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Total edge irregularity strength of some cycle related graphs

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Abstract

An edge irregular total k-labeling $f : V \cup E \rightarrow \{1, 2, ..., k\}$ of a graph $G = (V, E)$ is a labeling of vertices and edges of G in such a way that for any two different edges uv and $u'v'$, their weights $f(u) + f(uv) + f(v)$ and $f(u') + f(u'v') + f(v')$ are distinct.The total edge irregularity strength $tes(G)$ is defined as the minimum k for which the graph G has an edge irregular total k-labeling. In this paper, we determine the total edge irregularity strength of new classes of graphs $C_m@C_n$, $P^*_{m,n}$ and $C^*_{m,n}$ and hence we extend the validity of the conjecture tes(G) = max $\left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\}$ for some more graphs.

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1. Introduction

Throughout this paper, G is a simple graph, V and E are the sets of vertices and edges of G, with cardinalities $|V|$ and $|E|$ respectively. A labeling of a graph is a map that carries graph elements to the numbers. A labeling is called a vertex labeling, an edge labeling or a total labeling, if the domain of the map is the vertex set, the edge set, or the union of vertex and edge sets respectively.

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Baca et al. in [2] started to investigate the total edge irregularity strength of a graph, an invariant analogous to the irregularity strength for total labeling. For a graph $G = (V(G), E(G))$, the weight of an edge $e = xy$ under a total labeling ξ is $wt_\xi(e) = \xi(x) + \xi(e) + \xi(y)$. For a graph G we define a labeling $\xi : V(G) \cup E(G) \to \{1, 2, \cdots, k\}$ to be an edge irregular total k-labeling of the graph G if for every two different edges xy and $x'y'$ of G one has $wt_\xi(xy) \neq wt_\xi(x'y')$. The total edge irregular strength, $tes(G)$, is defined as the minimum k for which G has an edge irregular total k-labeling. In [3], we can find that

$$tes(G) \geq \max\left\{\left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil\right\},$$

where $\Delta(G)$ is the maximum degree of G, and also there are determined the exact values of the total edge irregularity strength for paths, cycles, stars, wheels and friendship graphs. Recently Ivanco and Jendrol [6] proved that for any tree T,

$$tes(T) = \max\left\{\left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil\right\}.$$

Moreover, they posed a conjecture that for an arbitrary graph G different from K_5 and the maximum degree $\Delta(G)$,

$$tes(G) = \max\left\{\left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil\right\}.$$

The Ivanco and Jendrol's conjecture has been verified for complete graphs and complete bipartite graphs in [7], for categorical product of cycle and path in [1] and [12], for corona product of paths with some graphs in [11]. In[8], Jeyanthi et al. verified the conjecture for disjoint union of double wheel graphs. In[5], Indra et al. verified the conjecture for generalized uniform theta graph.

Motivated by the papers [9, 10], we define three new classes of graphs and extend the validity for the conjecture for some more families of graphs. We define the graph $C_m@C_n$, $n \geq 3$, $m \geq 3$ as follows. Denote the vertex set of $C_m@C_n$ by $V(C_m@C_n) = \{u_i \mid 1 \leq i \leq n\} \cup \{v_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ and the edge set of $C_m@C_n$ by $E(C_m@C_n) = \{e_i = u_i u_{i+1} \mid 1 \leq i \leq n, u_{n+1} = u_1\} \cup \{e_{i1} = u_i v_{i1} | 1 \leq i \leq n\} \cup \{e_{i(j+1)} = v_{ij} v_{i(j+1)} | 1 \leq i \leq n, 1 \leq j \leq m, v_{i(m+1)} = v_{i1}\}$. In $C_m@C_n$, $|V(C_m@C_n)| = n(m+1)$ and $|E(C_m@C_n)| = n(m+2)$. The graph $C_3@C_9$ is shown in Figure 1.

We introduce another new class of graph $P^*_{m,n}$. The graph $P^*_{m,n}$, $m \geq 3$, $n \geq 2$ is defined as follows: denote the vertex set of $P^*_{m,n}$ by $V(P^*_{m,n}) = \{v_i \mid 1 \leq i \leq n+1\} \cup \{v_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m-2\}$ and the edge set of $P^*_{m,n}$ by $E(P^*_{m,n}) = \{e_i = v_i v_{i+1} \mid 1 \leq i \leq n\} \cup \{e_{ij} = v_{i(j-1)} v_{ij} | 1 \leq i \leq n, 2 \leq j \leq m-2\} \cup \{e_{i1} = v_i v_{i1}) | 1 \leq i \leq n\} \cup \{e_{i(m-1)} = v_{i(m-2)} v_{i1} | 1 \leq i \leq n\}$. In $P^*_{m,n}$, $|V(P^*_{m,n})| = mn - n + 1$ and $|E(P^*_{m,n})| = nm$. The graph $P^*_{4,7}$ is shown in Figure 2. In $P^*_{m,n}$, $m \geq 3$, $n \geq 2$, identifying the vertices v_1 and v_{n+1} we obtain the new class of graph denoted by $C^*_{m,n}$.

In this paper, we determine the total edge irregularity strength of these new classes of graphs $C_m@C_n$, $P^*_{m,n}$ and $C^*_{m,n}$ and hence we extend the validity of the conjecture

$$tes(G) = max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\}$$

for some more families of graphs.

2. Main Results

In the following theorem we describe an optimal edge irregular total labeling for the graph $C_m@C_n$.

Theorem 2.1. *For any integers $m \geq 3$, $n \geq 3$, $tes(C_m@C_n) = \lceil\frac{n(m+2)+2}{3}\rceil$.*

Proof. The vertex set of $C_m@C_n$ is $\{u_i \mid 1 \leq i \leq n\} \cup \{v_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ and the edge set of $C_m@C_n$ is $\{e_i = u_iu_{i+1} \mid 1 \leq i \leq n, u_{n+1} = u_1\} \cup \{e_{i1} = u_iv_{i1} | 1 \leq i \leq n\} \cup \{e_{i(j+1)} = v_{ij}v_{i(j+1)} | 1 \leq i \leq n, 1 \leq j \leq m, v_{i(m+1)} = v_{i1}\}$. Since $tes(G) \geq max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\}$, it is enough to prove that $tes(C_m@C_n) \leq \left\lceil\frac{n(m+2)+2}{3}\right\rceil$.
Let $k = \left\lceil\frac{n(m+2)+2}{3}\right\rceil$.
We construct an edge-irregular total labeling l as follows:

$$\begin{aligned}
l(u_i) &= \begin{cases} 1 & \text{if } i = 1 \\ \lceil\frac{m+3}{2}\rceil & \text{if } i = 2 \\ m+5 & \text{if } i = 3 \\ k - (\lfloor\frac{n-6}{2}\rfloor - (i-3)m) & \text{if } 4 \leq i < \lfloor\frac{n}{2}\rfloor \\ k & \text{if } \lfloor\frac{n}{2}\rfloor \leq i \leq \lfloor\frac{n}{2}\rfloor + 2 \\ k - \lfloor\frac{i-\lfloor\frac{n}{2}\rfloor}{2}\rfloor m & \text{if } \lfloor\frac{n}{2}\rfloor + 3 \leq i \leq n-2 \\ \lceil\frac{3m+7}{2}\rceil & \text{if } i = n-1 \\ 3+m & \text{if } i = n, \end{cases} \\
l(v_{1j}) &= \lfloor\tfrac{j+4}{3}\rfloor \text{ for } 1 \leq j \leq m, \\
l(v_{2j}) &= \lfloor\tfrac{m+j+5}{3}\rfloor \text{ for } 1 \leq j \leq m, \\
l(v_{ij}) &= \begin{cases} \lfloor\frac{m+3+(2i-3)(m+2)}{3}\rfloor & \text{for } 1 \leq j \leq m, 3 \leq i \leq \lceil\frac{n}{2}\rceil \\ \lceil\frac{m+4+(n-1)(m+2)}{3}\rceil & \text{for } 1 \leq j \leq m, i = \lceil\frac{n}{2}\rceil + 1 \\ \lfloor\frac{m+3+(2n-2i+2)(m+2)}{3}\rfloor & \text{for } 1 \leq j \leq m, \lceil\frac{n}{2}\rceil + 2 \leq i \leq n, \end{cases}
\end{aligned}$$

$$l(e_i) = \begin{cases} m+3-\lceil\frac{m+3}{2}\rceil & \text{if } i=1 \\ 2m+3-\lceil\frac{m+3}{2}\rceil & \text{if } i=2 \\ 3m+7-k+\lfloor\frac{n-6}{2}\rfloor m = & \text{if } i=3 \\ 4m+4i-2k+2\lfloor\frac{n-6}{2}\rfloor m & \text{if } 4\le i<\lfloor\frac{n}{2}\rfloor-1 \\ m+\lfloor\frac{n}{2}\rfloor m+\lfloor\frac{n-6}{2}\rfloor m+4\lfloor\frac{n}{2}\rfloor-4-2k & \text{if } i=\lfloor\frac{n}{2}\rfloor-1 \\ 2\lfloor\frac{n}{2}\rfloor m+4\lfloor\frac{n}{2}\rfloor-m-2k & \text{if } i=\lfloor\frac{n}{2}\rfloor \\ mn+2n-m+1-2k & \text{if } i=\lfloor\frac{n}{2}\rfloor+1 \\ 2mn-m+4n-2\lfloor\frac{n}{2}\rfloor m-4\lfloor\frac{n}{2}\rfloor-2k-2 & \text{if } i=\lfloor\frac{n}{2}\rfloor+2 \\ 2mn+4n+2m+2im-4i+6-2k-\lfloor\frac{i-\lfloor\frac{n}{2}\rfloor}{2}\rfloor m+\lfloor\frac{i+1-\lfloor\frac{n}{2}\rfloor}{2}\rfloor m & \text{if } \lfloor\frac{n}{2}\rfloor+3\le i\le n-3 \\ 6m+14-k+\lfloor\frac{n-2-\lfloor\frac{n}{2}\rfloor}{2}\rfloor m-\lceil\frac{3m+7}{2}\rceil & \text{if } i=n-2 \\ 3m+7-\lceil\frac{3m+7}{2}\rceil & \text{if } i=n-1 \\ m+2 & \text{if } i=n, \end{cases}$$

$$l(e_{1j}) = \begin{cases} 1 & \text{if } j=1 \\ j+2-\lfloor\frac{j+3}{3}\rfloor-\lfloor\frac{j+4}{3}\rfloor & \text{if } 2\le j\le m \\ m+2-\lfloor\frac{m+4}{3}\rfloor & \text{if } j=m+1, \end{cases}$$

$$l(e_{ij}) = \begin{cases} j+2+(2i-3)(m+2)-2\lfloor\frac{(m+3)+(2i-3)(m+2)}{3}\rfloor & \text{if } 2\le j\le m+1, 3\le i\le\lceil\frac{n}{2}\rceil \\ j+3+(n-1)(m+2)-2\lceil\frac{m+4+(n-1)(m+2)}{3}\rceil & \text{if } 2\le j\le m+1, i=\lceil\frac{n}{2}\rceil+1 \\ j+2+(2n-2i+2)(m+2)-2\lfloor\frac{(m+3)+(2n-2i+2)(m+2)}{3}\rfloor & \text{if } 2\le j\le m+1, \lceil\frac{n}{2}\rceil+2\le i\le n, \end{cases}$$

$$l(e_{2j}) = \begin{cases} m+4+j-\lfloor\frac{m+4+j}{3}\rfloor-\lfloor\frac{m+5+j}{3}\rfloor & \text{if } 2\le j\le m \\ 2m+5-\lfloor\frac{2m+5}{3}\rfloor-\lfloor\frac{m+6}{3}\rfloor & \text{if } j=m+1. \end{cases}$$

For n is odd,

$$l(e_{i1}) = \begin{cases} m+5-\lfloor\frac{m+6}{3}\rfloor-\lceil\frac{m+3}{2}\rceil & \text{if } i=2 \\ 2m+4-\lfloor\frac{4m+9}{3}\rfloor & \text{if } i=3 \\ mi+4i-3-k+\lfloor\frac{n-6}{2}\rfloor m-\lfloor\frac{m+3+(2i-3)(m+2)}{3}\rfloor & \text{if } 4\le i<\lfloor\frac{n}{2}\rfloor \\ 2mi+4i-3m-3-k-\lfloor\frac{m+3+(2i-3)(m+2)}{3}\rfloor & \text{if } \lfloor\frac{n}{2}\rfloor\le i\le\lfloor\frac{n}{2}\rfloor+1 \\ 4+(n-1)(m+2)-k-\lceil\frac{m+4+(n-1)(m+2)}{3}\rceil & \text{if } i=\lfloor\frac{n}{2}\rfloor+2 \\ 3+(2n-2i+2)(m+2)-k+\lfloor\frac{i-\lfloor\frac{n}{2}\rfloor}{2}\rfloor m-\lfloor\frac{m+3+(2n-2i+2)(m+2)}{3}\rfloor & \text{if } \lfloor\frac{n}{2}\rfloor+3\le i\le n-2 \\ 4m+11-\lceil\frac{3m+7}{2}\rceil-\lfloor\frac{m+3+4(m+2)}{3}\rfloor & \text{if } i=n-1 \\ m+4-\lfloor\frac{3m+7}{3}\rfloor & \text{if } i=n. \end{cases}$$

For n is even,

$$l(e_{i1}) = \begin{cases} m+5-\lfloor\frac{m+6}{3}\rfloor-\lceil\frac{m+3}{2}\rceil & \text{if } i=2 \\ 2m+4-\lfloor\frac{4m+9}{3}\rfloor & \text{if } i=3 \\ mi+4i-3-k+\lfloor\frac{n-6}{2}\rfloor m-\lfloor\frac{m+3+(2i-3)(m+2)}{3}\rfloor & \text{if } 4\le i<\frac{n}{2} \\ 2mi+4i-3m-3-k+\lfloor\frac{m+3+(2i-3)(m+2)}{3}\rfloor & \text{if } i=\frac{n}{2} \\ 4+(n-1)(m+2)-k-\lceil\frac{m+4+(n-1)(m+2)}{3}\rceil & \text{if } i=\frac{n}{2}+1 \\ 3+(2n-2i+2)(m+2)-k-\lfloor\frac{i-\lfloor\frac{n}{2}\rfloor}{2}\rfloor m-\lfloor\frac{m+3+(2n-2i+2)(m+2)}{3}\rfloor & \text{if } \frac{n}{2}+2\le i\le n-2 \\ 4m+11-\lceil\frac{3m+7}{2}\rceil-\lfloor\frac{m+3=4(m+2)}{3}\rfloor & \text{if } i=n-1 \\ m+4-\lfloor\frac{3m+7}{3}\rfloor & \text{if } i=n. \end{cases}$$

Now $max\{\{l(v) \mid v\in V(C_m@C_n)\}\cup\{l(e) \mid e\in E(C_m@C_n)\}\} = k$ and l is a function

from $V(C_m@C_n) \cup E(C_m@C_n)$ into $\{1, 2, ..., k\}$.
The weights of the edges are given by

$$w(e_{ij}) = \begin{cases} m+4 & \text{if } i = 1 \\ 2+(2i-1)(m+2) & \text{if } 2 \leq i \leq \lceil \frac{n}{2} \rceil \\ 3+(n-1)(m+2) & \text{if } i = \lceil \frac{n}{2} \rceil + 1 \\ 2+(2n-2i+2)(m+2) & \text{if } \lceil \frac{n}{2} \rceil + 2 \leq i \leq n, \end{cases}$$

$$w(e_i) = \begin{cases} j+2 & \text{if } i = 1 \\ j+2+(2i-3)(m+2) & \text{if } 2 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ j+3+(n-1)(m+2) & \text{if } i = \lfloor \frac{n}{2} \rfloor + 1 \\ j+2+(2n-2i+2)(m+2) & \text{if } \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n. \end{cases}$$

The weights of the edges of E under total labeling l form a set of consecutive integers from 3 to $n(m+2)+2$ and no two edges have the same weight. Hence $tes(C_m@C_n) = \lceil \frac{n(m+2)+2}{3} \rceil$. $\square$

In the following theorem we describe an optimal edge irregular total labeling for the graph $P^*_{m,n}$.

Theorem 2.2. *For any integers* $m \geq 3, n \geq 2$, $tes(P^*_{m,n}) = \lceil \frac{nm+2}{3} \rceil$.

Proof. The vertex set of $P^*_{m,n}$ is $\{v_i \mid 1 \leq i \leq n+1\} \cup \{v_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m-2\}$ and the edge set of is $\{e_i = v_iv_{i+1} \mid 1 \leq i \leq n\} \cup \{e_{ij} = v_{i(j-1)}v_{ij} | 1 \leq i \leq n, 2 \leq j \leq m-2\} \cup \{e_{i1} = v_iv_{i1} | 1 \leq i \leq n\} \cup \{e_{i(m-1)} = v_{i(m-2)}v_{i+1} | 1 \leq i \leq n\}$. Since $tes(G) \geq max \left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\}$, it is enough to prove that $tes(P^*_{m,n}) \leq \lceil \frac{nm+2}{3} \rceil$.
We construct an edge-irregular total labeling l as follows:

$$l(v_i) = \begin{cases} 1 & \text{if } i = 1, i = n+1 \\ \lfloor \frac{m+5}{3} \rfloor & \text{if } i = 2 \\ \lfloor \frac{(2i-2)m+1}{3} \rfloor & \text{if } 3 \leq i \leq \lfloor \frac{n+2}{2} \rfloor \\ \lceil \frac{nm+2}{3} \rceil & \text{if } i = \lfloor \frac{n+2}{2} \rfloor + 1, \lfloor \frac{n+2}{2} \rfloor + 2 \\ \lfloor \frac{(2n-2i+5)m+1}{3} \rfloor & \text{if } \lfloor \frac{n+2}{2} \rfloor + 3 \leq i \leq n, \end{cases}$$

$$l(v_{ij}) = \begin{cases} \lfloor \frac{j+2}{3} \rfloor & \text{if } i = 1, 1 \leq j \leq m-2 \\ \lfloor \frac{m+5}{3} \rfloor & \text{if } i = 2, 1 \leq j \leq m-2 \\ \lfloor \frac{(2i-2)m+1}{3} \rfloor & \text{if } 3 \leq i \leq \lfloor \frac{n+2}{2} \rfloor, 1 \leq j \leq m-2 \\ \lfloor \frac{(2n-2i+3)m+1}{3} \rfloor & \text{if } \lfloor \frac{n+2}{2} \rfloor + 1 \leq i \leq n, 1 \leq j \leq m-2, \end{cases}$$

$$l(e_i) = \begin{cases} m+1-\lfloor \frac{m+5}{3} \rfloor & \text{if } i = 1 \\ 2m+2-\lfloor \frac{m+5}{3} \rfloor - \lfloor \frac{4m+1}{3} \rfloor & \text{if } i = 2 \\ 2im-2m+2-\lfloor \frac{(2i-2)m+1}{3} \rfloor - \lfloor \frac{2im+1}{3} \rfloor & \text{if } 3 \leq i \leq \lfloor \frac{n+2}{2} \rfloor - 1 \\ (2\lfloor \frac{n+2}{2} \rfloor - 4)m+2-\lceil \frac{mn+2}{3} \rceil - \lfloor \frac{(2\lfloor \frac{n+2}{2} \rfloor - 4)m+1}{3} \rfloor & \text{if } i = \lfloor \frac{n+2}{2} \rfloor \\ (2n-2\lfloor \frac{n+2}{2} \rfloor + 1)m+2-\lfloor \frac{(2n-2\lfloor \frac{n+2}{2} \rfloor + 1)m+1}{3} \rfloor & \text{if } i = \lfloor \frac{n+2}{2} \rfloor + 1 \\ (2n-2i+3)m+1-\lfloor \frac{(2n-2i+5)m+1}{3} \rfloor - \lfloor \frac{(2n-2i+3)m+1}{3} \rfloor & \text{if } \lfloor \frac{n+2}{2} \rfloor + 2 \leq i \leq n, \end{cases}$$

$$l(e_{i1}) = \begin{cases} 1 & \text{if } i = 1 \\ m + 3 - 2\lfloor\frac{m+5}{3}\rfloor & \text{if } i = 2 \\ (2i-3)m + 3 - 2\lfloor\frac{(2i-2)m+1}{3}\rfloor & \text{if } 3 \le i \le \lfloor\frac{n+2}{2}\rfloor \\ (2n - 2\lfloor\frac{n+2}{2}\rfloor)m + m + 1 - \lceil\frac{mn+2}{3}\rceil - \lfloor\frac{(2n-2\lfloor\frac{n+2}{2}\rfloor+1)m+1}{3}\rfloor & \text{if } i = \lfloor\frac{n+2}{2}\rfloor + 1 \\ (2n - 2\lfloor\frac{n+2}{2}\rfloor - 2)m + m + 1 - \lceil\frac{mn+2}{3}\rceil - \lfloor\frac{(2n-2\lfloor\frac{n+2}{2}\rfloor-1)m+1}{3}\rfloor & \text{if } i = \lfloor\frac{n+2}{2}\rfloor + 2 \\ (2n-2i+2)m + m + 1 - \lfloor\frac{(2n-2i+3)m+1}{3}\rfloor - \lfloor\frac{(2n-2i+1)m+1}{3}\rfloor & \text{if } \lfloor\frac{n+2}{2}\rfloor + 3 \le i \le n, \end{cases}$$

$$l(e_{ij}) = \begin{cases} m + 1 - \lfloor\frac{m}{3}\rfloor - \lfloor\frac{m+5}{3}\rfloor & \text{if } i = 1, j = m-1 \\ j + 2 - \lfloor\frac{j+2}{3}\rfloor - \lfloor\frac{j+1}{3}\rfloor & \text{if } i = 1, 2 \le j \le m-2 \\ m + 2 + j - 2\lfloor\frac{m+5}{3}\rfloor & \text{if } i = 2, 2 \le j \le m-2 \\ 2m + 1 - \lfloor\frac{m+5}{3}\rfloor - \lfloor\frac{4m+1}{3}\rfloor & \text{if } i = 2, j = m-1 \\ (2i-3)m + 2 + j - 2\lfloor\frac{(2i-2)m+1}{3}\rfloor & \text{if } 3 \le i \le \lfloor\frac{n+2}{2}\rfloor - 2, 2 \le j \le m-2 \\ (2i-3)m + m + 1 - \lfloor\frac{(2i-2)m+1}{3}\rfloor - \lceil\frac{nm+2}{3}\rceil & \text{if } i = \lfloor\frac{n+2}{2}\rfloor - 1, \lfloor\frac{n+2}{2}\rfloor, 2 \le j \le m-2 \\ (2i-3)m + m + 1 - \lfloor\frac{(2i-2)m+1}{3}\rfloor - \lfloor\frac{2im+1}{3}\rfloor & \text{if } 3 \le i \le \lfloor\frac{n+2}{2}\rfloor, j = m-1 \\ (2n-2i+2)m + 2 + m - j - 2\lfloor\frac{(2n-2i+5)m+1}{3}\rfloor & \text{if } \lfloor\frac{n+2}{2}\rfloor + 1 \le i \le n, 2 \le j \le m-2 \\ (2n-2i+2)m + 3 - \lfloor\frac{(2n-2i+5)m+1}{3}\rfloor - \lfloor\frac{(2n-2i+3)m+1}{3}\rfloor & \text{if } \lfloor\frac{n+2}{2}\rfloor + 1 \le i \le n, j = m-1. \end{cases}$$

Now $\max\{\{l(v) \mid v \in V(P^*_{m,n})\} \cup \{l(e) \mid e \in E(P^*_{m,n})\}\} = \lceil\frac{mn+2}{3}\rceil$ and l is a function from $V(P^*_{m,n}) \cup E(P^*_{m,n})$ into $\{1, 2, ..., \lceil\frac{mn+2}{3}\rceil\}$.

The weights of the edges are given by

$$w(e_i) = \begin{cases} m + 2 & \text{if } i = 1 \\ (2i-2)m + 2 & \text{if } 2 \le i \le \lfloor\frac{n+2}{2}\rfloor \\ (2n-2i+3)m + 2 & \text{if } \lfloor\frac{n+2}{2}\rfloor + 1 \le i \le n, \end{cases}$$

$$w(e_{ij}) = \begin{cases} j + 2 & \text{if } i = 1, 1 \le j \le m-1 \\ (2i-3)m + 2 + j & \text{if } 2 \le i \le \lfloor\frac{n+2}{2}\rfloor, 1 \le j \le m-1 \\ (2n-2i+2)m + 2 + m - j & \text{if } \lfloor\frac{n+2}{2}\rfloor + 1 \le i \le n, 1 \le j \le m-1. \end{cases}$$

The weights of the edges of E under total labeling l form a set of consecutive integers from 3 to $nm + 2$ and no two edges have the same weight. Hence $tes(P^*_{m,n}) = \lceil\frac{nm+2}{3}\rceil$. □

Consider n copies of the graph C_m and label the vertices in the i^{th} copy of C_m as $v_i, v_{i1}, v_{i2}, \ldots, v_{i(m-2)}, v_{i+1}$ for $1 \le i \le n$. Then using the labelings as in $P^*_{m,n}$,we get the following result.

Corollary 2.1. *[2] For any integers $m \ge 3$, $n \ge 3$, $tes(nC_m) = \lceil\frac{nm+2}{3}\rceil$.*

In the graph $P^*_{m,n} \cup P^*_{m,r}$, labeling the vertices in $P^*_{m,r}$ by

$$\{v_i \mid n+1 \le i \le n+r\} \cup \{v_{ij} \mid n+1 \le i \le n+r-1, 1 \le j \le m-2\},$$

we obtain the following result.

Corollary 2.2. *For any integers $m \geq 3$, $n \geq 3$, $tes(P^*_{m,n} \cup P^*_{m,r}) = \lceil \frac{m(n+r)+2}{3} \rceil$.*

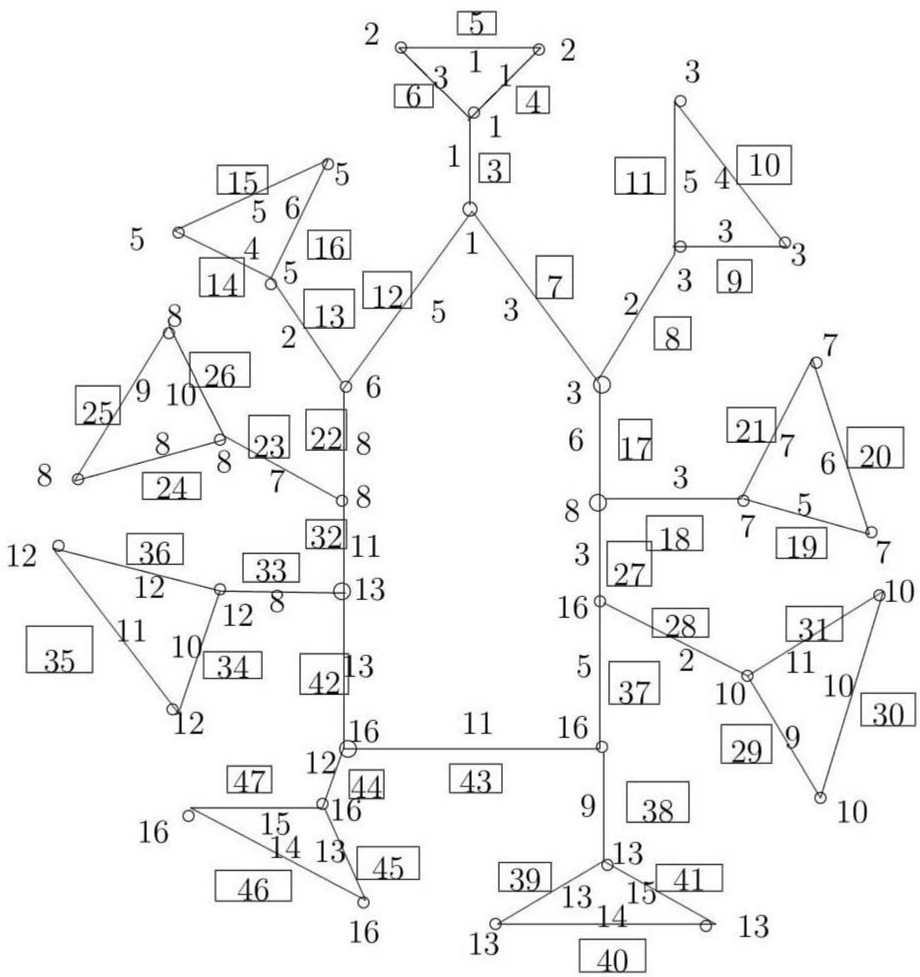

Figure 1. $C_3@C_9$

3. Open Problem

In $C_m@C_n$, we take n copies of C_m. Instead of considering same cycles, consider n cycles with different lengths $m_1, m_2, \ldots, m_n$ and denote the new graph by C'. Prove that

$$tes(C') = \left\lceil \frac{\sum_{i=1}^{n} m_i + 2}{3} \right\rceil.$$

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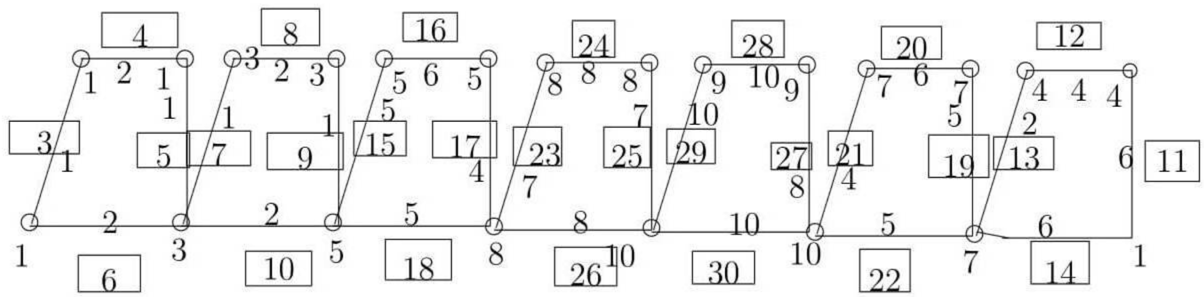

Figure 2. $P^*_{4,7}$

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