Indonesian Journal of Combinatorics **3** (2) (2019), 79–94

INDONESIAN JOURNAL OF COMBINATORICS

On the total vertex irregularity strength of comb product of two cycles and two stars

Rismawati Ramdani

Department of Mathematics, Faculty of Science and Technology, UIN Sunan Gunung Djati Bandung, Indonesia

rismawatiramdani@uinsgd.ac.id

Abstract

Let $G = (V(G), E(G))$ be a graph and k be a positive integer. A total k-labeling of G is a map $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$. The vertex weight v under the labeling f is denoted by $w_f(v)$ and defined by $w_f(v) = f(v) + \sum_{uv \in E(G)} f(uv)$. A total k-labeling of G is called vertex irregular if there are no two vertices with the same weight. The total vertex irregularity strength of G, denoted by $tvs(G)$, is the minimum k such that G has a vertex irregular total k-labeling. This labelings were introduced by Bača, Jendroľ, Miller, and Ryan in 2007. Let G and H be two connected graphs. Let o be a vertex of H. The comb product between G and H, denoted by $G \rhd_o H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and grafting the i-th copy of H at the vertex o to the i-th vertex of G. In this paper, we determine the total vertex irregularity strength of comb product of two cycles and two stars.

Keywords: total vertex irregular labeling, total vertex irregularity strength, comb product, cycle, star
Mathematics Subject Classification : 05C78
DOI: 10.19184/ijc.2019.3.2.2

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A total labeling $f : V \cup E \to \{1, 2, \ldots, k\}$ is called a *vertex irregular total k-labeling* of G if every two distinct vertices x and y in $V(G)$ satisfy $wt(x) \neq wt(y)$, where $wt(x) = f(x) + \sum\limits_{xz \in E(G)} f(xz)$. The *total vertex irregularity strength* of G, denoted by $\text{tvs}(G)$, is the minimum k for which G has a vertex irregular total k-

Received: 31 May 2019, Revised: 26 Sep 2019, Accepted: 25 Nov 2019.

labeling. In [1], Bača et al. gave the bounds for a graph G with minimum degree $\delta(G)$ and maximum degree $\Delta(G)$ by the following form:

$$\lceil(|V(G)| + \delta(G))/(\Delta(G) + 1)\rceil \leq \mathrm{tvs}(G) \leq |V(G)| + \Delta(G) - 2\delta + 1. \quad (1)$$

In [6], Przybylo proved that $tvs(G) < 32|V(G)|/\delta(G)+8$ in general and $tvs(G) < 8|V(G)|/r+3$ for $r-$regular graphs.

Ramdani et al. in [7], gave an upper bound on the total vertex irregularity strength for $\bigcup_{i=1}^{m} G_i$ as follows.
Let G_i be an r-regular graph, for $i = 1, 2, \ldots, m$. Then

$$tvs\left(\bigcup_{i=1}^{m} G_i\right) \leq \sum_{i=1}^{m} tvs(G_i) - \left\lfloor \frac{m-1}{2} \right\rfloor. \quad (2)$$

In the same paper, Ramdani et al. obtained the exact value of the total vertex irregularity strength for disjoint union of arbitrary r-regular graphs G_i, for $i = 1, 2, \ldots, m$, if there is a vertex irregular total $(tvs(G_i))$-labeling of G_i such that the vertex-weight function

$$w_{f_i}(v_{ia}) : V(G_i) \to \{r+1, r+2, \cdots, (r+1)tvs(G_i) - 1\}$$

is a bijection for every $i = 1, 2, \ldots, m$, which is

$$tvs\left(\bigcup_{i=1}^{m} G_i\right) = \sum_{i=1}^{m} tvs(G_i) - m + 1. \quad (3)$$

In [3], Nurdin proved that

$$tvs(G) \geq \max\left\{\left\lceil \frac{\delta + n_\delta}{\delta + 1} \right\rceil, \left\lceil \frac{\delta + n_\delta + n_{\delta+1}}{\delta + 2} \right\rceil, \cdots, \left\lceil \frac{\delta + \sum_{i=\delta}^{\Delta} n_i}{\Delta + 1} \right\rceil\right\} \quad (4)$$

for connected graph G having n_i vertices of degree $i(i = \delta, \delta+1, \delta+2, \cdots, \Delta)$, where δ and Δ are the minimum and the maximum degree of G, respectively.

In [8], Ramdani and Ramdhani obtained the exact value of the total vertex irregularity strength of comb product between cycles C_n and C_4, as follows.

$$tvs(C_n \rhd_o C_4) = n + 1, \text{ for } n \geq 3. \quad (5)$$

Some other results of the total vertex irregularity strength of graphs can be found in [2], [4], [5], [9], and [10].

2. Main Results

In this paper we determine the total vertex irregularity strength of some comb product graphs.

Let o be a vertex of H. The comb product between G and H, denoted by $G \rhd_o H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and grafting the i-th copy of H at the vertex o to the i-th vertex of G.

An illustration of comb product graph is given in Figure 1.

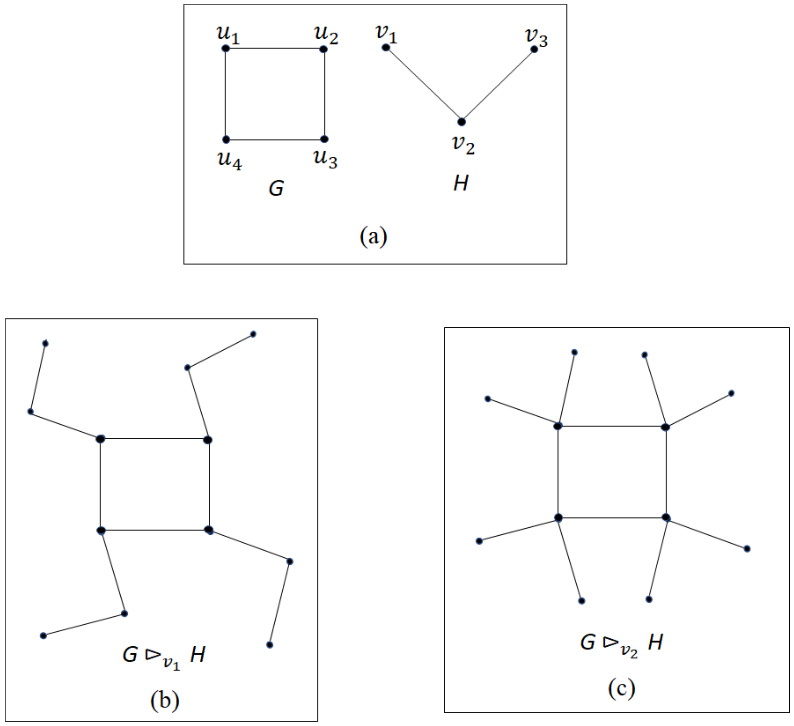

Figure 1. (a) Two graphs G and H; (b) The comb product graph $G \rhd_{v_1} H$; (c) The comb product graph $G \rhd_{v_2} H$

The first result gives the total vertex irregularity strength of comb product between two cycles.

Theorem 2.1. *Let C_m and C_n be cycles with order m and n respectively. Then, for $m \geq 3$, $n \geq 3$, and for every vertex o in C_m,*

$$tvs(C_m \rhd_o C_n) = \left\lceil \frac{m(n-1)+2}{3} \right\rceil.$$

Proof. Let $V(C_m \rhd_o C_n) = \{u_{i,j} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and

$$\begin{aligned} E(C_m \rhd_o C_n) &= \{u_{i,j}u_{i,j+1} \mid 1 \leq i \leq m,\ 1 \leq j \leq n-1\} \cup \{u_{i,n}u_{i,1} \mid 1 \leq i \leq m\} \\ &\quad \cup \{u_{i,1}u_{i+1,1} \mid 1 \leq i \leq m-1\} \cup \{u_{m,1}u_{1,1}\}. \end{aligned}$$

An illustration of $C_m \rhd_o C_n$ with o be a vertex in C_n can be seen in Figure 2.

The $C_m \rhd_o C_n$ graphs have $m(n-1)$ vertices with degree $\delta = 2$ and m vertices with degree $\Delta = 4$. So, by using Inequality (4), we have

$$tvs(C_m \rhd_o C_n) \geq \max\left\{ \left\lceil \tfrac{2+m(n-1)}{3} \right\rceil, \left\lceil \tfrac{2+m(n-1)+m}{5} \right\rceil \right\} = \left\lceil \tfrac{2+m(n-1)}{3} \right\rceil.$$

So, we have

$$tvs(C_m \rhd_o C_n) \geq \left\lceil \frac{m(n-1)+2}{3} \right\rceil. \tag{6}$$

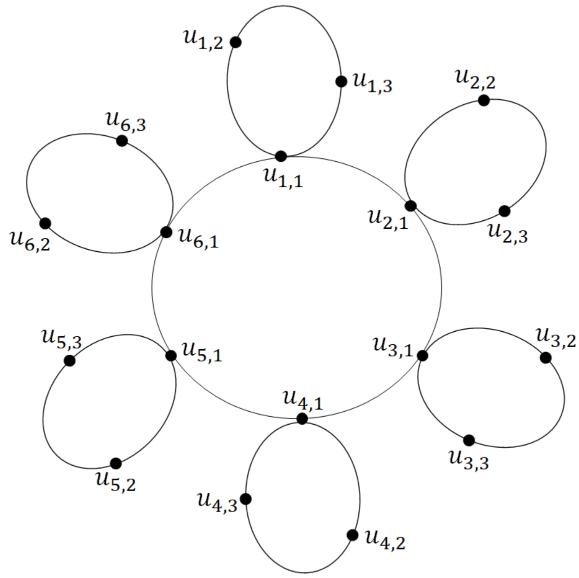

Figure 2. Comb product graph $C_m \rhd_o C_n$ with $m = 6$ and $n = 3$

Next, we will show that

$$tvs(C_m \rhd_o C_n) \leq \left\lceil \frac{m(n-1)+2}{3} \right\rceil.$$

Define a total labeling $f : V(C_m \rhd_o C_n) \cup E(C_m \rhd_o C_n) \to \left\{1, 2, \ldots, \left\lceil \frac{m(n-1)+2}{3} \right\rceil \right\}$ as follows.

$$f(u_{i,1}) = \left\lceil \frac{m(n-1)+2}{3} \right\rceil, \text{ for } 1 \leq i \leq m;$$

$$f(u_{i,j}) = \left\lceil \frac{i(n-1)-(n-j)}{3} \right\rceil \text{ for } 1 \leq i \leq m,\ 2 \leq j \leq n;$$

$$f(u_{i,j}u_{i,j+1}) = \left\lceil \frac{i(n-1)-(n-j-2)}{3} \right\rceil \text{ for } 1 \leq i \leq m,\ 1 \leq j \leq n-1;$$

$$f(u_{i,n}u_{i,1}) = \left\lceil \frac{i(n-1)+2}{3} \right\rceil \text{ for } 1 \leq i \leq m;$$

$$f(u_{i,1}u_{i+1,1}) = \left\lceil \frac{m(n-1)+2}{3} \right\rceil \text{ for } 1 \leq i \leq m-1;$$

$$f(u_{m,1}u_{11}) = \left\lceil \frac{m(n-1)+2}{3} \right\rceil.$$

From the labeling above, we have the weight of each vertex of $C_m \rhd_o C_n$ as follows.

1. For $1 \leq i \leq m$ and $j = 1$, we devided the formula into three cases.

   (a) Case 1 : For $i = 1$ and $j = 1$,

$$\begin{aligned} w_f(u_{i,j}) &= w_f(u_{1,1}) \\ &= f(u_{1,1}) + f(u_{1,1}u_{2,1}) + f(u_{1,1}u_{m,1}) + f(u_{1,1}u_{1,2}) + f(u_{1,n}u_{1,1}) \\ &= \left\lceil \tfrac{m(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{m(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{m(n-1)+2)}{3} \right\rceil + \left\lceil \tfrac{n-1-(n-1-2)}{3} \right\rceil \\ &\quad + \left\lceil \tfrac{n-1+2}{3} \right\rceil \\ &= 3\left\lceil \tfrac{m(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{2}{3} \right\rceil + \left\lceil \tfrac{n+1}{3} \right\rceil. \end{aligned}$$

   (b) Case 2 : For $2 \leq i \leq m-1$ and $j = 1$,

$$\begin{aligned} w_f(u_{i,j}) &= w_f(u_{i,1}) \\ &= f(u_{i,1}) + f(u_{i,1}u_{i+1,1}) + f(u_{i-1,i}u_{i,1}) + f(u_{i,1}u_{i,2}) + f(u_{i,n}u_{i,1}) \\ &= \left\lceil \tfrac{m(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{m(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{m(n-1)+2)}{3} \right\rceil + \left\lceil \tfrac{i(n-1)-(n-1-2)}{3} \right\rceil \\ &\quad + \left\lceil \tfrac{i(n-1)+2}{3} \right\rceil \\ &= 3\left\lceil \tfrac{m(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{i(n-1)-(n-3)}{3} \right\rceil + \left\lceil \tfrac{i(n-1)+2}{3} \right\rceil. \end{aligned}$$

   (c) Case 3 : For $i = m$ and $j = 1$,

$$\begin{aligned} w_f(u_{i,j}) &= w_f(u_{m,1}) \\ &= f(u_{m,1}) + f(u_{m,1}u_{1,1}) + f(u_{m-1,i}u_{m,1}) + f(u_{m,1}u_{m,2}) + f(u_{m,n}u_{m,1}) \\ &= \left\lceil \tfrac{m(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{m(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{m(n-1)+2)}{3} \right\rceil + \left\lceil \tfrac{m(n-1)-(n-1-2)}{3} \right\rceil \\ &\quad + \left\lceil \tfrac{m(n-1)+2}{3} \right\rceil \\ &= 3\left\lceil \tfrac{m(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{m(n-1)-(n-3)}{3} \right\rceil + \left\lceil \tfrac{m(n-1)+2}{3} \right\rceil. \end{aligned}$$

   So, we have the general formula of $w_f(u_{i,j})$, for $1 \leq i \leq m$ and $j = 1$, as follows,

$$w_f(u_{i,j}) = w_f(u_{i,1}) = 3\left\lceil \frac{m(n-1)+2}{3} \right\rceil + \left\lceil \frac{i(n-1)-(n-3)}{3} \right\rceil + \left\lceil \frac{i(n-1)+2}{3} \right\rceil. \quad (7)$$

   for $1 \leq i \leq m$.

2. For $1 \leq i \leq m$ and $2 \leq j \leq n$, we devided the formula into two cases.
   (a) Case 1 : For $1 \leq i \leq m$ and $2 \leq j \leq n-1$,

$$
\begin{aligned}
w_f(u_{i,j}) &= f(u_{i,j}) + f(u_{i,j}u_{i,j+1}) + f(u_{i,j-1}u_{i,j}) \\
&= \left\lceil \tfrac{i(n-1)-(n-j)}{3} \right\rceil + \left\lceil \tfrac{i(n-1)-(n-j-2)}{3} \right\rceil + \left\lceil \tfrac{i(n-1)-(n-(j-1)-2)}{3} \right\rceil \\
&= \left\lceil \tfrac{i(n-1)-(n-j)}{3} \right\rceil + \left\lceil \tfrac{i(n-1)-(n-j-2)}{3} \right\rceil + \left\lceil \tfrac{i(n-1)-(n-j-1)}{3} \right\rceil \\
&= \left\lceil \tfrac{i(n-1)-(n-j)}{3} \right\rceil + \left\lceil \tfrac{i(n-1)-(n-j)+2}{3} \right\rceil + \left\lceil \tfrac{i(n-1)-(n-j)+1}{3} \right\rceil \\
&= \begin{cases}
\left(\tfrac{i(n-1)-(n-j)}{3}\right) + \left(\tfrac{i(n-1)-(n-j)+3}{3}\right) + \left(\tfrac{i(n-1)-(n-j)+3}{3}\right) \\ \text{for } i(n-1)-(n-j) \equiv 0 \; mod\, 3; \\[1ex]
\left(\tfrac{i(n-1)-(n-j)+2}{3}\right) + \left(\tfrac{i(n-1)-(n-j)+2}{3}\right) + \left(\tfrac{i(n-1)-(n-j)+2}{3}\right) \\ \text{for } i(n-1)-(n-j) \equiv 1 \; mod\, 3; \\[1ex]
\left(\tfrac{i(n-1)-(n-j)+1}{3}\right) + \left(\tfrac{i(n-1)-(n-j)+4}{3}\right) + \left(\tfrac{i(n-1)-(n-j)+1}{3}\right) \\ \text{for } i(n-1)-(n-j) \equiv 2 \; mod\, 3;
\end{cases} \\
&= \left(\tfrac{3i(n-1)-(n-j)+6}{3}\right) \\
&= i(n-1)-(n-j)+2.
\end{aligned}
$$

(b) Case 2 : For $1 \leq i \leq m$ and $j = n$,

$$\begin{aligned}
w_f(u_{i,n}) &= f(u_{i,n}) + f(u_{i,n}u_1) + f(u_{i,n-1}u_{i,n}) \\
&= \left\lceil \tfrac{i(n-1)-(n-n)}{3} \right\rceil + \left\lceil \tfrac{i(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{i(n-1)-(n-(n-1)-2)}{3} \right\rceil \\
&= \left\lceil \tfrac{i(n-1)}{3} \right\rceil + \left\lceil \tfrac{i(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{i(n-1)+1)}{3} \right\rceil \\
&= \begin{cases} \left(\tfrac{i(n-1)}{3}\right) + \left(\tfrac{i(n-1)+3}{3}\right) + \left(\tfrac{i(n-1)+3}{3}\right) \\ \text{for } i(n-1) \equiv 0 \, mod \, 3; \\[2ex] \left(\tfrac{i(n-1)+2}{3}\right) + \left(\tfrac{i(n-1)+2}{3}\right) + \left(\tfrac{i(n-1)+2}{3}\right) \\ \text{for } i(n-1) \equiv 1 \, mod \, 3; \\[2ex] \left(\tfrac{i(n-1)+1}{3}\right) + \left(\tfrac{i(n-1)+4}{3}\right) + \left(\tfrac{i(n-1)+1}{3}\right) \\ \text{for } i(n-1) \equiv 2 \, mod \, 3; \end{cases} \\
&= \left(\tfrac{3i(n-1)+6}{3}\right) \\
&= i(n-1) + 2.
\end{aligned}$$

So, we have the general formula of $w_f(u_{i,j})$, for $1 \leq i \leq m$ and $2 \leq j \leq n$, as follows,

$$w_f(u_{i,j}) = i(n-1) - (n-j) + 2. \tag{8}$$

It will be shown that there are no two vertices with the same weight.

(a) It will be shown that $w_f(u_{i,1}) \neq w_f(u_{k,1})$ for $i \neq k$ and $1 \leq i, k \leq m$.
Let $i = k + 1$. It will be shown that $w_f(u_{i,1}) > w_f(u_{k,1})$.

i. For $n = 3$,

$$\begin{aligned}
w_f(u_{i,1}) &= 3\left\lceil \tfrac{m(3-1)+2}{3} \right\rceil + \left\lceil \tfrac{i(3-1)-(3-3)}{3} \right\rceil + \left\lceil \tfrac{i(3-1)+2}{3} \right\rceil \\
&= 3\left\lceil \tfrac{m(3-1)+2}{3} \right\rceil + \left\lceil \tfrac{(k+1)(3-1)-(3-3)}{3} \right\rceil + \left\lceil \tfrac{(k+1)(3-1)+2}{3} \right\rceil \\
&= 3\left\lceil \tfrac{2m+2}{3} \right\rceil + \left\lceil \tfrac{2k+2}{3} \right\rceil + \left\lceil \tfrac{2k+4}{3} \right\rceil \\
&> 3\left\lceil \tfrac{2m+2}{3} \right\rceil + \left\lceil \tfrac{2k}{3} \right\rceil + \left\lceil \tfrac{2k+2}{3} \right\rceil \\
&= w_f(u_{k,1}).
\end{aligned}$$

ii. For $n \geq 4$,

$$\begin{aligned}
w_f(u_{i,1}) &= 3\left\lceil \tfrac{m(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{i(n-1)-(n-3)}{3} \right\rceil + \left\lceil \tfrac{i(n-1)+2}{3} \right\rceil \\
&= 3\left\lceil \tfrac{m(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{(k+1)(n-1)-(n-3)}{3} \right\rceil + \left\lceil \tfrac{(k+1)(n-1)+2}{3} \right\rceil \\
&= 3\left\lceil \tfrac{m(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{k(n-1)+(n-1)-(n-3)}{3} \right\rceil + \left\lceil \tfrac{(k(n-1)+(n-1)+2)}{3} \right\rceil \\
&\geq 3\left\lceil \tfrac{m(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{k(n-1)+(4-1)-(n-3)}{3} \right\rceil + \left\lceil \tfrac{k(n-1)+(4-1)+2}{3} \right\rceil \\
&= 3\left\lceil \tfrac{m(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{k(n-1)+(3)-(n-3)}{3} \right\rceil + \left\lceil \tfrac{k(n-1)+(3)+2}{3} \right\rceil \\
&= 3\left\lceil \tfrac{m(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{k(n-1)-(n-3)}{3} \right\rceil + \left(\tfrac{3}{3}\right) + \left\lceil \tfrac{k(n-1)+2}{3} \right\rceil + \left(\tfrac{3}{3}\right) \\
&= 3\left\lceil \tfrac{m(n-1)+2}{3} \right\rceil + \left\lceil \tfrac{k(n-1)-(n-3)}{3} \right\rceil + \left\lceil \tfrac{k(n-1)+2}{3} \right\rceil + 2 \\
&= w_f(u_{k,1}) + 2 \\
&> w_f(u_{k,1}).
\end{aligned}$$

So, it has been proven that $w_f(u_{i,1}) \neq w_f(u_{k,1})$ for every $i \neq k$ and $1 \leq i, k \leq m$.

(b) It will be shown that $w_f(u_{i,j}) \neq w_f(u_{k,l})$ for $i \neq k$ or $j \neq l$, $1 \leq i, k \leq m$ and $2 \leq j, l \leq n$.

i. For $i > k$ and $j = l$,

$$\begin{aligned}
w_f(u_{i,j}) &= i(n-1) - (n-j) + 2 \\
&> k(n-1) - (n-j) + 2 \\
&= k(n-1) - (n-l) + 2 \\
&= w_f(u_{k,l}).
\end{aligned}$$

So, we have $w_f(u_{i,j}) > w_f(u_{k,l})$ for $i > k$ and $j = l$.

ii. For $i > k$ and $j > l$,

$$\begin{aligned}
w_f(u_{i,j}) &= i(n-1) - (n-j) + 2 \\
&> k(n-1) - (n-l) + 2 \\
&= w_f(u_{k,l}).
\end{aligned}$$

So, we have $w_f(u_{i,j}) > w_f(u_{k,l})$ for $i > k$ and $j > l$.

iii. For $i > k$ and $j < l$,

$$\begin{aligned}
w_f(u_{i,j}) &= i(n-1) - (n-j) + 2 \\
&\geq (k+1)(n-1) - (n-j) + 2 \\
&= k(n-1) + (n-1) - n + j + 2.
\end{aligned}$$

Since $j \geq 2$,

$$\begin{aligned}
k(n-1) + (n-1) - n + j + 2 &\geq k(n-1) + (n-1) - n + 2 + 2 \\
&= k(n-1) + n - n + 3.
\end{aligned}$$

Since $l \leq n$,

$$\begin{aligned} k(n-1)+n-n+3 &\geq k(n-1)+l-n+3. \\ &= k(n-1)-(n-l)+3 \\ &> k(n-1)-(n-l)+2 \\ &= w_f(u_{k,l}). \end{aligned}$$

So, we have $w_f(u_{i,j}) > w_f(u_{k,l})$ for $i > k$ and $j < l$.

So, it has been proven that $w_f(u_{i,j}) \neq w_f(u_{k,l})$ for every $i \neq k$ or $j \neq l$ for $1 \leq i, k \leq m$ and $2 \leq j, l \leq n$.

(c) It will be shown that $w_f(u_{i,1}) \neq w_f(u_{k,l})$ for $1 \leq i, k \leq m$ and $2 \leq l \leq n$.
Since $i \geq 1$, we have

$$\begin{aligned} w_f(u_{i,1}) &= 3\left\lceil \frac{m(n-1)+2}{3} \right\rceil + \left\lceil \frac{i(n-1)-(n-3)}{3} \right\rceil + \left\lceil \frac{i(n-1)+2}{3} \right\rceil. \\ &\geq 3\left\lceil \frac{m(n-1)+2}{3} \right\rceil + \left\lceil \frac{1(n-1)-(n-3)}{3} \right\rceil + \left\lceil \frac{1(n-1)+2}{3} \right\rceil. \\ &\geq 3\left\lceil \frac{m(n-1)+2}{3} \right\rceil + \left\lceil \frac{2}{3} \right\rceil + \left\lceil \frac{n+1}{3} \right\rceil. \end{aligned}$$

Since $3\left\lceil \frac{m(n-1)+2}{3} \right\rceil \geq 3\left(\frac{m(n-1)+2}{3}\right)$ and $n \geq 3$, we have

$$\begin{aligned} w_f(u_{i,1}) &\geq 3\left\lceil \frac{m(n-1)+2}{3} \right\rceil + \left\lceil \frac{2}{3} \right\rceil + \left\lceil \frac{n+1}{3} \right\rceil. \\ &\geq 3\left(\frac{m(n-1)+2}{3}\right) + \left\lceil \frac{2}{3} \right\rceil + \left\lceil \frac{3+1}{3} \right\rceil. \\ &\geq (m(n-1)+2)+1+2. \\ &\geq m(n-1)+5. \end{aligned}$$

So, we have an inequality

$$w_f(u_{i,1}) \geq m(n-1)+5, \tag{9}$$

On the other hand, Since $1 \leq k \leq m$ and $2 \leq l \leq n$,

$$\begin{aligned} w_f(u_{k,l}) &= k(n-1)-n+l+2 \\ &\leq m(n-1)-n+n+2 \\ &\leq m(n-1)+2. \\ &< m(n-1)+5. \end{aligned}$$

So that, for $1 \leq k \leq m$ and $2 \leq l \leq n$, we have an inequality as follows,

$$w_f(u_{k,l}) < m(n-1)+5 \tag{10}$$

From Inequatily (9) and (10), we have $w_f(u_{k,l}) < w_f(u_{i,1})$ for $1 \leq i, k \leq m$ and $2 \leq l \leq n$.

From the three points above, we can conclude that from the labeling f, there are no two vertices with the same weight. So, f is a vertex irregular total $\left\lceil \frac{m(n-1)+2}{3} \right\rceil$-labeling of $C_m \rhd_o C_n$.

So we have Inequality (11),

$$tvs(C_m \rhd_o C_n) \leq \left\lceil \frac{m(n-1)+2}{3} \right\rceil. \tag{11}$$

By using Inequalities (6) and (11), we have an equation as follows,

$$tvs(C_m \rhd_o C_n) = \left\lceil \frac{m(n-1)+2}{3} \right\rceil. \tag{12}$$

□

The next theorem provides the total vertex irregularity strength of comb product between two stars.

Theorem 2.2. *Let S_m and S_n be stars with order $m+1$ and $n+1$ respectively, and o be the center vertex of S_m. Then, for $m \geq 2$ and $n \geq 2$,*

$$tvs(S_m \rhd_o S_n) = \left\lceil \frac{n(m+1)+1}{2} \right\rceil.$$

Proof. Let

$$V(S_m \rhd_o S_n) = \{u_{i,j} \mid 1 \leq i \leq m+1,\ 1 \leq j \leq n+1\}$$

and

$$E(S_m \rhd_o S_n) = \{u_{i,1}u_{m+1,1} \mid 1 \leq i \leq m\} \cup \{u_{i,1}u_{i,j} \mid 1 \leq i \leq m+1, 2 \leq j \leq n+1.\}$$

An illustration of $S_m \rhd_o S_n$ with o be the center vertex in S_n can be seen in Figure 3.

The $S_m \rhd_o S_n$ graphs have $n(m+1)$ vertices with degree $\delta = 1$, m vertices with degree $n+1$, and one vertex with degree $n+m$. So, by using Inequality (4), we have

$$tvs(S_m \rhd_o S_n) \geq \max\left\{ \left\lceil \frac{n(m+1)+1}{2} \right\rceil, \left\lceil \frac{mn+n+2}{n+1} \right\rceil, \left\lceil \frac{mn+n+3}{n+m+1} \right\rceil \right\} = \left\lceil \frac{n(m+1)+1}{2} \right\rceil.$$

So, we have

$$tvs(S_m \rhd_o S_n) \geq \left\lceil \frac{n(m+1)+1}{2} \right\rceil. \tag{13}$$

Next, we will show that

$$tvs(S_m \rhd_o S_n) \leq \left\lceil \frac{n(m+1)+1}{2} \right\rceil.$$

Define a total labeling $f : V(S_m \rhd_o S_n) \cup E(S_m \rhd_o S_n) \to \left\{1, 2, \ldots, \left\lceil \frac{n(m+1)+1}{2} \right\rceil \right\}$ as follows.

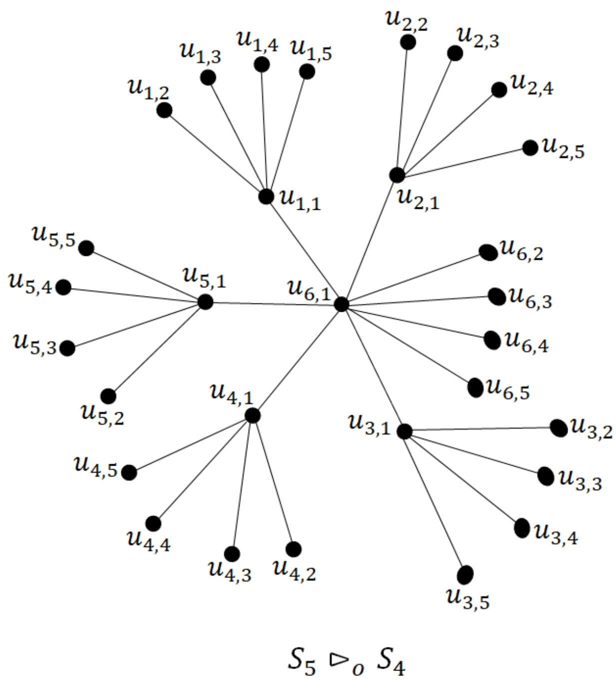

Figure 3. Comb product graph $S_m \rhd_o S_n$ with $m = 5$ and $n = 4$

$$f(u_{i,1}) = \left\lceil \frac{n(m+1)+1}{2} \right\rceil \quad \text{for } 1 \le i \le m+1;$$

$$f(u_{i,j}) = \left\lceil \frac{n(i-1)+j-1}{2} \right\rceil, \quad \text{for } 1 \le i \le m+1,\ 2 \le j \le n+1;$$

$$f(u_{i,1}u_{m+1,1}) = \left\lceil \frac{n(m+1)+1}{2} \right\rceil \quad \text{for } 1 \le i \le m;$$

$$f(u_{i,1}u_{i,j}) = \left\lceil \frac{n(i-1)+j}{2} \right\rceil \quad \text{for } 1 \le i \le m+1,\ 2 \le j \le n+1.$$

From the labeling above, we have the weight of vertices of $S_m \rhd_o S_n$ as follows.

1. For $1 \leq i \leq m$ and $j = 1$,

$$\begin{aligned} w_f(u_{i,j}) &= w_f(u_{i,1}) \\ &= f(u_{i,1}) + \sum_{j=2}^{n} f(u_{i,1}u_{i,j}) \\ &= \left\lceil \frac{n(m+1)+1}{2} \right\rceil + \sum_{j=2}^{n} \left\lceil \frac{n(i-1)+j}{2} \right\rceil \\ &= \begin{cases} 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{n(i-1)}{2}\right) + \frac{n^2+4n-1}{4} & \text{for } n \text{ is odd and } i \text{ is odd;} \\ 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{n(i-1)+1}{2}\right) + \frac{n^2+2n+1}{4} & \text{for } n \text{ is odd and } i \text{ is even;} \\ 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{n(i-1)}{2}\right) + \frac{n^2+4n}{4} & \text{for } n \text{ is even.} \end{cases} \end{aligned}$$

2. For $i = m+1$ and $j = 1$,

$$\begin{aligned} w_f(u_{i,j}) &= w_f(u_{m+1,1}) \\ &= f(u_{m+1,1}) + \sum_{j=2}^{n} f(u_{m+1,1}u_{m+1,j}) + \sum_{i=1}^{m} f(u_{i,1}u_{m+1,1}) \\ &= \left\lceil \frac{n(m+1)+1}{2} \right\rceil + \sum_{j=2}^{n} \left\lceil \frac{n((m+1)-1)+j}{2} \right\rceil + \sum_{i=1}^{m} \left\lceil \frac{n(m+1)+1}{2} \right\rceil \\ &= \begin{cases} (m+1)\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{nm+1}{2}\right) + \frac{n^2+2n+1}{4} & \text{for } n \text{ is odd and } m \text{ is odd;} \\ (m+1)\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{nm}{2}\right) + \frac{n^2+4n-1}{4} & \text{for } n \text{ is odd and } m \text{ is even;} \\ (m+1)\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{nm}{2}\right) + \frac{n^2+4n}{4} & \text{for } n \text{ is even.} \end{cases} \end{aligned}$$

3. For $1 \leq i \leq m+1$ and $2 \leq j \leq n+1$,

$$\begin{aligned} w_f(u_{i,j}) &= f(u_{i,j}) + f(u_{i,1}u_{i,j}) \\ &= \left\lceil \frac{n(i-1)+j-1}{2} \right\rceil + \left\lceil \frac{n(i-1)+j}{2} \right\rceil \\ &= \begin{cases} \left(\frac{n(i-1)+j-1}{2}\right) + \left(\frac{n(i-1)+j+1}{2}\right) & \text{for } n(i-1)+j \text{ is odd;} \\ \left(\frac{n(i-1)+j}{2}\right) + \left(\frac{n(i-1)+j}{2}\right) & \text{for } n(i-1)+j \text{ is even;} \end{cases} \\ &= n(i-1)+j. \end{aligned}$$

It will be shown that there are no two vertices with the same weight.

1. It will be shown that $w_f(u_{i,1}) \neq w_f(u_{k,1})$ for $i \neq k$ and $1 \leq i, k \leq m$.
   Let $i = k+1$. It will be shown that $w_f(u_{i,1}) > w_f(u_{k,1})$.
   (a) For i is odd, (then, k is even), we have

$$\begin{aligned} w_f(u_{i,1}) &= 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{n(i-1)}{2}\right) + \frac{n^2+4n-1}{4} \\ &= 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{n(k)}{2}\right) + \frac{n^2+4n-1}{4}. \\ &= 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{nk-(n-1)}{2}\right) + \frac{n(n-1)}{2} + \frac{n^2+2n+1}{4} + \frac{2n-2}{4}. \\ &= 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{n(k-1)+1}{2}\right) + \frac{n^2+2n+1}{4} + \frac{n(n-1)}{2} + \frac{2n-2}{4}. \\ &= w_f(u_{k,1}) + \frac{n(n-1)}{2} + \frac{2n-2}{4}. \end{aligned}$$

   Since $n > 2$, then $\frac{n(n-1)}{2} + \frac{2n-2}{4} > 1$. So,

$$w_f(u_{i,1}) > w_f(u_{k,1}). \quad (14)$$

   (b) For i is even, (then, k is odd), we have

$$\begin{aligned} w_f(u_{i,1}) &= 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{n(i-1)+1}{2}\right) + \frac{n^2+2n+1}{4} \\ &= 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{n(k)+1}{2}\right) + \frac{n^2+2n+1}{4}. \\ &= 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{nk+1}{2}\right) - \frac{2n^2}{4} + \frac{n^2+4n-1}{4} + \frac{2n^2+2}{4} - \frac{2n}{4}. \\ &= 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{nk}{2}\right) + \frac{n}{2} - \frac{n^2}{2} + \frac{n^2+4n-1}{4} + \frac{n^2+1}{2} - \frac{n}{2}. \\ &= 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{nk}{2}\right) - \frac{n^2}{2} + \frac{n^2+4n-1}{4} + \frac{n^2+1}{2}. \\ &= 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{nk-n}{2}\right) + \frac{n^2+4n-1}{4} + \frac{n^2+1}{2}. \\ &= 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{n(k-1)}{2}\right) + \frac{n^2+4n-1}{4} + \frac{n^2+1}{2}. \\ &= w_f(u_{k,1}) + \frac{n^2+1}{2}. \end{aligned}$$

   Since $n > 2$, then $\frac{n^2+1}{2} > 2$. So,

$$w_f(u_{i,1}) > w_f(u_{k,1}). \quad (15)$$

   So, it has been proven that $w_f(u_{i,1}) \neq w_f(u_{k,1})$ for $i \neq k$ and $1 \leq i, k \leq m$.
2. It will be shown that $w_f(u_{i,1}) \neq w_f(u_{m+1,1})$ for $1 \leq i \leq m$.
   For $1 \leq i \leq m$, we have Inequality (16),

$$w_f(u_{i,1}) \leq \begin{cases} 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{n(m-1)}{2}\right) + \frac{n^2+4n-1}{4} & \text{for } n \text{ is odd and } m \text{ is odd;} \\ 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{n(m-1)+1}{2}\right) + \frac{n^2+2n+1}{4} & \text{for } n \text{ is odd and } m \text{ is even;} \\ 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{n(m-1)}{2}\right) + \frac{n^2+4n}{4} & \text{for } n \text{ is even.} \end{cases} \quad (16)$$

On the other hand, since $m > 2$, we have Inequality (17),

$$w_f(u_{m+1,1}) > \begin{cases} 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{nm+1}{2}\right) + \frac{n^2+2n+1}{4} & \text{for } n \text{ is odd and } m \text{ is odd;} \\ 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{nm}{2}\right) + \frac{n^2+4n-1}{4} & \text{for } n \text{ is odd and } m \text{ is even;} \\ 2\left\lceil \frac{n(m+1)+1}{2} \right\rceil + n\left(\frac{nm}{2}\right) + \frac{n^2+4n}{4} & \text{for } n \text{ is even.} \end{cases} \quad (17)$$

From Inequality (16) and (17), we have

$$w_f(u_{m+1,1}) > w_f(u_{i,1}) \quad (18)$$

for every $1 \leq i \leq m$.

3. It will be shown that $w_f(u_{i,j}) \neq w_f(u_{k,l})$ for $i \neq k$ or $j \neq l$, $1 \leq i,k \leq m+1$ and $2 \leq j,l \leq n+1$.
   (a) For $i > k$ and $j = l$,

   $$\begin{aligned} w_f(u_{i,j}) &= n(i-1)+j \\ &> n(k-1)+j \\ &= n(k-1)+l \\ &= w_f(u_{k,l}). \end{aligned}$$

   So, we have $w_f(u_{i,j}) > w_f(u_{k,l})$ for $i > k$ and $j = l$.

   (b) For $i > k$ and $j > l$,

   $$\begin{aligned} w_f(u_{i,j}) &= n(i-1)+j \\ &> n(k-1)+j \\ &> n(k-1)+l \\ &= w_f(u_{k,l}). \end{aligned}$$

   So, we have $w_f(u_{i,j}) > w_f(u_{k,l})$ for $i > k$ and $j > l$.

   (c) For $i > k$ and $j < l$,

   $$\begin{aligned} w_f(u_{i,j}) &= n(i-1)+j \\ &\geq n((k+1)-1)+j \\ &= n(k-1)+n+j. \end{aligned}$$

   Since $j \geq 2$,

   $$n(k-1)+n+j \geq n(k-1)+n+2.$$

   Since $n+1 \geq l$,

   $$\begin{aligned} n(k-1)+n+2 &\geq n(k-1)+l+1. \\ &> n(k-1)+l \\ &= w_f(u_{k,l}). \end{aligned}$$

   So, we have $w_f(u_{i,j}) > w_f(u_{k,l})$ for $i > k$ and $j < l$.

So, it has been proven that $w_f(u_{i,j}) \neq w_f(u_{k,l})$ for every $i \neq k$ or $j \neq l$ for $1 \leq i, k \leq m+1$ and $2 \leq j, l \leq n+1$.

4. It will be shown that $w_f(u_{i,1}) \neq w_f(u_{k,l})$ for $1 \leq i, k \leq m+1$ and $2 \leq l \leq n+1$. Since $k \leq m+1$ and $l \leq n+1$, we have

$$\begin{aligned} w_f(u_{k,l}) &= n(k-1)+l \\ &\leq n((m+1)-1)+(n+1) \\ &= n(m+1)+1 \\ &= 2\left(\tfrac{n(m+1)+1}{2}\right) \\ &\leq 2\left\lceil\tfrac{n(m+1)+1}{2}\right\rceil. \end{aligned}$$

So, for $1 \leq k \leq m+1$ and $2 \leq l \leq n+1$, we have Inequality (19),

$$w_f(u_{k,l}) \leq 2\left\lceil\frac{n(m+1)+1}{2}\right\rceil. \tag{19}$$

On the other hand, for $1 \leq i \leq m$, we have

$$w_f(u_{i,1}) = \begin{cases} 2\left\lceil\frac{n(m+1)+1}{2}\right\rceil + n\left(\frac{n(i-1)}{2}\right) + \frac{n^2+4n-1}{4} & \text{for } n \text{ is odd and } i \text{ is odd;} \\ 2\left\lceil\frac{n(m+1)+1}{2}\right\rceil + n\left(\frac{n(i-1)+1}{2}\right) + \frac{n^2+2n+1}{4} & \text{for } n \text{ is odd and } i \text{ is even;} \\ 2\left\lceil\frac{n(m+1)+1}{2}\right\rceil + n\left(\frac{n(i-1)}{2}\right) + \frac{n^2+4n}{4} & \text{for } n \text{ is even.} \end{cases}$$

Since $i \geq 1$, we have

$$w_f(u_{i,1}) \geq \begin{cases} 2\left\lceil\frac{n(m+1)+1}{2}\right\rceil + \frac{n^2+4n-1}{4} & \text{for } n \text{ is odd and } i \text{ is odd;} \\ 2\left\lceil\frac{n(m+1)+1}{2}\right\rceil + n\left(\frac{1}{2}\right) + \frac{n^2+2n+1}{4} & \text{for } n \text{ is odd and } i \text{ is even;} \\ 2\left\lceil\frac{n(m+1)+1}{2}\right\rceil + \frac{n^2+4n}{4} & \text{for } n \text{ is even.} \end{cases}$$

Since $n > 1$, then $\frac{n^2+4n-1}{4} > 1$, $n\left(\frac{1}{2}\right)+\frac{n^2+2n+1}{4} > 1$, and $\frac{n^2+4n}{4} > 1$. So, we have inequality as follows.

$$w_f(u_{i,1}) > 2\left\lceil\frac{n(m+1)+1}{2}\right\rceil. \tag{20}$$

From Inequality (19), (20), and (18), we have $w_f(u_{k,l}) < w_f(u_{i,1}) < w_f(u_{m+1,1})$ for $1 \leq i \leq m$, $1 \leq k \leq m+1$, and $2 \leq l \leq n+1$.

From the three points above, there are no two vertices with the same weight. So, f is a vertex irregular total $\left\lceil\frac{n(m+1)+1}{2}\right\rceil$-labeling of $S_m \rhd_o S_n$. So, we have inequality

$$tvs(S_m \rhd_o S_n) \leq \left\lceil\frac{n(m+1)+1}{2}\right\rceil. \tag{21}$$

From Inequality (13) and (21), we have Equation (22),

$$tvs(S_m \triangleright_o S_n) = \left\lceil \frac{n(m+1)+1}{2} \right\rceil . \tag{22}$$

□

Acknowledgement

The research for this article was supported by Litapdimas Grant.

References

[1] M. Bača, S. Jendroľ, M. Miller, and J. Ryan, On irregular total labellings, *Discrete Math.* **307** (2007), 1378–1388.

[2] P. Majerski, and J. Przybylo, Total vertex irregularity strength of dense graphs, *J. Graph Theory* **76** (1) (2014), 34–41.

[3] Nurdin, E. T. Baskoro, A. N. M. Salman, and N. N. Goas, On the total vertex irregularity strength of trees, *Discrete Math.* **310** (2010), 3043–3048.

[4] Nurdin, E. T. Baskoro, A. N. M. Salman, and N.N. Goas, On the total vertex irregular labeling for several types of trees, *Utilitas Math.* **83** (2010), 277–290.

[5] Nurdin, A. N. M. Salman, N. N. Goas, and E. T. Baskoro, On the total vertex-irregular strength of a disjoint of t copies of a path, *J. Combin. Math.* **71** (2009), 227–233.

[6] J. Przybylo, Liniear bound on the irregularity strength and the total vertex irregularity strength of graphs, *SIAM J. Discrete Math.* **23** (2009), 511–516

[7] R. Ramdani, A. N. M. Salman, H. Assiyatun, and A. Semaničová-Feňovcičová, M. Bača, On the total irregularity strength of disjoint union of arbitrary graphs, *Math. Rep.* **18(68)**, 4 (2016), 469–482.

[8] R. Ramdani and M. A. Ramdhani, Total vertex irregularity strength of comb product of two cycles, *MATEC Web Conf.* **197**, 01007 (2018).

[9] K. Wijaya and Slamin, Total vertex irregular labeling of wheels, fans, suns, and friendship graphs, *J. Combin. Math. Combin. Comput.* **65** (2008), 103–112.

[10] K. Wijaya, Slamin, Surahmat, and S. Jendroľ, Total vertex irregular labeling of complete bipartit graphs, *J. Combin. Math. Combin. Comput.* **55** (2005), 129–136.