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Locating-chromatic number of the edge-amalgamation of trees

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Abstract

The investigation on the locating-chromatic number for graphs was initially studied by Chartrand *et al.* on 2002. This concept is in fact a special case of the partition dimension for graphs. Even though this topic has received much attention, the current progress is still far from satisfaction. We can define the locating-chromatic number of a graph G as the smallest integer k such that there exists a proper k-coloring on the vertex-set of G such that all vertices have distinct coordinates (color codes) with respect to this coloring. Not like the metric dimension of any tree which is completely solved, the locating-chromatic number for most types of trees are still open. In this paper, we study the locating-chromatic number of trees. In particular, we give lower and upper bounds of the locating-chromatic number of trees formed by an edge-amalgamation of the collection of smaller trees. We also show that the bounds are tight.

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1. Introduction

The topic of locating-chromatic number of graphs was introduced by Chartrand *et al.* [5] on 2002. They determined the locating-chromatic numbers of some well-known classes of graphs,

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i.e., paths, cycles, and double stars. They also characterized all graphs of order n with locating-chromatic number n, *i.e.* multipartite complete graphs. This topic has received much attention. Inspired by Chartrand *et al.*, other authors have determined the locating-chromatic numbers of some well-known classes of graphs. But the results are still limited. In particular for trees, the locating-chromatic number for most types of trees are still open. Some classes of trees with their locating-chromatic numbers known are amalgamations of stars and firecrackers by Asmiati *et al.* [1, 2], homogeneous lobsters and binary trees by Syofyan *et al.* [6, 7], and complete n-arry trees by Welyyanti *et al.* [9]. Furthermore, all trees on n vertices with locating-chromatic number 3 or $n - t$ where $2 \leq t < \frac{n}{2}$ have been successfully characterized, see [4] and [8], respectively. In this paper, our aim is to determine the locating-chromatic number of the edge-amalgamation of trees. We then estimate the locating-chromatic numbers for some structures of trees obtained by the edge-amalgamation of trees.

Throughout this paper, we only deal with connected graphs. Let $G = (V, E)$ be a connected graph. For $u, v \in V(G)$, let $d(u, v)$ denote the *distance* between u and v. A k*-coloring* of G is a function $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for any two adjacent vertices u and v. In other words, c is a partition Π of $V(G)$ into color classes $C_1, C_2, \ldots, C_k$, where the vertices of C_i are colored by i for $1 \leq i \leq k$. The *color code* of vertex u in G, denoted by $c_\Pi(u)$, is defined to be the ordered k-tuple $(d(u, C_1), d(u, C_2), \ldots, d(u, C_k))$, where $d(u, C_i) = \min\{d(u, x)|x \in C_i\}$ for $1 \leq i \leq k$. If any two distinct vertices of G have distinct color codes, then c is called a *locating* k*-coloring* of G. Moreover, the least integer k such that there is a locating-coloring in G is called the *locating-chromatic number* of G, denoted by $\chi_L(G)$.

The following two results are natural consequences and showed in [5].

Lemma 1.1. *Let G be a connected non-trivial graph. Let c be a locating coloring of G and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for every $w \in V(G) \setminus \{u, v\}$, then $c(u) \neq c(v)$.*

Corollary 1.1. *If G is a connected graph containing a vertex adjacent to k leaves of G, then $\chi_L(G) \geq k + 1$.*

2. Main Results

For $i = 1, 2, \ldots, t$, let T_i be a tree with a fixed edge e_{o_i} called the *terminal edge*. The *edge-amalgamation* of all these trees $T_i s$, denoted by Edge-Amal$\{T_i; e_{o_i}\}$, is a tree formed by taking all these trees T_is and identifying their terminal edges. In this section, we will derive the (lower and upper) bounds for the locating-chromatic number of the edge-amalgamation of trees.

Let T be a tree. A *stem* is a vertex in T that is adjacent to a leaf. A *pendant edge* is an edge in T incident to a leaf in a tree. For any vertices u and v in T, we denote by ${}_uP_v$ the unique path connecting u and v. Let $u \in V(T)$ and define $N(u) = \{x \in V(T)|d(u, x) = 1\}$. For a k-locating-coloring c of T, we denote $c(N(u)) = \{c(v)|v \in N(u)\}$.

For $i = 1, 2, \ldots, t$, let T_i be a tree with a chosen terminal edge $e_{o_i} = s_i l_i$, where s_i is a stem and l_i is a leaf. For any stem z of a tree T_i we denote $N_p(z)$ is the set of pendant vertices adjacent to stem z. Let m_i be the number of pendant edges adjacent to stem s_i and $r_i = max\{|N_p(z)| z \textit{ is a stem of } T_i\}$. Next, in Edge-Amal$\{T_i; e_{o_i}\}$, we denote $s = s_i$ and $l = l_i$.

Theorem 2.1. *Let Edge-Amal$\{T_i; e_{o_i}\}$ be an edge-amalgamation of t disjoint trees T_i. Then, max$\{r_i + 1, 2 + \sum_{i=1}^{t}(m_i - 1)\} \leq \chi_L($Edge-Amal$\{T_i; e_{o_i}\}) \leq 2 + \sum_{i=1}^{t}(\chi_L(T_i) - 2)$.*

Proof. For $i = 1, 2, \ldots, t$, let $\chi_L(T_i) = k_i$. Let c_i be a k_i-locating coloring of T_i such that $c_i(s_i) = 1$ and $c_i(l_i) = 2$. Define $A = \{v \in V(T_i) | c_i(v) = 1, \forall i \in [1, t]\}$ and $B = \{v \in V(T_i) | c_i(v) = 2, \forall i \in [1, t]\}$. Now, define $c : V(\text{Edge-Amal}\{T_i; e_{o_i}\}) \to \{1, 2, \ldots, 2 + \Sigma_{i=1}^{t}(k_i - 2)\}$ as follows

$$c(x) = \begin{cases} 1, & \text{if } x \in A \\ 2, & \text{if } x \in B \\ c_1(x), & \text{if } x \in V(T_1) \\ c_i(x) + \Sigma_{j=2}^{i}(k_{j-1} - 2), & \text{if } x \in V(T_i) \setminus (A \cup B), \text{for all } i > 1. \end{cases}$$

Since the coloring c preserves the locating coloring in every tree $T_1, T_2, \ldots, T_t$, two vertices u and v where $c(u) = c(v)$ and $c(N(u)) = c(N(v))$ only occur for two cases below.

1. $u, v \in V(T_i)$ for some i.
   Then, their color codes are distinguished by the k_i-locating coloring c_i of T_i. Therefore, these vertices are also distinguished by c.
2. $u \in V(T_i)$ and $v \in V(T_j)$ for some $i \neq j$.
   Let $c(u) = c(v) = 1$. Since c_i is a k_i-locating coloring and by the definition of the coloring c, there exists integer $p \neq 1, 2$ such that $c(x) = p$ for some $x \in N^2(s)$ and $x \in T_i$. Thus, we have:
   $$d_T(u, C_p) \leq d_T(u, s), \tag{1}$$
   and
   $$d_T(v, s) + 1 \leq d_T(v, C_p) \leq d_T(v, s) + 2. \tag{2}$$
   Similarly, consider the subtree T_j. Since c_j is a k_j-locating coloring and by the definition of the coloring c, there exists integer $q \neq 1, 2$ and $q \neq p$ such that $c(y) = q$ for some $y \in N^2(s)$ and $y \in T_j$. Thus, we have:
   $$d_T(v, C_q) \leq d_T(v, s), \tag{3}$$
   and
   $$d_T(u, s) + 1 \leq d_T(u, C_q) \leq d_T(u, s) + 2. \tag{4}$$
   Now, if $d_T(u, C_p) = d_T(v, C_p)$ then from Eqs (1), (2), (3) and (4), we have that:
   $$d_T(v, C_q) < d_T(v, s) + 1 \leq d_T(v, C_p) = d_T(u, C_p) \leq d_T(u, s) < d_T(u, C_q). \tag{5}$$
   Thus, we have that $d_T(u, C_q) \neq d_T(v, C_q)$. Therefore, the color codes of u and v are different. A similar argument holds for the case $c(u) = c(v) = 2$.

Thus, all vertices of the Edge-Amal$\{T_i; e_{o_i}\}$ have distinct color codes. We conclude that

$$\chi_L(\text{Edge-Amal}\{T_i; e_{o_i}\}) \leq 2 + \sum_{i=1}^{t}(k_i - 2).$$

Next, since there is a stem adjacent to *max*$\{r_i, 1+\sum_{i=1}^{t}(m_i-1)\}$ leaves, by Corollary 1.1

$$\chi_L(\text{Edge-Amal}\{T_i; e_{o_i}\}) \geq \textit{max}\{r_i+1, 2+\sum_{i=1}^{t}(m_i-1)\}.$$

□

The following two theorems show the existence of trees formed by an edge-amalgamation operation with the locating-chromatic number equals to the lower or upper bounds of Theorem 2.1. Furthermore, in Theorem 2.4, we give the example of trees formed by an edge-amalgation operation with the locating-chromatic number lies in between upper and lower bounds of Theorem 2.1.

Theorem 2.2. *If* $\chi_L(T_i) = k_i$ *and* $m_i = k_i - 1$ *for any* i, *then* $\chi_L(Edge - Amal(T_i; e_{o_i})) = 2 + \sum_{i=1}^{t}(\chi_L(T_i) - 2)$.

Proof. By using the locating-coloring c in proof Theorem 2.1, we have $\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) \leq 2 + \sum_{i=1}^{t}(k_i - 2)$.

Next, since there are $1 + \sum_{i=1}^{t}(k_i - 2)$ leaves adjacent to a stem in Edge-Amal$(T_i; e_{o_i})$, by Lemma 1.1 $\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) \geq 2 + \sum_{i=1}^{t}(k_i - 2)$. So, we conclude that

$$\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) = 2 + \sum_{i=1}^{t}(k_i - 2).$$

□

Let G_{w_i} be a tree having a pendant e_{o_i} as depicted in Figure 1, where $w_i \geq 2$.

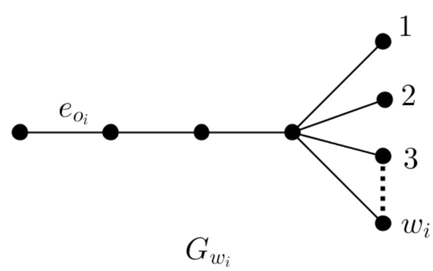

Figure 1. A tree G_{w_i} where $w_i \geq 2$.

Theorem 2.3. *For* $i = 1, 2, \ldots, t$, *let* $T_i = G_{w_i}$. *If* $t \leq$ *max*$\{w_i \mid i \in [1, t]\}$, *then*

$$\chi_L(\textit{Edge-Amal}(T_i; e_{o_i})) = \textit{max}\{w_i + 1 \mid i \in [1, t]\}.$$

Proof. Let $r = max\{w_i \mid i \in [1,t]\}$. Since there are r leaves adjacent to a stem in Edge-Amal$(T_i; e_{o_i})$, by Lemma 1.1 $\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) \geq r+1$.

Now, let $T_i = G_{w_i}$ such that $w_1 \leq w_2 \leq \ldots \leq w_t$. We denote x_i, y_i, z_{ij} the non stem vertex, the stem adjacent to w_i leaves, and all leaves adjacent to y_i, respectively.

Define a coloring $c : V(\text{Edge-Amal}\{T_i; e_{o_i}\}) \to \{1, 2, \ldots, r+1\}$ as follows

$$c(u) = \begin{cases} 1, & \text{if } u = s \\ 2, & \text{if } u = l \text{ or } u = x_i \text{ for } 1 \leq i \leq t-1 \text{ and } i \neq 2 \\ 3, & \text{if } u = x_2 \\ i, & \text{if } u = y_i \\ j, & \text{if } u = z_{ij} \text{ and } i \neq j \\ r+1, & \text{if } u = z_{ij} \text{ and } i = j. \end{cases}$$

By this coloring, any two vertices u and v satisfying $c(u) = c(v)$ and $c(N(u)) = c(N(v))$ only occur for the pair of vertices s and y_i for $w_1 = 2$, and the pair of vertices l and x_1. Their color codes are distinguished by the last ordinate (their distances to a vertex in the color class $r+1$). Hence, all vertices have distinct color codes. So, $\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) \leq max\{r_i + 1\}$. □

Let H_m be a tree having a pendant e_{o_i} as depicted in Figure 2, where $m \geq 3$.

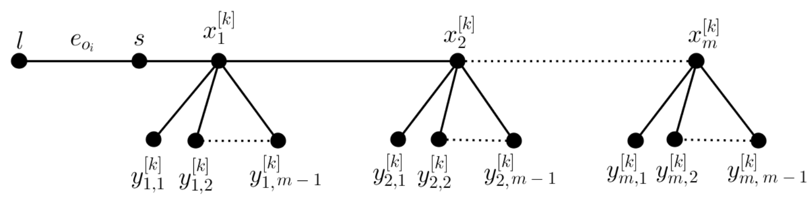

Figure 2. A tree H_m where $m \geq 3$.

Theorem 2.4. *For $i = 1, 2, \ldots, t$, let $T_i = H_m$. We have that χ_L(Edge-Amal$(T_i; e_{o_i})) = m+2$, if $2 \leq t \leq m$.*

Proof. Let $t \in [2, m]$. Then, there are tm stems and each is adjacent to m leaves in graph Edge-Amal$(T_i; e_{o_i})$. We suppose that $\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) = m+1$. Then, there are $m+1$ possibilities to coloring all stems and their neighbors in Edge-Amal$(T_i; e_{o_i})$. Since $t \geq 2$, there are at least two stems having the same color. Therefore, the color codes of these stems are the same, a contradiction to $\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) = m+1$. So,

$$\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) \geq m+2.$$

Next, we define a coloring $c : V(\text{Edge-Amal}\{T_i; e_{o_i}\}) \to \{1, 2, \ldots, m+2\}$ as follows:

$$c(u) = \begin{cases} 1, & \text{if } u = s \\ 2, & \text{if } u = l \\ i+k, & \text{if } u = x_i^{[k]} \text{ for } 1 \leq i \leq m-1, \text{ and } 1 \leq k \leq t \\ k, & \text{if } u = x_m^{[k]} \text{ for } 1 \leq k \leq t \\ j+k-1, & \text{if } u = y_{i,j}^{[k]} \text{ and } j \neq (i+k) \bmod m, \text{ for } 1 \leq i \leq m-1, \\ & 1 \leq j \leq m-1, \text{ and } 1 \leq k \leq t \\ m+k-1, & \text{if } u = y_{i,j}^{[k]} \text{ and } j = (i+k) \bmod m, \text{ for } 1 \leq i \leq m-1, \\ & 1 \leq j \leq m-1, \text{ and } 1 \leq k \leq t \\ j+k, & \text{if } u = y_{m,j}^{[k]} \text{ for } 1 \leq k \leq t. \end{cases}$$

Note that all the colors above in modulo $m+2$. We will show that $\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) \leq m+2$. Let u and v be any two vertices with $c(u) = c(v)$. Then, by the coloring c, $c(N(u)) \neq c(N(v))$ because the $m-1$ neighbors colors of u are permutation of $m-1$ neighbors colors of v in modulo $m+2$. Hence, all vertices in Edge-Amal$(T_i; e_{o_i})$ have distinct color codes. So, $\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) \leq m+2$. □

From Theorem 2.3, we shows the exact value of locating-chromatic number for some classes of trees. First, we give definition of some classes of trees and their locating-chromatic number, *i.e.* double stars, homogeneous caterpillars, and homogeneous lobsters. A double star, denoted by $S_{m,n}$ where $n \geq m \geq 1$, is the graph consisting of two stars $K_{1,n}$ and $K_{1,m}$ together with an edge joining their centers. Chartrand *et al.* [5] have proved $\chi_L(S_{m,n}) = n+1$. The homogeneous caterpillar $C(m,n)$ is the graph consisting of m stars $K_{1,n}$ by linking the centers from each stars. Asmiati *et al.* [3] showed that the locating-chromatic number of homogeneous caterpillar is $n+1$ for $1 \leq m \leq n+1$, and $n+2$ for $m > n+1$. The homogeneous lobster $Lb(m,n)$ is the graph obtained by attaching the centers of stars $K_{1,n}$ to each leaf of $C(m,n)$. Syofyan *et al.* [6] showed that the locating-chromatic number of the homogeneous lobster is $n+1$ if $m = 1$, $n+2$ for $2 \leq m \leq 3(n=2)+1$, or $n+3$ for $m > 3(n+2)+1$.

Based on Theorem 2.3 and the locating-chromatic numbers of double stars, homogeneous caterpillars, and homogeneous lobsters, we have the locating-chromatic number of edge-amalgamation of these trees as follows. The terminal edge in each tree is chosen from the edges incident to a stem having maximum leaves.

Corollary 2.1. *For* $i = 1, 2, \ldots, t$, *let* $T_i = S_{m,n}$. *Then,* $\chi_L(\textit{Edge-Amal}(T_i; e_{o_i})) = t(n-1)+1$, *if* $n \geq m \geq 1$.

Corollary 2.2. *For* $i = 1, 2, \ldots, t$, *let* $T_i = C(m,n)$. *If* $1 \leq m \leq n+1$, *then* $\chi_L(\textit{Edge-Amal}(T_i; e_{o_i})) = t(n-1)+1$.

Corollary 2.3. *For* $i = 1, 2, \ldots, t$, *let* $T_i = Lb(m,n)$. *If* $m = 1$, *then* $\chi_L(\textit{Edge-Amal}(T_i; e_{o_i})) = t(n-1)+1$.

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