\ \\ \frac{(2n+1)m+2}{2} - n, \text{for } m \text{ even.} \end{cases}$$

Proof. For the lower bound follows the result which is formulated in Theorem 2.1. Here we only discuss the upper bound in the theorem. Consider again the diagram of graph $P_m + \overline{K_n}$ as is shown in Figure 12. Now we introduce the following labeling function for $P_m + \overline{K_n}$.

$f(y_j) = j + 1, j = 1, 2, \ldots, n,$

and for $i = 1, 2, \ldots, m,$

$$f(x_i) = \begin{cases} \frac{(2n+1)i+1}{2} - n, \text{for } i \text{ odd} \\ \\ \frac{(2n+1)i+2}{2} - n, \text{for } i \text{ even.} \end{cases}$$

Observe that, by a simple calculation, for $1 \leq i \leq m-1$, we have $f(x_{i+1}) - f(x_i) = n + 1$ if i is odd, and $f(x_{i+1}) - f(x_i) = n$ if i is even. Thus, for every $i, 1 \leq i \leq m-2$, we have $f(x_{i+2}) = f(x_i) + 2n + 1$. This implies that

$$\begin{aligned} f'(x_{i+2}y_j) = &\quad f'(x_iy_j) + 2n + 1, \text{ with } i = 1, 2, \ldots, m-2, j = 1, 2, \ldots, n, \\ f'(x_{i+1}x_i) = &\qquad\qquad f'(x_ix_{i-1}) + 2n + 1 \text{ with } i = 1, 2, \ldots, m-1, \end{aligned} \tag{1}$$

where $f'(xy) = f(x) + f(y), xy \in E(P_m + \overline{K_n})$.

Furthermore, we see also that $f'(x_1y_j) = f(x_1) + f(y_j) = 1 + (j+1) = j + 2$, $f'(x_2y_j) = f(x_2)+f(y_j) = (n+1)+(j+1) = n+2+j$, and $f'(x_1x_2) = f(x_1)+f(x_2) = 1+(n+2) = n+3$. So, here we have

$$\{f'(x_1y_j), f'(x_1x_2), f'(x_2y_j) : j = 1, 2, \ldots, n\} = \{3, 4, \ldots, n+2, n+3, n+4, \ldots, 2n+2\},$$

which are all distinct.

Moreover, since we have Eqn (1), then we may conclude that all induced edge labels of $P_m+\overline{K_n}$ are distinct, and therefore f is irregular labeling. The largest vertex label is $f(x_m) = \frac{(2n+1)m+1}{2} - n$ if m is odd, or $f(x_m) = \frac{(2n+1)m+2}{2} - n$ if m is even. Hence, the theorem is proved. □

We inform here that for all $n \geq 3$ and $3 \leq m \leq 6$, our bound meets the bound proposed in [1]. Therefore, our result can be considered as an extension result of the formulated result in [1].

3. Conclusion

We derived partial solutions for open problems and conjectures proposed in [1]. The remaining problems are still worth to be observed. Especially for the irregularity strength of $P_m + \overline{K_n}$, we see that the gap between the lower and the upper bounds is big enough. It seems to be possible to narrow the gap.

Acknowledgement

The authors thank to LP2M Undiksha for the fund allocated for this work under contract no. SP DIPA-024.01.2.400987/2019. The authors also thank to Monik, Pande, dan Satya for the discussions during the preparation of this results. I also express my gratitude to anonymous referees for their useful notions.

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