Locating-chromatic number of the edge-amalgamation of trees

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Abstract

The investigation on the locating-chromatic number for graphs was initially studied by Chartrand et al. on 2002. This concept is in fact a special case of the partition dimension for graphs. Even though this topic has received much attention, the current progress is still far from satisfaction. We can define the locating-chromatic number of a graph $G$ as the smallest integer $k$ such that there exists a proper $k$-coloring on the vertex-set of $G$ such that all vertices have distinct coordinates (color codes) with respect to this coloring. Not like the metric dimension of any tree which is completely solved, the locating-chromatic number for most types of trees are still open. In this paper, we study the locating-chromatic number of trees. In particular, we give lower and upper bounds of the locating-chromatic number of trees formed by an edge-amalgamation of the collection of smaller trees. We also show that the bounds are tight.

Keywords: locating-chromatic number, tree, edge-amalgamation

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1. Introduction

The topic of locating-chromatic number of graphs was introduced by Chartrand et al. [5] on 2002. They determined the locating-chromatic numbers of some well-known classes of graphs,
i.e., paths, cycles, and double stars. They also characterized all graphs of order $n$ with locating-chromatic number $n$, i.e., multipartite complete graphs. This topic has received much attention. Inspired by Chartrand et al., other authors have determined the locating-chromatic numbers of some well-known classes of graphs. But the results are still limited. In particular for trees, the locating-chromatic number for most types of trees are still open. Some classes of trees with their locating-chromatic numbers known are amalgamations of stars and firecrackers by Asmiati et al. [1, 2], homogeneous lobsters and binary trees by Syofyan et al. [6, 7], and complete $n$-ary trees by Welyyanti et al. [9]. Furthermore, all trees on $n$ vertices with locating-chromatic number 3 or $n-t$ where $2 \leq t < \frac{n}{2}$ have been successfully characterized, see [4] and [8], respectively. In this paper, our aim is to determine the locating-chromatic number of the edge-amalgamation of trees. We then estimate the locating-chromatic numbers for some structures of trees obtained by the edge-amalgamation of trees.

Throughout this paper, we only deal with connected graphs. Let $G = (V, E)$ be a connected graph. For $u, v \in V(G)$, let $d(u, v)$ denote the distance between $u$ and $v$. A $k$-coloring of $G$ is a function $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for any two adjacent vertices $u$ and $v$. In other words, $c$ is a partition $\Pi$ of $V(G)$ into color classes $C_1, C_2, \ldots, C_k$, where the vertices of $C_i$ are colored by $i$ for $1 \leq i \leq k$. The color code of vertex $u$ in $G$, denoted by $c_\Pi(u)$, is defined to be the ordered $k$-tuple $(d(u, C_1), d(u, C_2), \ldots, d(u, C_k))$, where $d(u, C_i) = \min\{d(u, x) | x \in C_i\}$ for $1 \leq i \leq k$. If any two distinct vertices of $G$ have distinct color codes, then $c$ is called a locating $k$-coloring of $G$. Moreover, the least integer $k$ such that there is a locating-coloring in $G$ is called the locating-chromatic number of $G$, denoted by $\chi_L(G)$.

The following two results are natural consequences and showed in [5].

**Lemma 1.1.** Let $G$ be a connected non-trivial graph. Let $c$ be a locating coloring of $G$ and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for every $w \in V(G) \setminus \{u, v\}$, then $c(u) \neq c(v)$.

**Corollary 1.1.** If $G$ is a connected graph containing a vertex adjacent to $k$ leaves of $G$, then $\chi_L(G) \geq k + 1$.

2. Main Results

For $i = 1, 2, \ldots, t$, let $T_i$ be a tree with a fixed edge $e_{o_i}$ called the terminal edge. The edge-amalgamation of all these trees $T_i$s, denoted by Edge-Amal$\{T_i; e_{o_i}\}$, is a tree formed by taking all these trees $T_i$s and identifying their terminal edges. In this section, we will derive the (lower and upper) bounds for the locating-chromatic number of the edge-amalgamation of trees.

Let $T$ be a tree. A stem is a vertex in $T$ that is adjacent to a leaf. A pendant edge is an edge in $T$ incident to a leaf in a tree. For any vertices $u$ and $v$ in $T$, we denote by $uP_v$ the unique path connecting $u$ and $v$. Let $u \in V(T)$ and define $N(u) = \{x \in V(T) | d(u, x) = 1\}$. For a $k$-locating-coloring $c$ of $T$, we denote $c(N(u)) = \{c(v) | v \in N(u)\}$.

For $i = 1, 2, \ldots, t$, let $T_i$ be a tree with a chosen terminal edge $e_{o_i} = s_i l_i$, where $s_i$ is a stem and $l_i$ is a leaf. For any stem $z$ of a tree $T_i$ we denote $N_p(z)$ is the set of pendant vertices adjacent to stem $z$. Let $m_i$ be the number of pendant edges adjacent to stem $s_i$ and $r_i = \max\{|N_p(z)| | z \text{ is a stem of } T_i\}$. Next, in Edge-Amal$\{T_i; e_{o_i}\}$, we denote $s = s_i$ and $l = l_i$. 
Theorem 2.1. Let Edge-Amal\{T_i; e_o\} be an edge-amalgamation of t disjoint trees T_i. Then,\[
\max\{r_i + 1, 2 + \sum_{i=1}^{t}(m_i - 1)\} \leq \chi_L(Edge-Amal\{T_i; e_o\}) \leq 2 + \sum_{i=1}^{t}(\chi_L(T_i) - 2).
\]

Proof. For i = 1, 2, . . . , t, let \(\chi_L(T_i) = k_i\). Let \(c_i\) be a \(k_i\)-locating coloring of \(T_i\) such that \(c_i(s_i) = 1\) and \(c_i(l_i) = 2\). Define \(A = \{v \in V(T_i)|c_i(v) = 1, \forall i \in [1, t]\}\) and \(B = \{v \in V(T_i)|c_i(v) = 2, \forall i \in [1, t]\}\). Now, define \(c : V(Edge-Amal\{T_i; e_o\}) \rightarrow \{1, 2, \ldots , 2 + \sum_{i=1}^{t}(k_i - 2)\}\) as follows:

\[
c(x) = \begin{cases} 
1, & \text{if } x \in A \\
2, & \text{if } x \in B \\
n(x), & \text{if } x \in V(T_i) \\
n(x) + \sum_{j=2}^{k_{i-1}}(k_j - 2), & \text{if } x \in V(T_i) \setminus (A \cup B), \text{for all } i > 1
\end{cases}
\]

Since the coloring \(c\) preserves the locating coloring in every tree \(T_1, T_2, \ldots , T_t\), two vertices \(u\) and \(v\) where \(c(u) = c(v)\) and \(c(N(u)) = c(N(v))\) only occur for two cases below.

1. \(u, v \in V(T_i)\) for some \(i\).
   Then, their color codes are distinguished by the \(k_i\)-locating coloring \(c_i\) of \(T_i\). Therefore, these vertices are also distinguished by \(c\).
2. \(u \in V(T_i)\) and \(v \in V(T_j)\) for some \(i \neq j\).
   Let \(c(u) = c(v) = 1\). Since \(c_i\) is a \(k_i\)-locating coloring and by the definition of the coloring \(c\), there exists integer \(p \neq 1, 2\) such that \(c(x) = p\) for some \(x \in N^2(s)\) and \(x \in T_i\). Thus, we have:

\[
d_T(u, C_p) \leq d_T(u, s),
\]

and

\[
d_T(v, s) + 1 \leq d_T(v, C_p) \leq d_T(v, s) + 2.
\]

Similarly, consider the subtree \(T_j\). Since \(c_j\) is a \(k_j\)-locating coloring and by the definition of the coloring \(c\), there exists integer \(q \neq 1, 2\) and \(q \neq p\) such that \(c(y) = q\) for some \(y \in N^2(s)\) and \(y \in T_j\). Thus, we have:

\[
d_T(v, C_q) \leq d_T(v, s),
\]

and

\[
d_T(u, s) + 1 \leq d_T(u, C_q) \leq d_T(u, s) + 2.
\]

Now, if \(d_T(u, C_p) = d_T(v, C_p)\) then from Eqs (1), (2), (3) and (4), we have that:

\[
d_T(v, C_q) < d_T(v, s) + 1 \leq d_T(v, C_p) = d_T(u, C_p) \leq d_T(u, s) < d_T(u, C_q).
\]

Thus, we have that \(d_T(u, C_q) \neq d_T(v, C_q)\). Therefore, the color codes of \(u\) and \(v\) are different. A similar argument holds for the case \(c(u) = c(v) = 2\).

Thus, all vertices of the Edge-Amal\{T_i; e_o\} have distinct color codes. We conclude that

\[
\chi_L(Edge-Amal\{T_i; e_o\}) \leq 2 + \sum_{i=1}^{t}(k_i - 2).
\]
Next, since there is a stem adjacent to \( \max\{r_i, 1 + \sum_{i=1}^{t}(m_i - 1)\} \) leaves, by Corollary 1.1

\[
\chi_L(\text{Edge-Amal}\{T_i; e_{o_i}\}) \geq \max\{r_i + 1, 2 + \sum_{i=1}^{t}(m_i - 1)\}. 
\]

\[
\Box
\]

The following two theorems show the existence of trees formed by an edge-amalgamation operation with the locating-chromatic number equals to the lower or upper bounds of Theorem 2.1. Furthermore, in Theorem 2.4, we give the example of trees formed by an edge-amalgamation operation with the locating-chromatic number lies in between upper and lower bounds of Theorem 2.1.

**Theorem 2.2.** If \( \chi_L(T_i) = k_i \) and \( m_i = k_i - 1 \) for any \( i \), then \( \chi_L(\text{Edge-Amal}(T_i; e_{o_i})) = 2 + \sum_{i=1}^{t}(\chi_L(T_i) - 2) \).

**Proof.** By using the locating-coloring \( c \) in proof Theorem 2.1, we have \( \chi_L(\text{Edge-Amal}(T_i; e_{o_i})) \leq 2 + \sum_{i=1}^{t}(k_i - 2) \).

Next, since there are \( 1 + \sum_{i=1}^{t}(k_i - 2) \) leaves adjacent to a stem in \( \text{Edge-Amal}(T_i; e_{o_i}) \), by Lemma 1.1 \( \chi_L(\text{Edge-Amal}(T_i; e_{o_i})) \geq 2 + \sum_{i=1}^{t}(k_i - 2) \). So, we conclude that

\[
\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) = 2 + \sum_{i=1}^{t}(k_i - 2). 
\]

\[
\Box
\]

Let \( G_{w_i} \) be a tree having a pendant \( e_{o_i} \) as depicted in Figure 1, where \( w_i \geq 2 \).

![Figure 1. A tree \( G_{w_i} \) where \( w_i \geq 2 \).](image)

**Theorem 2.3.** For \( i = 1, 2, \ldots, t \), let \( T_i = G_{w_i} \). If \( t \leq \max\{w_i \mid i \in [1, t]\} \), then

\[
\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) = \max\{w_i + 1 \mid i \in [1, t]\}. 
\]
Proof. Let \( r = \max \{ w_i \mid i \in [1, t] \} \). Since there are \( r \) leaves adjacent to a stem in \( \text{Edge-Amal}(T_i; e_{o_i}) \), by Lemma 1.1 \( \chi_L(\text{Edge-Amal}(T_i; e_{o_i})) \geq r + 1 \).

Now, let \( T_i = G_{w_i} \) such that \( w_1 \leq w_2 \leq \ldots \leq w_t \). We denote \( x_i, y_i, z_{ij} \) the non stem vertex, the stem adjacent to \( w_i \) leaves, and all leaves adjacent to \( y_i \), respectively.

Define a coloring \( c : V(\text{Edge-Amal}(T_i; e_{o_i})) \rightarrow \{ 1, 2, \ldots, r + 1 \} \) as follows

\[
c(u) = \begin{cases} 
1, & \text{if } u = s \\
2, & \text{if } u = l \text{ or } u = x_i \text{ for } 1 \leq i \leq t - 1 \text{ and } i \neq 2 \\
3, & \text{if } u = x_1 \\
i, & \text{if } u = y_i \\
j, & \text{if } u = z_{ij} \text{ and } i \neq j \\
r + 1, & \text{if } u = z_{ij} \text{ and } i = j.
\end{cases}
\]

By this coloring, any two vertices \( u \) and \( v \) satisfying \( c(u) = c(v) \) and \( c(N(u)) = c(N(v)) \) only occur for the pair of vertices \( s \) and \( y_i \) for \( w_1 = 2 \), and the pair of vertices \( l \) and \( x_1 \). Their color codes are distinguished by the last ordinate (their distances to a vertex in the color class \( r + 1 \)). Hence, all vertices have distinct color codes. So, \( \chi_L(\text{Edge-Amal}(T_i; e_{o_i})) \leq \max \{ r_i + 1 \} \).

Let \( H_m \) be a tree having a pendant \( e_{o_i} \) as depicted in Figure 2, where \( m \geq 3 \).

![Figure 2. A tree \( H_m \) where \( m \geq 3 \).](image)

**Theorem 2.4.** For \( i = 1, 2, \ldots, t \), let \( T_i = H_m \). We have that \( \chi_L(\text{Edge-Amal}(T_i; e_{o_i})) = m + 2 \), if \( 2 \leq t \leq m \).

**Proof.** Let \( t \in [2, m] \). Then, there are \( tm \) stems and each is adjacent to \( m \) leaves in graph \( \text{Edge-Amal}(T_i; e_{o_i}) \). We suppose that \( \chi_L(\text{Edge-Amal}(T_i; e_{o_i})) = m + 1 \). Then, there are \( m + 1 \) possibilities to coloring all stems and their neighbors in \( \text{Edge-Amal}(T_i; e_{o_i}) \). Since \( t \geq 2 \), there are at least two stems having the same color. Therefore, the color codes of these stems are the same, a contradiction to \( \chi_L(\text{Edge-Amal}(T_i; e_{o_i})) = m + 1 \). So,

\[
\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) \geq m + 2.
\]

Next, we define a coloring \( c : V(\text{Edge-Amal}(T_i; e_{o_i})) \rightarrow \{ 1, 2, \ldots, m + 2 \} \) as follows:
Note that all the colors above in modulo $m + 2$. We will show that $\chi_L(\text{Edge-Amal}(T; e_o)) \leq m + 2$. Let $u$ and $v$ be any two vertices with $c(u) = c(v)$. Then, by the coloring $c$, $c(N(u)) \neq c(N(v))$ because the $m - 1$ neighbors colors of $u$ are permutation of $m - 1$ neighbors colors of $v$ in modulo $m + 2$. Hence, all vertices in $\text{Edge-Amal}(T; e_o)$ have distinct color codes. So, $\chi_L(\text{Edge-Amal}(T; e_o)) \leq m + 2$. □

From Theorem 2.3, we shows the exact value of locating-chromatic number for some classes of trees. First, we give definition of some classes of trees and their locating-chromatic number, i.e. double stars, homogeneous caterpillars, and homogeneous lobsters. A double star, denoted by $S_{m,n}$ where $n \geq m \geq 1$, is the graph consisting of two stars $K_{1,n}$ and $K_{1,m}$ together with an edge joining their centers. Chartrand et al. [5] have proved $\chi_L(S_{m,n}) = n + 1$. The homogeneous caterpillar $C(m, n)$ is the graph consisting of $m$ stars $K_{1,n}$ by linking the centers from each stars. Asmiati et al. [3] showed that the locating-chromatic number of homogeneous caterpillar is $n + 1$ for $1 \leq m \leq n + 1$, and $n + 2$ for $m > n + 1$. The homogeneous lobster $Lb(m, n)$ is the graph obtained by attaching the centers of stars $K_{1,n}$ to each leaf of $C(m, n)$. Syofyan et al. [6] showed that the locating-chromatic number of the homogeneous lobster is $n + 1$ if $m = 1$, $n + 2$ for $2 \leq m \leq 3(n = 2) + 1$, or $n + 3$ for $m > 3(n + 2) + 1$.

Based on Theorem 2.3 and the locating-chromatic numbers of double stars, homogeneous caterpillars, and homogeneous lobsters, we have the locating-chromatic number of edge-amalgamation of these trees as follows. The terminal edge in each tree is chosen from the edges incident to a stem having maximum leaves.

**Corollary 2.1.** For $i = 1, 2, \ldots, t$, let $T_i = S_{m,n}$. Then, $\chi_L(\text{Edge-Amal}(T; e_o)) = t(n - 1) + 1$, if $n \geq m \geq 1$.

**Corollary 2.2.** For $i = 1, 2, \ldots, t$, let $T_i = C(m, n)$. If $1 \leq m \leq n + 1$, then $\chi_L(\text{Edge-Amal}(T; e_o)) = t(n - 1) + 1$.

**Corollary 2.3.** For $i = 1, 2, \ldots, t$, let $T_i = Lb(m, n)$. If $m = 1$, then $\chi_L(\text{Edge-Amal}(T; e_o)) = t(n - 1) + 1$.

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