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The total disjoint irregularity strength of some certain graphs

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Abstract

Under a totally irregular total k-labeling of a graph $G = (V, E)$, we found that for some certain graphs, the edge-weight set $W(E)$ and the vertex-weight set $W(V)$ of G which are induced by $k = ts(G)$, $W(E) \cap W(V)$ is a nonempty set. For which k, a graph G has a totally irregular total labeling if $W(E) \cap W(V) = \emptyset$? We introduce the total disjoint irregularity strength, denoted by $ds(G)$, as the minimum value k where this condition satisfied. We provide the lower bound of $ds(G)$ and determine the total disjoint irregularity strength of cycles, paths, stars, and complete graphs.

Keywords: total disjoint irregularity strength, total irregularity strength, irregular total labeling
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1. Introduction

Let G be a finite, simple, and undirected graph with the vertex set V and the edge set E. Let $f : V \cup E \to \{1, 2, \cdots, k\}$ be a total k-labeling. Under f, the *weight of a vertex* $v \in V$ is $w(v) = f(v) + \sum_{uv \in E} f(uv)$ and the *weight of an edge* $uv \in E$ is $w(uv) = f(u) + f(uv) + f(v)$. If all the vertex (or edge)-weights are distinct then f is called a vertex (or edge) irregular total

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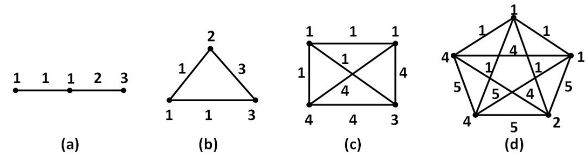

Figure 1. Totally disjoint irregular total labeling of (a) P_3, (b) C_3, (c) K_4, and (d) K_5

k-labeling and the minimum value k such that G has a vertex (or edge) irregular total k-labeling is called the total vertex (or edge) irregularity strength, denoted by $tvs(G)$ (or $tes(G)$), respectively. This parameters were introduced by Baca *et al.* [2]. They gave the boundary of $tes(G)$ and $tvs(G)$ and determined that for n vertices, $tvs(C_n) = tes(C_n) = \lceil \frac{n+2}{3} \rceil$, $tes(P_n) = \lceil \frac{n+1}{3} \rceil$, $tvs(S_n) = tes(S_n) = \lceil \frac{n+1}{2} \rceil$, and $tvs(K_n) = 2$.

Later, Jendrol *et al.* [7] provided a better lower bound of $tes(G)$ and determined that $tes(K_5) = 5$ and $tes(K_n) = \left\lceil \frac{n^2-n+4}{6} \right\rceil$, for $n \neq 5$. For any tree T, Ivanco and Jendrol [6] proved that $tes(T)$ is equal to the lower bound. Nurdin *et al.* [9] gave the lower bound for tvs for any graph G.

Recently, Marzuki *et al.* [8] combined the properties of $tes(G)$ and $tvs(G)$ and gave new parameter called the total irregularity strength, denoted by $ts(G)$. It is the minimum value k for which G has a totally irregular total k-labeling. They proved that the lower bound $ts(G) \geq max\{tes(G), tvs(G)\}$ is sharp for C_n and P_n except for P_2 or P_5. In [14], we proved that for $n \neq 2$, $ts(K_n) = tes(K_n)$. In [5], Indriati *et al.* proved that for $n \geq 3$, $ts(S_n) = tvs(S_n)$. For further reading, one can see [1], [3], [4], [5], and [10] - [13]. All these results showed that the lower bound is sharp.

Observing $ts(G)$, for the vertex weight-set $W(V)$ and the edge weight-set $W(E)$ under a totally irregular total labeling on G, $W(V) \cap W(E) \neq \emptyset$. Considering this condition, we define a new parameter called the total disjoint irregularity strength. *A totally disjoint irregular total k-labeling* of a graph G as a total labeling $f : V \cup E \to \{1, 2, \cdots, k\}$ which satisfies: (i) for any two vertices $x \neq y \in V$, $w(x) \neq w(y)$; (ii) for any two edges $x_1y_1 \neq x_2y_2 \in E$, $w(x_1y_1) \neq w(x_2y_2)$; (iii) $W(V) \cap W(E) = \emptyset$; where $W(V)$ (and $W(E)$) is the vertex (and edge) weight- set, respectively. The minimum value k such that a graph G has a totally disjoint irregular total labeling is called *the total disjoint irregularity strength* of a graph G, denoted by $ds(G)$. Thus, for any graph G,

$$ds(G) \geq ts(G). \tag{1}$$

For instance, Fig. 1 shows a totally disjoint irregular total labeling of P_3, C_3, K_4, and K_5.

In this paper, we determine ds of cycles, paths, stars, and complete graphs.

2. Main Results

Let $G = (V, E)$ be a connected graph. For G has a totally irregular total k-labeling $f : V \cup E \to \{1, 2, \cdots, k\}$, we need $|V| + |E|$ distinct weights. Let $\delta = \delta(G)$ (or $\Delta = \Delta(G)$) be the minimum (or maximum) degree of vertex in G, respectively. Let n_i be the number of vertices of degree i, where $i = \delta, \delta + 1, \cdots, \Delta$. Then $|V| = \sum_{i=\delta}^{\Delta} n_i$. Now, assume that $\delta = 1$. Let f be a optimal labeling with respect to $ds(G)$. Then the maximum weight has to be at least $|E|+|V|+1$. The maximum vertex weight is the sum of $\Delta+1$ labels and every edge weight is the sum of three labels imply that $k \geq \left\lceil \frac{|E|+|V|+1}{\Delta+1} \right\rceil$. Moreover, when $n_1 \leq \Delta$, $|V|+|E|$ distinct weights are only exist if $\left\lceil \frac{|E|+|V|+1}{\Delta+1} \right\rceil \geq \left\lceil \frac{|E|+n_1+n_2+1}{3} \right\rceil$. In the other hand, when $n_1 \geq \Delta$, we have $2n_1$ distinct weights depend on 2 labels, such that $|V|+|E|$ distinct weights are only exist if $\left\lceil \frac{|E|+|V|+1}{\Delta+1} \right\rceil \geq n_1$. Hence, the minimum value $k \geq \max\left\{n_1, \left\lceil \frac{|E|+n_1+n_2+1}{3} \right\rceil\right\}$. Next, assume that $\delta \neq 1$. For f is optimal then the minimum weight is at least 3. Then, $k \geq \left\lceil \frac{|E|+n_1+n_2+2}{3} \right\rceil$. Thus we have the lower bound of $ds(G)$.

Theorem 2.1. *Let $G=(V,E)$ be a connected graph. Let v be a pendant vertex and $n_i (i=1,2)$ be the number of vertices of degree i. Then*

$$ds(G) \geq \begin{cases} max\left\{n_1, \left\lceil \frac{|E|+n_1+n_2+1}{3} \right\rceil\right\}, & \text{if } v \in V; \\ \left\lceil \frac{|E|+n_1+n_2+2}{3} \right\rceil, & \text{if } v \notin V. \end{cases}$$

Our next results show that this lower bound is sharp.

Theorem 2.2. *Let $m_1 \geq 3$ and $m_2 \in \mathbb{N}$. Let C_{m_1} be a cycle with m_1 vertices and P_{m_2} be a path with m_2 vertices. Then*

$$\begin{aligned} ds(C_{m_1}) &= \left\lceil \frac{2m_1+2}{3} \right\rceil; \\ ds(P_{m_2}) &= \begin{cases} 3, & \text{for } m_2 = 3; \\ \left\lceil \frac{2m_2}{3} \right\rceil, & \text{otherwise}. \end{cases} \end{aligned}$$

To prove Theorem 2.2, we need this lemma.

Lemma 2.1. *For any integers y and $x_i, 1 \leq i \leq 2n$, let $\{x_i\}$ be a sequence. If the sum of 3 consecutive integers in $\{x_i\}$ is*

$$x_i + x_{i+1} + x_{i+2} = \begin{cases} y + 2i - 2, & \text{for } 1 \leq i \leq n-1; \\ y + 4n - 2i - 3, & \text{for } n \leq i \leq 2n-2; \end{cases}$$

then

$$x_i = \begin{cases} x_{2n-1}, & \text{for } i = n - 3j \text{ and } 1 \leq j \leq \left\lceil \frac{2n}{3} \right\rceil; \\ x_{2n-i+2}, & \text{for } i = n - 3j + 1 \text{ and } 1 \leq j \leq \left\lceil \frac{2n}{3} \right\rceil. \end{cases}$$

Proof. Set all equations above as a linear system leads to the solution which is required. □

Now, we are able to prove Theorem 2.2.

Proof. Let $C_{m_1} = \{v_1, e_1, v_2, e_2, \cdots, v_{m_1}, e_{m_1}\}$ and $P_{m_2} = \{v_1, e_1, v_2, e_2, \cdots, v_{m_2}\}$. Let $t(m_1) = \left\lceil \frac{2m_1+2}{3} \right\rceil$ and $t(m_2) = \left\lceil \frac{2m_2}{3} \right\rceil$. We divide the proof into two cases as follows:

Case 1. $m_2 = 3$

Suppose that $ds(P_3) = 2$. Since we need 5 distinct weight from 2 to 6, one endpoint (and its incidence edge) can be labeled by 1 to have smallest weight. In the other hand, maximum weight 6 only can occur when the rest vertices and edge are labeled by 2. Hence, there are one vertex and one edge of the same weight. Contrary to hypothesis. Thus, $ds(P_3) \geq 3$. By label P_3 as in Fig.1, we have $ds(P_3) = 3$.

Case 2. $m_1 \geq 3$ **and** $m_2 \neq 3$

It is trivial for $n_2 = 1$. For $n_1 \geq 3$ and $n_2 \geq 2$, by Theorem 2.1, $ds(C_{m_1}) \geq t(m_1)$ and $ds(P_{m_2}) \geq t(m_2)$. For the reverse inequality, we construct $f_i : V \cup E \to \{1, 2, \cdots, t(m_i)\}$, for $i = 1, 2$, as follows: Let $f_1^{m_1} = \{v_1^{m_1}, e_1^{m_1}, v_2^{m_1}, e_2^{m_1} \cdots v_{m_1}^{m_1}, e_{m_1}^{m_1}\}$ and $f_2^{m_2} = \{v_1^{m_2}, e_1^{m_2}, v_2^{m_2}, e_2^{m_2} \cdots v_{m_2}^{m_2}\}$ be the alternating vertex (and edge) label-sets, where $f_1(v_i) = v_i^{m_1}$, $f_2(v_i) = v_i^{m_2}$, $f_1(e_i) = e_i^{m_1}$, $f_2(e_i) = e_i^{m_2}$, and $W(C_{m_1}) = \{w(v_i), w(e_i) \,|1 \leq i \leq m_1\}$ and $W(P_{m_2}) = \{w(v_1), w(e_1), w(v_2), w(e_2), \cdots, w(v_{m_2})\}$ be the alternating vertex (and edge) weight-sets of C_{m_1} and P_{m_2}, respectively. Let

$$d(m_i) = \begin{cases} t(m_i) - 1, & \text{for } m_1 \equiv j \bmod 3, j = 0, 1, m_2 \equiv j \bmod 3, j = 0, 2; \\ t(m_i), & \text{for } m_1 \equiv 2 \bmod 3, m_2 \equiv 1 \bmod 3. \end{cases}$$

We prove by induction on m_i. For the base step, it is true that for $f_1^3 = \{1, 1, 2, 3, 3, 1\}$, $f_1^4 = \{1, 2, 2, 3, 3, 4, 4, 1\}$, and $f_1^5 = \{1, 1, 2, 2, 3, 3, 4, 4, 4, 1\}$, we have $W(C_3) = \{3, 4, 6, 8, 7, 5\}$, $W(C_4) = \{4, 5, 7, 8, 10, 11, 9, 6\}$, and $W(C_5) = \{3, 4, 5, 7, 8, 10, 11, 12, 9, 6\}$ and for $f_2^2 = \{1, 1, 2\}$, $f_2^4 = \{1, 1, 1, 2, 2, 3, 3\}$, and $f_2^6 = \{1, 1, 1, 3, 4, 4, 4, 2, 3, 1, 3\}$, we have $W(P_2) = \{2, 4, 3\}$, $W(P_4) = \{2, 3, 4, 5, 7, 8, 6\}$, and $W(P_6) = \{2, 3, 5, 8, 11, 12, 10, 9, 6, 7, 4\}$ such that for $i = 1, 2$, f_i is a totally disjoint irregular total $t(m_i)$- labeling, $ds(C_{m_1}) = t(m_1)$ for $m_1 \in \{3, 4, 5\}$ and $ds(P_{m_2}) = t(m_2)$ for $m_2 \in \{2, 4, 6\}$, where the maximum weight is $w(e_{d(m_i)})$.

For the inductive step, we assume that for all k_1 and k_2, f_1 is a totally disjoint irregular total t_1-labeling of C_{k_1} and f_2 is a totally disjoint irregular total t_2-labeling of P_{k_2}, where

$$e_{d(k_i)}^{k_i} = \begin{cases} t_i - 1, & \text{for } i = 2, k_2 \equiv j \bmod 9,\ j \in \{1, 2, 8\}; \\ t_i, & \text{for } i \in \{1, 2\}, k_2 \equiv j \bmod 9,\ j \in \{0, 3, 4, 5, 6, 7\}; \end{cases}$$

$$v_{d(k_i)+1}^{k_i} = \begin{cases} t_i - 1, & \text{for } i \in \{1, 2\}, k_1 = 6, k_2 \equiv j \bmod 9,\ j \in \{5, 7, 8\}; \\ t_i, & \text{for } i \in \{1, 2\}, k_1 \neq 6, k_2 \equiv j \bmod 9,\ j \in \{0, 1, 2, 3, 4, 6\}; \end{cases}$$

and the maximum weight is $w(e_{d(k_i)})$.

Let $G_{k_1} \cong C_{k_1}$ and $G_{k_2} \cong P_{k_2}$. To prove that $ds(G_{(k_i)+3}) = t(k_i + 3) = ds(G_{k_i}) + 2$, we construct $G_{(k_i)+3}$ from G_{k_i} by subdividing $e_{d(k_i)}$ as described in Fig.2. Define $f_i^{(k_i)+3} \setminus \{v_{(k_i)+1}^{(k_i)+3}, e_{(k_i)+1}^{(k_i)+3}, v_{(k_i)+2}^{(k_i)+3}, e_{(k_i)+2}^{(k_i)+3}, v_{(k_i)+3}^{(k_i)+3}, e_{(k_i)+1}^{(k_i)+3}\} = f_i^{k_i}$. Setting $w(e_{d(k_i)}) = w(e_{d(k_i+3)-2})$ and $w(v_{d(k_i+3)+1}) = w(v_{d(k_i)+1})$, we have $a \neq b$ for $a, b \in W(G_{(k_i)+3}) \setminus \{v_{(k_i)+1}, e_{(k_i)+1}, v_{(k_i)+2}, e_{(k_i)+2}, v_{(k_i)+3}, e_{(k_i)+1}\}$. Moreover, $e_{d(k_i)}^{(k_i)+3} = e_{(k_i)+3}^{(k_i)+3} = e_{d(k_i)}^{(k_i)}$ and $v_{d(k_i)+1}^{(k_i)+3} = v_{(k_i)+1}^{(k_i)+3} = v_{d(k_i)+1}^{(k_i)}$. This is sufficient to apply Lemma 2.1.

Let $\{x_i \,|1 \leq i \leq 8\} = \{e_{d(k_i)}^{(k_i)+3}, v_{(k_i)+1}^{(k_i)+3}, e_{(k_i)+1}^{(k_i)+3}, v_{(k_i)+2}^{(k_i)+3}, e_{(k_i)+2}^{(k_i)+3}, v_{(k_i)+3}^{(k_i)+3}, e_{(k_i)+3}^{(k_i)+3}, v_{d(k_i)+1}^{(k_i)+3}\}$ and $y = w(e_{d(k_i)}) + 1$. Then, we have $v_{(k_i)+2}^{(k_i)+3} = e_{d(k_i)}^{(k_i)} + 2$, $e_{(k_i)+2}^{(k_i)+3} = v_{d(k_i)+1}^{(k_i)} + 2$, and $v_{(k_i)+3}^{(k_i)+3} - 1 = e_{(k_i)+1}^{(k_i)+3}$,

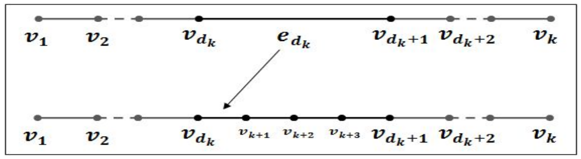

Figure 2. The construction of P_{k+3} from P_k

where

$$e_{(k_i)+1}^{(k_i)+3} = \begin{cases} 3, & \text{for } i=1, k_i=3; \\ t(k_1)+1, & \text{for } i=1, k_i \neq 3; \\ 2k_i - 2t(k_i)+3, & \text{for } i=2, k_2 \equiv 8 \bmod 9; \\ 2k_i - 2t(k_i)+2, & \text{for } i=2, k_2 \equiv j \bmod 9,\ j \in \{1,,2,5,7\}; \\ 2k_i - 2t(k_i)+1, & \text{for } i=2, k_2 \equiv j \bmod 9,\ j \in \{0,3,4,6\}. \end{cases}$$

Then, it can be checked that the maximum label is $ds(G_{k_i})+2 = ds(G_{(k_i)+3})$. We have completed the labeling f_i on $G_{(k_i)+3}$ and have proved that f_i is a totally disjoint irregular total $t(k_i)$-labeling. Thus, for any positive integer $m_1 \geq 3$ and $m_2 \in \mathbb{N}$, $ds(C_{m_1}) = \lceil \frac{2m_1+2}{3} \rceil$, $ds(P_{m_2}) = \lceil \frac{2m_2}{3} \rceil$, for $m_2 \neq 3$ and $ds(P_3) = 3$. □

Theorem 2.3. *Let $n \in \mathbb{N}$, $n \geq 3$ and S_n be a star with $n+1$ vertices, then $ds(S_n) = n$.*

Proof. Let $V(S_n) = \{v_i | 1 \leq i \leq n+1\}$ where v_{n+1} is the vertex of degree n. By Theorem 2.1, $ds(S_n) \geq n$. To prove the reverse inequality, we construct an irregular total labeling $f : V \cup E \rightarrow \{1, 2, \cdots, t\}$ by define $f(v_i) = i$ for $1 \leq i \leq n-1$, $f(v_n) = n-2$, $f(v_{n+1}) = n$, $f(v_i v_{n+1}) = 1$ for $1 \leq i \leq n-1$, and $f(v_n v_{n+1}) = 3$. Hence, we have $w(v_i) = i+1$ for $1 \leq i \leq n-1$, $w(v_n) = n+1$, $w(v_{n+1}) = 2n+2$, $w(v_i v_{n+1}) = n+i+1$ for $1 \leq i \leq n-1$, and $w(v_n v_{n+1}) = 2n+1$. See that $W(V) \cap W(E) = \emptyset$. Thus, f is a totally disjoint irregular total labeling and $ds(S_n) = n$ for $n \geq 3$. □

Next, by using our previous result in [14], we determine the exact value of $ds(K_n)$. For the convenient of reader, we provided the construction of totally irregular total labeling of K_n for $n \neq 5, 10, 12$ given in [14]. Let $\left\lceil \frac{n^2-n+4}{6} \right\rceil = t$ and $\lfloor \frac{n+1}{3} \rfloor = s$. We divide the vertex-set into 3 mutually disjoint subsets, say A, B, and C, where $A = \{a_i \,|1 \leq i \leq s\}$, $B = \{b_i \,|1 \leq i \leq n-2s\}$, and $C = \{c_i \,|1 \leq i \leq s\}$. Let $f : V \cup E \rightarrow \{1, 2, \cdots, t\}$ defined by:

$$\begin{aligned} f(a_i) &= 1, && \text{for } 1 \leq i \leq s; \\ f(b_i) &= \tbinom{s}{2} + s(i-1)+1, && \text{for } 1 \leq i \leq n-2s; \\ f(c_i) &= t, && \text{for } 1 \leq i \leq s; \\ f(a_i a_j) &= \tbinom{j-1}{2} + i, && \text{for } 1 \leq i < j \leq s; \\ f(a_i b_j) &= i, && \text{for } 1 \leq i \leq s, 1 \leq j \leq n-2s; \\ f(a_i c_j) &= s(i-1)+j, && \text{for } 1 \leq i, j \leq s; \\ f(b_i b_j) &= s(n-s-i-j+2) - \tbinom{s}{2} + \tbinom{j-1}{2} + i, && \text{for } 1 \leq i < j \leq n-2s; \\ f(b_i c_j) &= \tbinom{n-2s}{2} + s(n-s) - t + j + 1, && \text{for } 1 \leq i \leq n-2s, 1 \leq j \leq s; \\ f(c_i c_j) &= \tbinom{n}{2} - 2(t-1) - \tbinom{s-i+1}{2} + j - i, && \text{for } 1 \leq i < j \leq s. \end{aligned} \tag{2}$$

Theorem 2.4. *Let $n \in \mathbb{N}, n \notin \{i\,|6 \le i \le 59\} \cup \{61, 62, 65, 68, 71, 74\}$ and K_n be a complete graph with n vertices. Then*

$$ds(K_n) = \begin{cases} n, & \textit{for } n \le 5; \\ \left\lceil \frac{n^2-n+4}{6} \right\rceil, & \textit{otherwise.} \end{cases}$$

Proof. By 1 and Theorem 2.1, $ds(K_n) \ge ts(K_n)$. Let $t = \left\lceil \frac{n^2-n+4}{6} \right\rceil$. For the reverse inequality, we divide the proof into three cases as follows:

Case 1. $n \le 5$

It is obvious for $n \le 3$. Now, suppose that $ds(K_4) = 3$. We need 10 distinct weight with minimum weight 3. We can label 2 vertices and one edge by label 1. In the other hand, the maximum weight should be 12. Labeling 3 edges and one vertex by label 3 implies that there are 2 edges with the same weight 7. Contrary to hypothesis. Thus, $ds(K_4) \ge 4$. To prove the upper bound for $n = 4$ or 5, we define f as in Fig. 1. Therefore, we have the exact value of $ds(K_n)$ for $n \le 5$.

Case 2. $n = 77$ **or** $n \ge 80$

Consider that under the totally irregular total t- labeling of K_n in (2), the maximum edge weight is $w(c_{s-1}c_s) = \binom{n}{2} + 2$ and minimum vertex-weight is $w(a_1) = \frac{s(s^2-1)}{6} + n$. It follow $w(c_{s-1}c_s) < w(a_1)$ implies vertex-weight set and edge weight set are disjoint. Thus, $ds(K_n) = t$ for $n = 77$ or $n \ge 80$.

Case 3. $n \in \{60, 63, 64, 66, 67, 69, 70, 72, 73, 75, 76, 78, 79\}$

Consider that under the totally irregular total t- labeling of K_n provided in (2), we met condition where the minimum vertex-weight $w(a_1)$ is equal to the weight of an edge connecting vertices in (B, C) or (C, C). Then, we modify f. Let $E(K_n) = \{e_i\,|1 \le i \le n(n-1)/2\}$. Let $e_p \in E(K_n)$ be an edge where $w(a_1) = w(e_p)$. Since $t \equiv 2 \bmod 3$, then $f(e_{n(n-1)/2}) = f(c_{s-1}c_s) = t - 1$. It implies that we can change $f(e_i)$ by $f(e_i + 1)$, for $p \le i \le n(n-1)/2$ without changing the maximum label such that $W(V(K_n)) \cap W(E(K_n)) = \emptyset$. It complete the proof. □

Open Problem

1. For $n \in \mathbb{N}, n \in \{i\,|6 \le i \le 59\} \cup \{61, 62, 65, 68, 71, 74\}$, find the exact value of $ds(K_n)$.
2. For any graph G, find $ds(G)$.

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