Super local edge anti-magic total coloring of paths and its derivation

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Abstract

Suppose $G(V,E)$ be a graph and suppose $u,v,x$ be vertices of graph $G$. A bijection $f : V \cup E \rightarrow \{1, 2, 3, \ldots, |V(G)| + |E(G)|\}$ is called super local edge antimagic total coloring if for any adjacent edges $uv$ and $vx$ in $E(G)$, $w(uv) \neq w(vx)$, which $w(uv) = f(u) + f(uv) + f(v)$ for $f(V) = \{1, 2, 3, \ldots, |V(G)|\}$. By giving $G$ a labeling $f$, we denotes the minimum number of distinct weight of edges needed in $G$ as $\gamma_{\text{sleat}}(G)$. In this study, we proved the $\gamma_{\text{sleat}}$ of paths and its derivation.

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1. Introduction

For a graph $G$, a bijection $f : V \cup E \rightarrow \{1, 2, 3, \ldots, |V(G)| + |E(G)|\}$ assigns each edges to a particular distinct number from 1 up to $|V(G)| + |E(G)|$. Suppose $u, v, x$ be vertices of graph $G$. Weight of an edge $w(uv)$ is defined as $w(uv) = f(u) + f(uv) + f(v)$ for every edge $uv$ in $G$. A bijection $f$ is called local edge antimagic total coloring if any adjacent edges $uv$ and $vx$, $w(uv) \neq w(vx)$. For every distinct weight, we denote them as distinct colors. The local edge antimagic total chromatic number of $G$, $\gamma_{\text{leat}}(G)$, is the minimum number of colors for edges taken over all colorings induced by local antimagic total labelings of $G$. If vertices of $G$ are assigned smaller labels, then we call $f$ as super local edge antimagic total coloring of $G$, and the minimum number of colors as $\gamma_{\text{sleat}}$. For convenience, we will use the abbreviation SLEAT as super local...
edge anti-magic total. A SLEAT colorable graph is a graph that admits SLEAT coloring. Let \( \gamma(G) \) be the edge chromatic number of \( G \). We can observe that \( \Delta(G) \leq \gamma(G) \), where \( \Delta(G) \) is a maximum degree in graph \( G \). A compilation of observation concludes that \( \Delta(G) \leq \gamma(G) \leq \gamma_{\text{sleat}}(G) \) for an arbitrary graph \( G \).

Ringel firstly introduced the term of antimagic labelings [6]. There are many researchs conducted in antimagic labelings and its variation based on the survey of Gallian [5]. One of many variations is local antimagic labelings. Arumugam et al. introduced local antimagic as vertex local antimagic edge labelings [3]. Followed by labels addition for the vertex, vertex local antimagic total labelings, one described in by Kurniawati et al. [8]. The analog rises, edge local antimagic total labelings, explained by Agustin et al. in [1] that includes path graph. If every vertex labels are smaller than edge labels, then it is a super local edge antimagic total labelings, which Agustin et al. [2] and Kurniawati et al. [7] describe in some other graph.

In this paper, we study super local edge anti-magic total coloring of path graph, path with edge(s) addition which forms unicycle, connecting two disjoint path, and amalgamation of star. If \( \gamma(G) = \gamma_{\text{sleat}}(G) \), then by the prior inequality, \( \gamma_{\text{leat}}(G) = \gamma_{\text{sleat}}(G) \). In other words, studies about local edge anti-magic total (LEAT) coloring in all graphs that satisfy \( \gamma(G) = \gamma_{\text{sleat}}(G) \) are not necessary.

2. Main Results

2.1. Path Graphs

First, we are going to establish SLEAT for paths. This theorem is used in every proceeding theorems.

**Theorem 2.1.** Let \( n \geq 2 \) be integer and \( P_n \) be a path with \( n \) vertices. \( \gamma_{\text{sleat}}(P_n) = 2 \).

**Proof.** Let \( V(P_n) = \{v_i|1 \leq i \leq n\} \) and \( E(P_n) = \{v_iv_{i+1}|1 \leq i \leq n-1\} \). Since \( \Delta(P_n) = 2 \), \( \gamma_{\text{sleat}}(P_n) \geq 2 \). To show \( \gamma_{\text{sleat}}(P_n) \leq 2 \), define \( f : V(P_n) \cup E(P_n) \rightarrow \{1, 2, ..., |V(P_n)|+|E(P_n)|\} \).

**Case 1.** \( n \) is odd.

Labels vertices and edges as follows

\[
f(v_i) = \begin{cases} \frac{n+2}{2}, & \text{if } i \text{ is even,} \\ \frac{n+2}{2}, & \text{if } i \text{ is odd,} \end{cases}
\]

\[
f(v_iv_{i+1}) = \begin{cases} 2n - i + 1, & \text{if } i \text{ is even,} \\ 2n - i - 1, & \text{if } i \text{ is odd.} \end{cases}
\]

Hence, we get

\[
w(v_iv_{i+1}) = f(v_i) + f(v_iv_{i+1}) + f(v_{i+1}) = \begin{cases} \frac{5n+1}{2} + 1, & \text{if } i \text{ is even,} \\ \frac{5n+1}{2} - 1, & \text{if } i \text{ is odd.} \end{cases}
\]
Case 2. $n$ is even.

Labels vertices and edges as follows

$$f(v_i) = \begin{cases} \frac{n+i}{2}, & \text{if } i = 1, \\ n, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i > 1, \end{cases}$$

$$f(v_i v_{i+1}) = \begin{cases} n + 1, & \text{if } i = 1, \\ 2n - i, & \text{if } i \text{ is even,} \\ 2n - i + 2, & \text{if } i > 1. \end{cases}$$

Hence, we get

$$w(v_i v_{i+1}) = f(v_i) + f(v_i v_{i+1}) + f(v_{i+1}) = \begin{cases} \frac{5n}{2} - 1, & \text{if } i \text{ is even,} \\ \frac{5n}{2} + 1, & \text{if } i \text{ is odd.} \end{cases}$$

Since in every case there is only two distinct $w$, therefore $f$ is a SLEAT labeling of $P_n$ with $\gamma_{sleat}(P_n) = 2$.

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2.2. Unicycle graph

2.2.1. Path with an edge addition

Now, we start to add variation by adding an edge for path, forming a unicycle graph. We want to exclude the possibility of forming a regular cycle.

**Theorem 2.2.** Let $n \geq 3$ be integer, $P_n$ be path, and $G = P_n + v_i v_j$, where $v_i, v_j \in V(P_n)$, with $i$ and $j$ integers $1 \leq i < j \leq n$ and, except $(i, j) = (1, n)$. $\gamma_{sleat}(G) = 3$.

**Proof.** Since $\Delta(G) = 3$, $\gamma_{sleat}(G) \geq 3$. To show that $\gamma_{sleat}(G) \leq 3$, suppose $g$ is a labeling of the unicycle graph and $f$ is a SLEAT coloring of $P_n$ given in the proof of Theorem 2.1. We divide the proof into 4 subcases.

**Case 1.** $n$ is odd.

**Subcase 1.1.** Either $i$ or $j$ is odd. Labels vertices and edges as follows

$$g(P_n) = f(P_n),$$

$$g(v_i v_j) = 2n.$$
Hence, we get
\[ w(v_i v_j) = g(v_i) + g(v_i v_j) + g(v_j) = \begin{cases} 3n + \frac{i+j}{2}, & \text{if } i \text{ and } j \text{ are odd,} \\ \frac{5n}{2} + \frac{i+j}{2}, & \text{if } i \text{ or } j \text{ is odd.} \end{cases} \]

Therefore, we get a minimum value of \( w(v_i v_j) \geq \frac{5n+1}{2} \). Now, we will show that \( w(v_i v_j) \) has different value with \( w(v_i v_{i+1}) \). It is clear that \( w(v_i v_{i+1}) \) has a maximum value of \( \frac{5n+1}{2} + 1 \). By the premise, \( i + j \) has the smallest possible value attainable \( i + j \geq 4 \). We observe that
\[ w(v_i v_j) \geq \frac{5n+4}{2} > \frac{5n+1}{2} + 1, \]
which implies that \( w(v_i v_j) \) is a third distinct weight.

**Sub-case 1.2.** \( i \) and \( j \) are even. Labels vertices and edges as follows
\[
\begin{align*}
g(V(P_n)) &= f(V(P_n)), \\
g(E(P_n)) &= f(E(P_n)) + 1, \\
g(v_i v_j) &= n + 1.
\end{align*}
\]
Since every labels of edges is increased by 1, we will have a new weight of
\[ w(v_i v_j) = g(v_i) + g(v_i v_j) + g(v_j), \]
\[
w(v_i v_{i+1}) = \begin{cases} \frac{5n+1}{2} + 2 & \text{for } i \text{ is even}, \\ \frac{5n+1}{2} & \text{for } i \text{ is odd.} \end{cases}
\]
and \( w(v_i v_j) = n + 1 + \frac{i+j}{2} \). Now, we will show that \( w(v_i v_j) \) has different value with \( w(v_i v_{i+1}) \). It is clear that \( w(v_i v_{i+1}) \) has a minimum value of \( \frac{5n+1}{2} \). By the premise, \( i + j \) has the largest possible value attainable \( i + j \leq 2n - 4 \). We observe that
\[ w(v_i v_j) \leq n + 1 + \frac{2n-4}{2} = 2n - 1 < \frac{5n+1}{2}, \]
and \( w(v_i v_j) \) is a third distinct weight.

**Case 2.** \( n \) is even.

**Subcase 2.1.** Either \( i \) or \( j \) is equal to 1 or even. Labels vertices and edges as follows
\[
\begin{align*}
g(P_n) &= f(P_n), \\
g(v_i v_j) &= 2n.
\end{align*}
\]
Hence, we get possible weights

$$w(v_i v_j) = \begin{cases} 
3n + \frac{i+j}{2} - 2, & \text{if } i \text{ and } j \text{ are even}, \\
\frac{7}{2}n + \frac{i+j}{2} - 1, & \text{if } i = 1 \text{ and } j \text{ is even}, \\
\frac{5}{2}n + \frac{i+j}{2} - \frac{3}{2}, & \text{if } i \text{ is even and } j \text{ is odd}, \\
\frac{3}{2}n + \frac{i+j}{2} - \frac{3}{2}, & \text{if } i \text{ is odd and } j \text{ is even}, \\
3n + \frac{i+j}{2} - \frac{3}{2}, & \text{if } i = 1 \text{ and } j \text{ is odd}.
\end{cases}$$

We will show that $w(v_i v_j)$ has different value with $w(v_i v_{i+1})$. It is clear that $w(v_i v_{i+1})$ has a maximum value of $\frac{5n}{2} + 1$. A minimum value for these possible weights is needed to be determined.

If both $i$ and $j$ are even, then $i + j$ has smallest value attainable $i + j \geq 2 + 4 = 6$, such that $w(v_i v_j) \geq 3n + 1$. If $i = 1$ and $j$ is even, then $j$ has smallest value attainable $j \geq 4$, such that $w(v_i v_j) \geq \frac{7}{2}n + 1 = \frac{5}{2}n + n + 1$. If $i$ is even and $j$ is odd, then $i + j$ smallest value attainable $i + j \geq 2 + 5 = 7$, such that $w(v_i v_j) \geq \frac{5}{2}n + 2$. If $i = 1$ and $j$ is odd, then $j$ smallest value attainable $j \geq 3$, such that $w(v_i v_j) \geq 3n + 1$.

It can be seen that $w(v_i v_j)$ has a minimum value of $\frac{5}{2}n + 2 \geq \frac{5}{2}n + 1$, since $n$ is even. It means $w(v_i v_j)$ is a third distinct weight.

**Subcase 2.2.** $i$ and $j$ are odd, and both of them does not equal to 1. Labels as follows

$$g(V(P_n)) = f(V(P_n)),
$$
$$g(E(P_n)) = f(E(P_n)) + 1,$$
$$g(v_i v_j) = n + 1.$$ since every labels of edges is incremented by 1, we will have a new weight of

$$w(v_i v_j) = g(v_i) + g(v_i v_j) + g(v_j),$$

$$w(v_i v_{i+1}) = \begin{cases} 
\frac{5n}{2}, & \text{if } i \text{ is even}, \\
\frac{5n}{2} + 2, & \text{if } i \text{ is odd}.
\end{cases}$$

and $w(v_i v_j) = n + \frac{i+j}{2}$. We will show that this $w(v_i v_j)$ has different value with $w(v_i v_{i+1})$. It is clear that $w(v_i v_{i+1})$ has a minimum value of $\frac{5n}{2}$. By the premise, $i + j$ has the largest attainable value $i + j \leq 2n - 4$. We observe that $w(v_i v_j) \leq n + \frac{2n - 4}{2} = 2n - 2 < \frac{5n}{2}$. Therefore, $w(v_i v_j)$ is a third distinct weight.

Since in every case $\gamma_{\text{sleat}}(G) \leq 3$, we can deduce that $g$ is SLEAT of $G$ with $\gamma_{\text{sleat}}(G) = 3$. \qed

2.2.2. Path with some edges addition

This method of edge addition can be done multiple times if it meets certain premises. We need to introduce a convenient even function, defined as follows

$$\text{even}(n) = \begin{cases} 
1, & \text{if } n \text{ is even}, \\
0, & \text{if } n \text{ is odd}.
\end{cases}$$
Theorem 2.3. Let \( n \geq 4 \) be integer, \( P_n \) be path, and \( G = P_n + \{v_{i-k}v_{j-k}|0 \leq k \leq p, 0 \leq p \leq \min\{i-1+\text{even}(n), j-i-1\}\} \) where \( v_{i-k}, v_{j-k} \in V(P_n) \), \( i \) and \( j \) satisfy \( 1 \leq i < j \leq n, i + j \) is odd, and \((i, j) \neq (1, n)\). \( \gamma_{\text{sleat}}(G) = 3 \)

Proof. Since \( \Delta(G) = 3 \), \( \gamma_{\text{sleat}}(G) \geq 3 \). To show that \( \gamma_{\text{sleat}}(G) \leq 3 \), suppose \( g \) is a labeling of \( G \) and \( f \) is a SLEAT coloring of \( P_n \) given in the proof of Theorem 2.1.

Labels vertices and edges as follows

\[
g(P_n) = f(P_n),
\]
\[
g(v_{i-k}v_{j-k}) = 2n + k.
\]

The weight of edge are the following

\[
w(v_{i-k}v_{j-k}) = g(v_{i-k}) + g(v_{i-k}v_{j-k}) + g(v_{j-k}) = \begin{cases} 
\frac{5}{2}n + \frac{i+j}{2}, & \text{if } n \text{ is odd, } i + j \text{ is odd,} \\
\frac{5}{2}n + \frac{i+j}{2} - \frac{3}{2}, & \text{if } n \text{ is even, } i + j \text{ is odd.}
\end{cases}
\]

We will show that \( w(v_{i-k}v_{j-k}) \) has different value with \( w(v_iv_{i+1}) \). It is clear that \( w(v_iv_{i+1}) \) has a maximum value of \( \frac{5n+1}{2} + 1 \). A minimum value for these possible weights is needed to be determined.

If both \( n \) and \( i + j \) are odd, then smallest value attainable \( i + j \geq 1 + 3 = 4 \), such that \( w(v_{i-k}v_{j-k}) \geq \frac{5}{2}n + 2 \). If \( n \) is even, \( i \neq 1 \), and \( i + j \) is odd, then smallest value attainable \( i + j \geq 2 + 5 = 7 \), such that \( w(v_{i-k}v_{j-k}) \geq \frac{5}{2}n + 2 \).

Since \( n \geq 4 \), \( w(v_{i-k}v_{j-k}) \) has a minimum value of \( \frac{5}{2}n + 2 \geq \frac{5n+1}{2} + 1 \). As a result, \( w(v_{i-k}v_{j-k}) \) is a third distinct weights. Therefore, \( g \) is a SLEAT of \( G \) with \( \gamma_{\text{sleat}}(G) = 3 \).

Figure 3: SLEAT coloring of \( G = P_7 + v_2v_5 \) and \( G = P_6 + v_3v_5 \)
2.3. Connecting disjoint paths

Another kind of graph that is observed in this study is some disjoint paths with addition of edge(s) that connects some paths. Two kinds of this family of graph are hedge graph and hedgerow graph. These two graph will be defined in the proceeding subsections.

2.3.1. Hedge graph

To start, we observe two disjoint paths with addition of one edge that connects the paths. Such graph is called hedge graph \(Hd\). Before giving the formal definition of hedge graph, we want to exclude the possibility of connection that forms a simple path. For arbitrary set \(A, B\), we use the cartesian products of the two sets \(A \times B = \{ (a, b) | a \in A, b \in B \}\) in order to have simple notations. Let \(P_m\) and \(P_n\) be path with \(V(P_m) = \{ v_k | 1 \leq k \leq m \}\) and \(V(P_n) = \{ u_l | 1 \leq l \leq n \}\). We define \(Hd(m, n, i, j)\) as follows

\[
V(Hd(m, n, i, j)) = V(P_m) \cup V(P_n),
\]
\[
E(Hd(m, n, i, j)) = E(P_m) \cup E(P_n) \cup \{ v_iu_j | 1 \leq i \leq m, 1 \leq j \leq n, (i, j) \notin (1, n) \times (1, m) \}.
\]

**Theorem 2.4.** For integers \(n \geq m \geq 3\), if \(G = Hd(m, n, i, j)\), then \(\gamma_{sleat}(G) = 3\).

**Proof.** Since \(\Delta(G) = 3\), \(\gamma_{sleat}(G) \geq 3\). To show that \(\gamma_{sleat}(G) \leq 3\), suppose \(g\) is a labeling of \(G\) and \(f\) is a SLEAT coloring of \(P_n\) given in the proof of Theorem 2.1.

**Case 1.** \(i \neq j\).

Add an edge \(v_1u_1\) to form a \(P_{m+n}\), apply labeling \(f\) in a path \(P_{m+n}\) from \(v_n\) to \(u_m\), then remove \(v_1u_1\). Now, \(g\) is a labelings that follows

\[
g(v_iu_j) = f(v_1u_1),
\]
\[
g(V(G)) = f(V(P_{m+n})),
\]
\[
g(E(G) \setminus \{ v_iu_j \}) = f(E(P_{m+n}) \setminus \{ v_1u_1 \}).
\]

If \(n + m\) is odd, we have the following weight

\[
w(v_i) = w(u_i) = \begin{cases} \frac{5n+1}{2} + 1, & \text{if } i \text{ is even}, \\ \frac{5n+1}{2} - 1, & \text{if } i \text{ is odd}. \end{cases}
\]

As for \(n + m\) is even, we have the following weight

\[
w(v_i) = w(u_i) = \begin{cases} \frac{5n}{2} - 1, & \text{if } i \text{ is even}, \\ \frac{5n}{2} + 1, & \text{if } i \text{ is odd}. \end{cases}
\]

We get \(g(v_1) + g(v_iu_j) + g(u_1) = w(v_i) + w(v_iu_{i+1}) = w(u_iu_{i+1})\) for even \(i\). Now, we can observe the partition \(\{ g(v_i) \} \) and \(\{ g(u_j) \} \) for arbitrary \(i\) and \(j\), to find that there is no pair of \((g(v_i), g(u_j))\) such that \(g(v_i) + g(u_j) = g(v_1) + g(u_1)\) which satisfies \(i \neq j\). Hence, \(w(v_iu_j) \neq w(v_iu_{i+1})\) and \(w(v_iu_j) \neq w(v_iu_{i+1})\) for even \(i\).
Next, we get $g(v_i) + g(v_iu_j) + g(u_1) - 2 = w(v_iu_{i+1}) = w(u_{i+1}u_{i+2})$ or $g(v_i) + g(v_iu_j) + g(u_1) + 2 = w(v_iu_{i+1}) = w(u_{i+1}u_{i+2})$ for odd $i$. Again, we can observe the partition $\{g(v_i)\}$ and $\{g(u_1)\}$ for arbitrary $i$ and $j$, to find that there is no pair of $(g(v_i), g(u_1))$ such that $g(v_i) + g(u_1) = g(v_i) + g(u_j) = g(v_i) + g(u_1) - 2$ or $g(v_i) + g(u_1) = g(v_i) + g(u_1) + 2$. Hence, $w(v_iu_j) \neq w(v_iu_{i+1})$ and $w(v_iu_j) \neq w(v_iu_{i+1})$ for odd $i$.

**Case 2.** $i = j$, and $n \neq m$.

Add an edge $v_nu_m$ to form a $P_{m+n}$, apply labeling $f$ to path $P_{m+n}$ from $v_1$ to $u_1$, then remove $v_nu_m$. Now, $g$ is a labeling that follows

$$g(v_iu_j) = f(v_nu_m),$$
$$g(V(G)) = f(V(P_{m+n})),
$$
$$g(E(G)\\{v_iu_j\}) = f(E(P_{m+n})\\{v_nu_m\}).$$

If $n + m$ is odd, we have the following weight

$$w(v_iu_{i+1}) = w(u_{i+1}u_{i+2}) = \begin{cases} \frac{5n+1}{2} + 1, & \text{if } i \text{ is even,} \\ \frac{5n+1}{2} - 1, & \text{if } i \text{ is odd.} \end{cases}$$

With the addition of $w(u_1u_2) = \frac{5n+1}{2} + 1$. As for $n + m$ is even, we have the following weight

$$w(v_iu_{i+1}) = w(u_{i+1}u_{i+2}) = \begin{cases} \frac{5n}{2} - 1, & \text{if } i \text{ is even,} \\ \frac{5n}{2} + 1, & \text{if } i \text{ is odd.} \end{cases}$$

We get $g(v_n) + g(v_iu_j) + g(u_m) = w(v_iu_{i+1}) = w(u_{i+1}u_{i+2}) = w(u_1u_2)$ for $n$ and $i$ that have the same parity. Now, we can observe the partition $\{g(v_i)\}$ and $\{g(u_i)\}$ for arbitrary $i$, to find that there is no pair of $(g(v_i), g(u_1))$ such that $g(v_i) + g(u_i) = g(v_n) + g(u_m)$. Hence, $w(v_iu_j)$ does not equal any of $w(v_iu_{i+1}), w(u_{i+1}u_{i+2}),$ or $w(u_1u_2)$.

Next, we get $g(v_n) + g(v_iu_j) + g(u_m) - 2$ or $g(v_1) + g(v_iu_j) + g(u_1) + 2$ to be equal with $w(v_{i+1}u_{i+2}) = w(u_{i+1}u_{i+2}) = w(v_1u_2)$ for $n$ and $i$ have different parity. Again, we can observe the partition $\{g(v_i)\}$ and $\{g(u_i)\}$ for arbitrary $i$, to find that there is no pair of $(g(v_i), g(u_i))$ such that $g(v_i) + g(u_i) = g(v_n) + g(u_m) - 2$ or $g(v_i) + g(u_i) = g(v_n) + g(u_m) + 2$. Hence, $w(v_iu_j)$ does not equal any of $w(v_{i+1}u_{i+2}), w(u_{i+1}u_{i+2}),$ or $w(v_1u_2)$.

**Case 3.** $i = j$, and $n = m$, except $n = 3$.

Since the graph with $i = 2$ is isomorphic with $i = n - 1$, it is only necessary to determine labels for $i \geq 3$.

Add an edge $v_1u_1$ to form a path $P_{m+n}$, apply labeling $f$ to $P_{m+n}$ from $v_n$ to $u_m$, then remove $v_1u_1$. If $n$ is odd, $g$ is a labeling that follows

$$g(v_iu_j) = f(u_1u_2).$$
\[ g(u_1u_2) = f(v_1u_1), \]
\[ g(V(G)) = f(V(P_{m+n})), \]
\[ g(E(G) \setminus \{v_iu_j \cup u_1u_2\}) = f(E(P_{m+n}) \setminus \{v_1u_1 \cup u_1u_2\}). \]

If \( n \) is even, \( g \) is a labeling that follows
\[ g(v_iu_j) = f(v_1v_2), \]
\[ g(v_1v_2) = f(v_1u_1), \]
\[ g(V(G)) = f(V(P_{m+n})), \]
\[ g(E(G) \setminus \{v_iu_j \cup v_1v_2\}) = f(E(P_{m+n}) \setminus \{v_1u_1 \cup v_1v_2\}). \]

By preceding labelings, we have the following weight
\[ w(v_{i+1}v_{i+2}) = w(u_{i+1}u_{i+2}) = \begin{cases} \frac{5n}{2} - 1, & \text{if } i \text{ is even,} \\ \frac{5n}{2} + 1, & \text{if } i \text{ is odd.} \end{cases} \]

In addition, if \( n \) is odd, \( w(v_1v_2) = w(v_{i+1}v_{i+2}) \) and \( w(u_1u_2) = w(v_iu_j) \). While if \( n \) is even, \( w(u_1u_2) = w(u_{i+1}u_{i+2}) \) and \( w(v_1v_2) = w(v_iu_j) \). By case 1, it is already proven that \( w(v_iu_j) \) is distinct.

**Case 4.** \( i = j = 2, \) and \( n = m = 3. \)

This is the only one specific graph that does not belong to any preceding case. Thus, we can simply do an enumeration for completing the theorem. Labels the graph \( G \) as in Figure 4.

Since every possible graph is covered, therefore \( \gamma_{sleat}(Hd(m, n, i, j)) = 3. \)

![Figure 4: SLEAT coloring of \( Hd(4, 3, 3, 2), \) \( Hd(5, 4, 2, 2), \) \( Hd(4, 4, 3, 3), \) and \( Hd(3, 3, 2, 2) \)](image-url)
2.3.2. Hedgerow

Suppose we have \( m \) disjoint paths with length of \( n \). Every neighbouring path is connected with an edge that is uniform through every two connected paths. Such graph is called hedgerow graph \( H_r \). Let \( mP_n \) be the disjoint union of \( m \) number of \( P_n \) with \( V(mP_n) = \{v^k_i | 1 \leq l \leq n, 1 \leq k \leq m\} \). Formally, we define hedgerow graph \( H_r(m, n, i, j) \) as follows

\[
V(H_r(m, n, i, j)) = V(mP_n),
\]

\[
E(H_r(m, n, i, j)) = E(mP_n) \cup \{v^k_i v^{k+1}_j | 1 \leq i \leq n, 1 \leq j \leq n, 1 \leq k \leq m - 1\}.
\]

We start an attempt to find hedgerow for \( m \geq 3 \). We only find the value of \( \gamma_{sleat} \) for hedgerow graph for some \((m, n, i, j)\).

**Theorem 2.5.** Let integer \( m \geq 2 \) and integer \( n \geq 2 \). For \( G \cong H_r(m, n, i, j) \),

1. If \( n \) is even, \( i \neq j \), and \( i \neq n - j + 1 \), then \( \gamma_{sleat}(G) = 3 \).
2. If \( m \) is odd, \( i = j \), and \( i \notin \{1, n\} \), then \( \gamma_{sleat}(G) = 4 \).

**Proof.** By Theorem 2.1, we have \( f \), SLEAT labelings of path graph. Suppose \( g \) is SLEAT labelings of the connected multiple path.

**Case 1.** \( n \) is even, \( i \neq j \), and \( i \neq n - j + 1 \)

Since \( \Delta(G) = 3 \), \( \gamma_{sleat}(G) \geq 3 \). Next, we will show that \( \gamma_{sleat}(G) \leq 3 \). Since the graph has reflexive form, if \( i = 1 \) we can redefine \((i, j) = (1, j^*) \) into \((i, j) = (j^*, 1)\), thus we only consider \( i \geq 2 \). Add edges \( v^k_i v^{k+1}_j \) for every integer \( k \in [1, m] \), forming \( P_{mn} \), apply \( f \) in a path from \( v^1_i \) to \( v^m_i \). Now, for every integer \( k \in [1, m - 1] \), \( g \) is a labeling that follows

\[
g(v^k_i v^{k+1}_j) = f(v^k_i v^{k+1}_j),
\]

\[
g(V(G)) = f(V(P_{mn})),
\]

\[
g(E(G \cup \{v^k_i v^{k+1}_j\})) = f(E(P_{mn} \cup \{v^k_i v^{k+1}_j\})).
\]

In the proof of Theorem 2.4, we have shown that \( w(v^k_i v^{k+1}_j) \) with two other weights in the paths, with restriction of \( i \neq n - j + 1 \). Hence, we have the weights

\[
w(v^k_i v^{k+1}_j) = \begin{cases} 
\frac{5}{2}nm - 1, & \text{if } i \text{ is even}, \\
\frac{5}{2}nm + 1, & \text{if } i \text{ is odd}.
\end{cases}
\]

Originally, \( f(v^k_i) + f(v^k_i v^{k+1}_j) + f(v^{k+1}_j) \) is constant for every integer \( k \in [1, m - 1] \). Since \( g(v^k_i) - g(v^k_i) \) and \( g(v^{k+1}_j) - g(v^{k+1}_j) \) are constant for the same \( k \), we have third distinct weight \( w(v^k_i v^{k+1}_j) \). Hence, \( \gamma_{sleat}(G) = 3 \).

**Case 2.** \( m \) is odd, \( i = j \), and \( i \notin \{1, n\} \)
Since $\Delta(G) = 4$, $\gamma_{sleat}(G) \geq 4$. Next, we will show that $\gamma_{sleat}(G) \leq 4$. Add edges $v_n^k v_n^{k+1}$ for every odd integer $k \in [1, m-2]$ and $v_1^k v_1^{k+1}$ for every even integer $k \in [2, m-1]$, forming $P_{mn}$, apply $f$ in a path from $v_1^1$ to $v_m^n$. Now, for every integer $k \in [1, m-1]$ and integer $k' \in [1, \frac{m-1}{2}]$, $g$ is a labeling that follows

$$
g(v_i^k v_i^{k+1}) = \begin{cases} 
f(v_i^k v_i^{k+1}) & \text{for } k \text{ is odd,} \\
f(v_i^k v_i^{k+1}) & \text{for } k \text{ is even,}
\end{cases}
$$

$$
g(V(G)) = f(V(P_{mn})),$$

$$
g(E(G\setminus \bigcup_k \{v_i^k v_i^{k+1}\})) = g(E(P_{mn}\setminus (\bigcup_k \{v_1^{2k'} v_1^{2k'+1}, v_n^{2k'-1} v_n^{2k'}\}))).$$

Since $f(v_1^{2k'} v_1^{2k'+1}) - f(v_n^{2k'-1} v_n^{2k'})$ is constant for every $k' \in [1, \frac{m-1}{2}]$, $w(v_1^{2k'} v_1^{2k'+1})$ is constant for every $k$, $w(v_n^{2k'-1} v_n^{2k'})$ is constant for every $k$, and $w(v_1^{2k'} v_1^{2k'+1}) \neq w(v_n^{2k'-1} v_n^{2k'})$. Hence, $\gamma_{sleat}(G) = 4$.

Therefore, the theorem holds.

The same procedure can also be applied if $i \in \{1, n\}$ for $G$, but it will not give an exact value of $\gamma_{sleat}(G)$ since $\gamma(G)$ is still lower than the number of the weights. Despite the uncertainty, we establish the corollary as follows.

**Corollary 2.1.** Let integer $n \geq 2$ and odd integer $m \geq 3$. If $G \cong Hr(m, n, i, i)$, then $3 \geq \gamma_{sleat}(G) \geq 4$.

It can be seen that the graph from the corollary is isomorphic with the comb product of $P_m$ with $P_{n-1}$. We revised a theorem that given by Kurniawati et al. in [7].

We attempted to find the value for some $(m, n, i, j)$ that did not satisfy preceding theorem premises, but it is not completed yet. It is left as an open problem.

**Open Problem 1.** Find the value of $\gamma_{sleat}(Hr(m, n, i, j))$ for remaining cases of $(m, n, i, j)$. 

Figure 5: SLEAT coloring of $Hr(4, 4, 4, 3)$ and $Hr(5, 4, 2, 2)$

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2.4. Amalgamation of Star

We proceed by finding SLEAT for star graph. Star graph is denoted as $S_m$ where $V(S_m) = \{c, v_i | 1 \leq i \leq m\}$ and $E(S_m) = \{cv_i | 1 \leq i \leq m\}$. Since every edge in the graph is adjacent, $\gamma(S_m) = m$. Since $\gamma(G) \leq \gamma_{\text{sleat}}(G) \leq |E(G)|$ for arbitrary graph $G$, it is clear that $\gamma_{\text{sleat}}(S_m) = m$.

If one want an example how to achieve such graph, labels as follows $f(v_i) = i, f(c) = m + 1, f(cv_i) = m + i + 1$. The weights is going to be consecutive by 2 as the common difference.

In this section, we study an amalgamation of star and graph which have certain $\gamma_{\text{sleat}}$.

**Theorem 2.6.** Suppose $G_1$ be a SLEAT colorable graph and $G_2 \cong \text{Amal}(S_m, G_1; c)$ which vertex $c$ is a center in $S_m$, for integer $m \geq 1$. $\gamma_{\text{sleat}}(G_2) \leq \gamma_{\text{sleat}}(G_1) + m$

**Proof.** Let $n_v = |V(G_1)|$ and $n_e = |E(G_1)|$. Suppose $g$ is the SLEAT labelings of $G_1$ and $f$ is the SLEAT labelings of $G_2$. Labels as follows:

\[
g(V(G_1)) = f(V(G_1)),
\]

\[
g(E(G_1)) = f(E(G_1)) + m,
\]

\[
g(v_i) = n_v + i,
\]

\[
g(cv_i) = n_v + n_e + m + i.
\]

Let $w_f$ is weights as a result of $f$. Now, we define $w'_f = w_f + m$. Next, we define $w_s$ as $w_s = g(v_i) + g(cv_i) + g(c) = 2n_v + n_e + m + 2i + f(c)$. Now, every distinct $w'_f$ and every distinct $w_s$ are weights that is generated from $g$. What is left to prove is $w'_f \neq w_s$ for arbitrary possible weights.

Let $p$ is the lowest $f(c)$ for arbitrary $c$. Since $p$ is adjacent with any $w'_f$, we can observe the upper bound for $w'_f$ by assuming $w'_f$ contains the largest vertex label $n_v$, and largest edge label $n_v + n_e + m$. As a result,

\[
w'_f \leq n_v + n_v + n_e + m + p = 2n_v + n_e + m + p
\]

Now, we can find the lower bound any $w_s$, by choosing the lowest possible $i = 1$, then

\[
w_s \geq 2n_v + n_e + m + p + 2
\]

It is clear that $2n_v + n_e + m + p + 2 \geq 2n_v + n_e + m + p$. Therefore, $w'_f \neq w_s$ for arbitrary possible weights. This holds the theorem.

The preceding theorem applies for any vertex $c$ in graph $G_1$. By choosing certain $c$, we have Corollary 2.2.

**Corollary 2.2.** Let $G_1$ is a SLEAT colorable graph and $G_2 \cong \text{Amal}(S_m, G_1; c)$ which vertex $c$ is a center in $S_m$, for integer $m \geq 1$. If $\text{deg}(c) = \Delta(G_2), \gamma_{\text{sleat}}(G_2) = \gamma_{\text{sleat}}(G_1) + m$. 

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3. Summary

Here we summarize our results in Table 1.

<table>
<thead>
<tr>
<th>Graph</th>
<th>Notation</th>
<th>$\gamma_{steat}$</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Path</td>
<td>$P_n$</td>
<td>2</td>
<td>$1 \leq i &lt; j \leq n$, and $(i, j) \neq (1, n)$</td>
</tr>
<tr>
<td>Path + edge</td>
<td>$P_n + v_i v_j$</td>
<td>3</td>
<td>$1 \leq i &lt; j \leq n$, $i + j$ is odd, $0 \leq p \leq \min{i-1+\text{even}(n), j-i-1}$ and $(i, j) \neq (1, n)$</td>
</tr>
<tr>
<td>Path + multiple edges</td>
<td>$P_n + {v_i-k v_j-k\mid 0 \leq k \leq p}$</td>
<td>3</td>
<td>$v_i \in V(P_n)$, $u_j \in V(P_m)$, and $(i, j) \notin (1, n) \times (1, m)$</td>
</tr>
<tr>
<td>Hedge graph</td>
<td>$Hd(m, n, i, j)$</td>
<td>3</td>
<td>$n$ is even, $i \neq j$, and $i \neq n-j+1$</td>
</tr>
<tr>
<td>Hedgerow graph</td>
<td>$Hr(m, n, i, j)$</td>
<td>3</td>
<td>$m$ is odd, $i = j$, and $i \notin {1, n}$</td>
</tr>
<tr>
<td>Star</td>
<td>$S_n$</td>
<td>$n$</td>
<td></td>
</tr>
<tr>
<td>Amalgamation of star</td>
<td>$Amal(G, S_n; c)$</td>
<td>at most $\gamma_{steat}(G) + n$</td>
<td>$c \in V(G), V(S_n)$ and $c$ is center in $S_n$</td>
</tr>
<tr>
<td>Amalgamation of star</td>
<td>$Amal(G, S_n; c)$</td>
<td>$\gamma_{steat}(G) + n$</td>
<td>$c \in V(G), V(S_n)$, $c$ is center in $S_n$, and $\deg(c) = \Delta(G)$</td>
</tr>
</tbody>
</table>

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References


