New families of star-supermagic graphs

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Abstract

A simple graph $G$ admits a $K_{1,n}$-covering if every edge in $E(G)$ belongs to a subgraph of $G$ isomorphic to $K_{1,n}$. The graph $G$ is $K_{1,n}$-supermagic if there exists a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, |V(G) \cup E(G)|\}$ such that for every subgraph $H'$ of $G$ isomorphic to $K_{1,n}$, $\sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$ is a constant and $f(V(G)) = \{1, 2, 3, \ldots, |V(G)|\}$. In such a case, $f$ is called a $K_{1,n}$-supermagic labeling of $G$. In this paper, we give a method how to construct $K_{1,n}$-supermagic graphs from the old ones.

Keywords: $K_{1,n}$-covering, $K_{1,n}$-supermagic labeling, $K_{1,n}$-supermagic graph

Mathematics Subject Classification: 05C78

DOI: 10.19184/ijc.2020.4.2.4

1. Introduction

In this paper, we consider finite and simple graphs $G$ with the vertex and edge sets $V(G)$ and $E(G)$, respectively. The number of vertices (edges) in the graph $G$ is called order (size) of $G$. Let $H$ be a given graph. An edge-covering of $G$ is a family of subgraphs $H_1, \ldots, H_k$ such that each edge in $E(G)$ belongs to at least one of the subgraphs $H_i$, $1 \leq i \leq k$. Then it is said that $G$ admits an $(H_1, \ldots, H_k)$-edgecovering. If every $H_i$, $1 \leq i \leq k$, is isomorphic to the graph $H$, then $G$ admits an $H$-covering. Suppose $G$ admits an $H$-covering. A total labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, |V(G) \cup E(G)|\}$ is called an $H$-magic labeling of $G$ if for every subgraph $H'$ of $G$ isomorphic to $H$, $\sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) = c_f$ is a constant. The constant $c_f$ is called magic constant of the labeling $f$. An $H$-magic labeling $f$ is called an

Received: 27 December 2019, Revised: 13 December 2020, Accepted: 17 December 2020.
$H$-supermagic labeling if $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$. A graph that admits $H$-(super)magic labelings is called $H$-(super)magic. In this paper, we consider such a labeling when $H$ is a star $K_{1,n}$.

The $H$-(super)magic labeling was first introduced and studied by Gutiérrez and Lladó [3] in 2005 where $H$-supermagic labelings for stars, complete bipartite graphs, paths and cycles are considered. In [7], Lladó and Moragas studied $C_n$-supermagic labeling of some graphs. They proved that the wheel $W_n$, the windmill $W(r, k)$, and the prism $C_n \times P_2$ are $C_h$-supermagic for some $h$. Cycles-supermagic labeling of chain graphs $kC_n$-path, triangle ladders $TL_n$, grids $P_m \times P_n$, for $n = 2, 3, 4, 5$, fans $F_n$, and books $B_n$ can be found in [8]. The complete results on these labelings can be found in [2].

For $H \cong P_2$, an $H$-supermagic graph is also called a super edge-magic graph. The notion of a super edge-magic graph was introduced by Enomoto at al [1] as a particular type of edge-magic graph given by Rosa [5]. For further information about (super) edge-magic graphs, see [2]. The $H$-magic labeling is related to a face-magic labeling of a plane graph introduced by Lih [6]. A total labeling $f$ of a plane graph is said to be face-magic if for every positive integer $s$, all $s$-sided faces have the same weight. The weight of a face under the labeling $f$ is the sum of labels carried by the edges and vertices surrounding it. Lih [6] allows different weights for different $s$. When a plane graph $G$ contains only $n$-sided faces then face-magic labeling of $G$ is also $C_n$-magic labeling. Other results about this labeling can be found in, for instance, [2].

In this paper, we give a method how to construct star-supermagic graphs from the old ones. Based on this, we have new families of star-supermagic graphs.

2. The Results

In this section, we propose a method for constructing new star-supermagic graphs from certain star-supermagic graphs. To do this, we need the following notations. The sum of all vertex and edge labels on $H$ (under a labeling $f$) is denoted by $\sum f(H)$. For any two integers $n < m$, the set of all consecutive integers from $n$ to $m$ is denoted by $[n, m]$. For any set $X \subseteq \mathbb{N}$, the set of natural numbers, we write $\Sigma X = \sum_{x \in X} x$. For any integer $k$, $X + k = \{x + k : x \in X\}$. Thus $k + [n, m]$ is the set of consecutive integers from $k + n$ to $k + m$. It is easy to check that $\Sigma(X + k) = k|X| + \Sigma X$. Furthermore, we also need the concept of a $k$-balanced set. $\mathbb{P} = \{X_1, X_2, \ldots, X_k\}$ is said to be an equipartition of a set of integers $X$ if $X_1, X_2, \ldots, X_k$ are non-empty disjoint subsets of $X$ whose union is $X$ and, for $i \in [1, k]$, $|X_i| = \frac{|X|}{k}$. The set $X$ is said to be $k$-balanced if there exists an equipartition $\mathbb{P} = \{X_1, X_2, \ldots, X_k\}$ of $X$ with the property that $\Sigma X_i = \frac{\Sigma X}{k}$, $i \in [1, k]$.

**Lemma 2.1.** For any positive integers $k$ and $m$, the set $X = [1, 2km]$ is $k$-balanced.

**Proof.** For every $i \in [1, k]$, define $A_i = [(i - 1)m + 1, im]$ and $B_i = km + A_{k+1-i}$. For every $i \in [1, k]$, let $C_i = A_i \cup B_i$. It can be checked that for $i \neq j$, $C_i \cap C_j = \emptyset$, $\bigcup_{i=1}^{k} C_i = X$, and for $i \in [1, k]$, $|C_i| = 2m$. So, $\mathbb{P} = \{C_1, C_2, \ldots, C_k\}$ is an equipartition of $X$. Furthermore, for every $i \in [1, k]$, it can be checked that $\sum C_i = m(2km + 1)$. Thus, $X$ is $k$-balanced.

For example, let $k = m = 3$, and thus $X = [1, 18]$. Then $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5, 6\}$, and $A_3 = \{7, 8, 9\}$. $B_1 = \{16, 17, 18\}$, $B_2 = \{13, 14, 15\}$, and $B_3 = \{10, 11, 12\}$. The
equipartition subsets of $X$ are $C_1 = \{1, 2, 3, 16, 17, 18\}$, $C_2 = \{4, 5, 6, 13, 14, 15\}$, and $C_3 = \{7, 8, 9, 10, 11, 12\}$. Here, $\sum C_1 = \sum C_2 = \sum C_3 = 57$.

**Corollary 2.1.** For any positive integers $k$, $m$, and $p$, the set $Y = [p + 1, 2km + p]$ is $k$-balanced.

**Proof.** An equipartition of $Y$ is $\{D_1, D_2, \ldots, D_k\}$, where $D_i = p + C_i$, $i \in [1, k]$, and $C_i$ is defined as in the proof of Lemma 2.1. \hfill $\square$

**Theorem 2.1.** Let $G$ be a graph with of order $p$ and size $q$ edges and admits a $K_{1, \Delta(G)}$-covering, where $\Delta(G)$ is maximum degree of $G$. Let $H$ be a graph formed from $G$ by attaching $m \geq 1$ pendants to every vertex $v$ of $G$ whose degree $\text{deg}(v) = \Delta(G)$. If $G$ is $K_{1, \Delta(G)}$-supermagic, then $H$ is $K_{1, \Delta(G)+m}$-supermagic.

**Proof.** Let $G$ be a $K_{1, \Delta(G)}$-supermagic graph with a $K_{1, \Delta(G)}$-supermagic labeling $f$. Let $v_1, v_2, \ldots, v_k$ be vertices of $G$ such that $\text{deg}(v_i) = \Delta(G)$, $i \in [1, k]$. Then, for every $i \in [1, k]$ we have

$$c_f = f(v_i) + \sum_{u \in N(v_i)} f(u) + \sum_{u \in N(v_i)} f(uv_i),$$

where $N(v_i) = \{u : uv_i \in E(G)\}$.

Next, define $H$ as a graph with

$$V(H) = V(G) \cup \{v_i^j : i \in [1, k], j \in [1, m]\},$$

$$E(H) = E(G) \cup \{v_i^jv_l^j : i \in [1, k], j \in [1, m]\}.$$ 

Thus, $H$ is a graph of order $p + km$ and size $q + km$. Additionally, $H$ is a graph with maximum degree $\Delta(G) + m$. Since $G$ admits a $K_{1, \Delta(G)}$-covering and based on how $H$ is constructed, then $H$ admits a $K_{1, \Delta(G)+m}$-covering.

Let $U_1 = [1, p]$, $U_2 = [p + 1, 2km + p]$, and $U_3 = [2km + p + 1, 2km + p + q]$. So, $U_1, U_2, U_3$ is a partition of $[1, 2km + p + q]$. By corollary 1, the set $U_2 = [p + 1, 2km + p]$ is $k$-balanced. For every $i \in [1, k]$, let $D_i$ be balanced subsets of $U_2$, where $D_i$ is defined as in the proof of Lemma 2.

Next, define a total labeling

$$g : V(H) \cup E(H) \rightarrow [1, p + q + 2km]$$

as follows.

$$g(x) = \begin{cases} f(x), & \text{for } x \in V(G), \\ 2km + f(x), & \text{for } x \in E(G). \end{cases}$$

Under the labeling $g$, $g(V(G)) = [1, p]$ and $g(E(G)) = [2km + p + 1, 2km + p + q]$. Label the remaining $2km$ pendant vertices and $2km$ pendant edges of $H$, as follows. For $i \in [1, k]$, label $\{v_i^j : j \in [1, m]\} \cup \{v_i^jv_l^j : j \in [1, m]\}$ with the elements of $D_i$ such that the label of $v_i^j$ less than the label of $v_i^jv_l^j$. Thus, under the labeling $g$, $g(V(H)) = [1, p + km]$ and $g(E(H)) = [km + p + 1, p + q + 2km].$
Next, we show that $g$ is a $K_{1,\Delta(G)+m}$-supermagic labeling of $H$. For every $i \in [1,k]$, 
\[
c_g = g(v_i) + \sum_{u \in N(v_i)} g(u) + \sum_{u \in N(v_i)} g(uv_i) \\
= g(v_i) + \sum_{u \in N(v_i) \cap V(G)} g(u) + \sum_{u \in N(v_i) \cap V(G)} g(uv_i) \\
+ \sum_{j=1}^{m} g(v_i^j) + \sum_{j=1}^{m} g(v_i^j) \\
= f(v_i) + \sum_{u \in N(v_i)} f(u) + \sum_{u \in N(v_i)} [2km + f(uv_i)] \\
+ \sum_i D_i \\
= c_f + (2k\Delta(G) + 2p + 1)m + 2km^2.
\]

Hence, $g$ is a $K_{1,\Delta(G)+m}$-supermagic labeling of $H$. So, $H$ is a $K_{1,\Delta(G)+m}$-supermagic graph. \hfill \square

Illustrations of Theorem 2.1 for case $\Delta(G) = 2$, $p = k = 5$, and $m = 1$ is given in Figure 1, and for case $\Delta(G) = 2$, $p = 7$, $k = 5$, and $m = 2$ is given in Figure 2.

![Figure 1](image1.png)

Figure 1. (a) The $K_{1,2}$-supermagic labeling of $C_5$ with the magic constant 25. (b) The $K_{1,3}$-supermagic labeling of the graph which is obtained by attaching a pendant to every vertex of $C_5$ with the magic constant 66.

![Figure 2](image2.png)

Figure 2. (a). A $K_{1,2}$-supermagic labeling of $P_7$ with the magic constant 34. (b) A $K_{1,4}$-supermagic labeling of a caterpillar which is formed by attaching two pendants to every vertices of $P_7$ except the pendants vertices with the magic constant 144.

In [3], Gutiérrez and Lladó proved the following results. The cycle $C_n$ is $P_t$-supermagic for any $t \in [2, n - 1]$ such that $gcd(n, t(t - 1)) = 1$, and $P_n$ is $P_h$-supermagic for every $h \in [2, n]$. 

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Proof.

Define a vertex labeling \( f_1 : V(G_k) \rightarrow [1, 3k + 5] \) as follows.

\[
\begin{align*}
  f_1(u) &= \begin{cases} 
    i, & \text{if } u = x_i, \ i \text{ is odd}, \\
    i, & \text{if } u = y_i, \ i \text{ is odd}, \\
    \frac{i}{3}(3k + 8 - i), & \text{if } u = x_i, \ i \text{ is even, } k \text{ is even}, \\
    \frac{i}{3}(4k + 9 - i), & \text{if } u = y_i, \ i \text{ is odd, } k \text{ is even}, \\
    \frac{i}{3}(4k + 10 - i), & \text{if } u = x_i, \ i \text{ is even, } k \text{ is odd}, \\
    \frac{i}{3}(3k + 8 - i), & \text{if } u = y_i, \ i \text{ is odd, } k \text{ is odd}, \\
    3k + 6 - i, & \text{if } u = c_i, \ i \in [1, k + 1].
  \end{cases}
\end{align*}
\]

In particular, they proved that the cycle \( C_n \) is \( P_3 \cong K_{1,2} \)-supermagic for any \( n > 3 \) such that \( \gcd(n, 6) = 1 \), and \( P_n \) is \( P_3 \)-supermagic for every \( n \geq 3 \). As a consequence of these results and Theorem 2.1, we have the following corollaries.

**Corollary 2.2.** For any \( n > 3 \) such that \( \gcd(n, 6) = 1 \), and \( m \geq 1 \), the corona product of \( C_n \) and \( mK_1 \), \( C_n \circ mK_1 \), is a \( K_{1,m+2} \)-supermagic graph.

**Corollary 2.3.** For \( n \geq 3 \) and \( m \geq 1 \), the caterpillar formed by attaching \( m \) pendant edges to every vertex of degree two of the path \( P_n \) is a \( K_{1,m+2} \)-supermagic graph.

The open problem related to the \( K_{1,m+2} \)-supermagic labeling of \( C_n \circ mK_1 \) is as follows.

**Problem 1.** For any \( n > 3 \) such that \( \gcd(n, 6) \neq 1 \), and \( m \geq 1 \), determine whether there is a \( K_{1,m+2} \)-supermagic labeling of \( C_n \circ mK_1 \).

In [4], Jeyanthi and Selvagopal proved the following results.

**Theorem 2.2.** [4] Let \( H_1, H_2, \ldots, H_n \) be \( n \) disjoint copies of star \( K_{1,n} \) and \( G_1 \) be the graph obtained by joining a new vertex to a pendant vertex of \( H_i \), \( i \in [1, n] \). Then \( G_1 \) is a \( K_{1,n} \)-supermagic graph.

**Theorem 2.3.** [4] Let \( H_1, H_2, \ldots, H_{n+1} \) be \( n + 1 \) disjoint copies of star \( K_{1,n} \) and \( G_2 \) be the graph obtained by joining a new vertex to the center vertex of \( H_i \), \( i \in [1, n + 1] \). Then \( G_2 \) is a \( K_{1,n+1} \)-supermagic graph.

Again, as a consequence of these results and Theorem 2.1, we have the following corollaries.

**Corollary 2.4.** For \( m \geq 1 \), the graphs \( G_1^* \) formed by attaching \( m \) pendant edges to every vertex of degree \( n \) of \( G_1 \) is a \( K_{1,n+m} \)-supermagic graph.

**Corollary 2.5.** For \( m \geq 1 \), the graphs \( G_2^* \) formed by attaching \( m \) pendant edges to every vertex of degree \( n + 1 \) of \( G_2 \) is a \( K_{1,n+m+1} \)-supermagic graph.

Next, we show the existence of a \( K_{1,n} \)-supermagic labeling of two classes of graphs for some integers \( n \). Let \( k \geq 1 \) be an integer. Let \( G_k \) be a graph with \( V(G_k) = \{x_i, y_i : i \in [1, k + 2]\} \cup \{c_i : i \in [1, k + 1]\} \) and \( E(G_k) = \{x_i c_i, y_i c_i : i \in [1, k + 1]\} \cup \{c_i x_{i+1}, c_i y_{i+1} : i \in [1, k + 1]\} \). Thus, \( G_k \) is a graph of order \( 3k + 5 \) and size \( 4k + 4 \), and it is obtained from a chain graph \( kC_4 \)-path by attaching two pendants to the vertices \( c_1 \) and \( c_{k+1} \), respectively.

**Theorem 2.4.** For every positive integer \( k \geq 1 \), the graph \( G_k \) is \( K_{1,4} \)-supermagic.

**Proof.** Define a vertex labeling \( f_1 : V(G_k) \rightarrow [1, 3k + 5] \) as follows.

\[
\begin{align*}
  f_1(u) &= \begin{cases} 
    i, & \text{if } u = x_i, \ i \text{ is odd}, \\
    i, & \text{if } u = y_i, \ i \text{ is odd}, \\
    \frac{i}{3}(3k + 8 - i), & \text{if } u = x_i, \ i \text{ is even, } k \text{ is even}, \\
    \frac{i}{3}(4k + 9 - i), & \text{if } u = y_i, \ i \text{ is odd, } k \text{ is even}, \\
    \frac{i}{3}(4k + 10 - i), & \text{if } u = x_i, \ i \text{ is even, } k \text{ is odd}, \\
    \frac{i}{3}(3k + 8 - i), & \text{if } u = y_i, \ i \text{ is odd, } k \text{ is odd}, \\
    3k + 6 - i, & \text{if } u = c_i, \ i \in [1, k + 1].
  \end{cases}
\end{align*}
\]
Next, for every $i \in [1, k + 1]$, define an edge labeling $f_2 : E(G_k) \rightarrow [1, 4k + 4]$ as follows.

$$f_2(u) = \begin{cases} 
2i - 1, & \text{if } u = x_ic_i, \\
2i, & \text{if } u = c_ix_{i+1}, \\
4k + 6 - 2i, & \text{if } u = y_ic_i, \\
4k + 5 - 2i, & \text{if } u = c_iy_{i+1}.
\end{cases}$$

For every $i \in [1, k+1]$, let $K_{1,4}^{(i)}$ be the sub-stars of $G_k$ with vertex set $V(K_{1,4}^{(i)}) = \{c_ix_i, x_{i+1}, y_i, y_{i+1}\}$ and edge set $E(K_{1,4}^{(i)}) = \{c_ix_i, c_ix_{i+1}, c_iy_i, c_iy_{i+1}\}$. It can be checked that for every $i \in [1, k + 1]$,

$$f_1(K_{1,4}^{(i)}) = f_1(c_i) + f_1(x_i) + f_1(y_i) + f_1(x_{i+1}) + f_1(y_{i+1}) = \left\lfloor \frac{1}{2}(13k + 31) \right\rfloor$$

and

$$f_2(K_{1,4}^{(i)}) = f_2(c_ix_i) + f_2(c_iy_i) + f_2(c_ix_{i+1}) + f_2(c_iy_{i+1}) = 8k + 10.$$ 

Finally, define a total labeling $f_3 : V(G_k) \cup E(G_k) \rightarrow [1, 7k + 9]$ as follows.

$$f_3(u) = \begin{cases} 
f_1(u), & \text{if } u \in V(G_k), \\
3k + 5 + f_2(u), & \text{if } u \in E(G_k).
\end{cases}$$

It is easy to verify that, for every $i \in [1, k + 1]$, $\sum f_3(K_{1,4}^{(i)}) = f_1(K_{1,4}^{(i)}) + 12k + 20 + f_2(K_{1,4}^{(i)}) = \left\lfloor \frac{1}{2}(53k + 91) \right\rfloor$.

As a direct consequence of this result and Theorem 2.1, we have the following corollary.

**Corollary 2.6.** For any integers $k \geq 1$ and $m \geq 1$, the graph formed by attaching $m$ pendants to every vertex of degree four of the graph $G_k$ is a $K_{1,m+4}$-supermagic graph.

Next, we consider of $K_{1,3}$-supermagic labelings of a ladder minus two edges. First, we define the ladder $L_n = P_n \times P_2$, $n \geq 3$, as a graph with vertex set $V(L_n) = \{x_i, y_i : i \in [1, n]\}$ and edge set $E(L_n) = \{x_ix_j, y_ix_j, y_jy_{j+1} : i, j \in [1, n - 1]\}$. For any integer $n \geq 3$, let $H_n = L_n - \{x_1y_1, x_ny_n\}$. Thus, $H_n$ is a graph with $V(H_n) = V(L_n)$ and $E(H_n) = E(L_n) - \{x_1y_1, x_ny_n\}$. In the following theorem, we show that $H_n$ is $K_{1,3}$-supermagic for every $n \geq 3$.

**Theorem 2.5.** For any integer $n \geq 3$, $H_n$ is $K_{1,3}$-supermagic.

**Proof.** Define a vertex labeling $g_1 : V(H_n) \rightarrow [1, 2n]$ as follows.

Case $n \equiv 0, 1 \mod 4$.

$$g_1(u) = \begin{cases} 
\left\lfloor \frac{1}{2}(3n + 3 - i) \right\rfloor, & \text{if } u = x_i, i \equiv 0 \mod 4, \\
\left\lfloor \frac{1}{2}(i + 1) \right\rfloor, & \text{if } u = x_i, i \equiv 1 \mod 4, \\
\left\lfloor \frac{1}{2}(4n + 2 - i) \right\rfloor, & \text{if } u = x_i, i \equiv 2 \mod 4, \\
\left\lfloor \frac{1}{2}(n + 2 + i) \right\rfloor, & \text{if } u = x_i, i \equiv 3 \mod 4, \\
\left\lfloor \frac{1}{2}(4n + 2 - i) \right\rfloor, & \text{if } u = y_i, i \equiv 0 \mod 4, \\
\left\lfloor \frac{1}{2}(n + 2 + i) \right\rfloor, & \text{if } u = y_i, i \equiv 1 \mod 4, \\
\left\lfloor \frac{1}{2}(3n + 3 - i) \right\rfloor, & \text{if } u = y_i, i \equiv 2 \mod 4, \\
\left\lfloor \frac{1}{2}(i + 1) \right\rfloor, & \text{if } u = y_i, i \equiv 3 \mod 4.
\end{cases}$$
Case \( n \equiv 2, 3 \mod 4 \).

\[
g_1(u) = \begin{cases} 
\frac{1}{2}(4n + 2 - i), & \text{if } u = x_i, \ i \equiv 0 \mod 4, \\
\frac{1}{2}(i + 1), & \text{if } u = x_i, \ i \equiv 1 \mod 4, \\
\frac{1}{2}(3n + 3 - i), & \text{if } u = x_i, \ i \equiv 2 \mod 4, \\
\frac{1}{2}(n + 2 + i), & \text{if } u = x_i, \ i \equiv 3 \mod 4, \\
\frac{1}{2}(3n + 3 - i), & \text{if } u = y_i, \ i \equiv 0 \mod 4, \\
\frac{1}{2}(n + 2 + i), & \text{if } u = y_i, \ i \equiv 1 \mod 4, \\
\frac{1}{2}(4n + 2 - i), & \text{if } u = y_i, \ i \equiv 2 \mod 4, \\
\frac{1}{2}(i + 1), & \text{if } u = y_i, \ i \equiv 3 \mod 4.
\end{cases}
\]

It is easy to verify that for \( i \in [2, n - 1], g_1(x_{i-1}) + g_1(x_i) + g_1(x_{i+1}) + g_1(y_i) = g_1(y_{i-1}) + g_1(y_i) + g_1(y_{i+1}) + g_1(x_i) \) is \( 4n + 3 \), if \( n \) is even and \( 4n + 4 \), if \( n \) is odd.

Next, define an edge labeling \( g_2 : E(H_n) \rightarrow [1, 3n - 4] \) as follows.

\[
g_2(u) = \begin{cases} 
\frac{1}{2}(i + 1), & \text{if } u = x_ix_{i+1}, \ i \text{ is odd}, \\
\frac{1}{2}(3n - 2 + i), & \text{if } u = x_ix_{i+1}, \ i \text{ is even}, \\
\frac{1}{2}(2n - 1 + i), & \text{if } u = y_iy_{i+1}, \ i \text{ is odd}, \\
\frac{1}{2}(n + i), & \text{if } u = y_iy_{i+1}, \ i \text{ is even}, \\
3n - 2 - i, & \text{if } u = x_iy_i, \ i \in [2, n - 1].
\end{cases}
\]

It can be checked that for \( i \in [2, n - 1], g_1(x_{i-1}x_i) + g_1(x_ix_{i+1}) + g_1(x_iy_i) = g_1(y_{i-1}y_i) + g_1(y_iy_{i+1}) + g_1(x_iy_i) \) is \( \frac{1}{2}(9n - 6) \), if \( n \) is even and \( \frac{1}{2}(9n - 7) \), if \( n \) is odd.

At last, define a total labeling \( g_3 : V(H_n) \cup E(H_n) \rightarrow [1, 5n - 4] \) as follows.

\[
g_3(u) = \begin{cases} 
g_1(u), & \text{if } u \in V(H_n), \\
2n + g_2(u), & \text{if } u \in E(H_n).
\end{cases}
\]

It is a routine procedure to check that \( g_3 \) is a \( K_{1,3} \)-supermagic labeling of \( H_n \) where for every subgraph \( H' \) of \( H_n \) isomorphic to \( K_{1,3} \), \( \sum f_3(H') \) is \( \frac{1}{2}(29n + 1) \).

By applying Theorem 2.1 to this result, we have the following result.

**Corollary 2.7.** For any integers \( n \geq 3 \) and \( m \geq 1 \), the graph formed by attaching \( m \) pendant edges to every vertex of degree three of the graph \( H_n \) is a \( K_{1,m+3} \)-supermagic graph.

**Problem 2.** Investigate the existence of \( K_{1,n} \)-supermagic labelings of other classes of graphs.

**Acknowledgement**

References


