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On locating-dominating number of comb product graphs

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Abstract

We consider a set $D \subseteq V(G)$ which dominate G and for every two distinct vertices $x, y \in V(G) \setminus D$, the open neighborhood of x and y in D are different. The minimum cardinality of D is called the *locating-dominating number* of G. In this paper, we determine an exact value of the locating-dominating number of comb product graphs of any two connected graphs of order at least two.

Keywords: comb product, locating-dominating number, locating-dominating sets
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1. Introduction

In this paper, all graphs are assumed to be connected, simple, finite, and undirected. For a graph G and a vertex $x \in V(G)$, we recall that the *open neighborhood* of x in G is defined as $N_G(x) = \{y \in V(G) | xy \in E(G)\}$. Now, we consider a subset S of $V(G)$. In case every vertex $x \in V(G) \backslash S$ satisfies $N_G(x) \cap S \neq \emptyset$, we say the set S as a *dominating set* of G. The *domination number* of G refers to the minimum cardinality of S, and denoted by $\gamma(G)$. The survey of this domination parameter can be detailed seen in [8, 9]. The concept of dominating set give us an information of a minimum set that can be the detector for every vertex which is adjacent to this

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set of vertices. But from this concept the detectors cannot distinguish every vertices in G. For this purpose, we will use the concept of location.

Now, we consider a dominating set S where for every two vertices $x, y \in V(G) \backslash S$, the open neighborhood of x and y in S are different. The set S then we called as a *locating-dominating set* of G. The *locating-dominating number*, denoted by $\lambda(G)$, is the minimum cardinality of locating-dominating sets of graph G. Therefore, by the definitions, it follows $\gamma(G) \leq \lambda(G)$. This concept was firstly introduced by Slater [16, 17].

In [15], it has been proven that determining the locating-dominating number of a graph is an NP-complete problem. There is no efficient algorithm to find the locating-dominating number of general graphs. However, Henning and Oellermann [10] have been characterized all graphs having locating-dominating number $n-1$ and $n-2$. Meanwhile, Caceres *et al.* [2] provided 16 non-isomorphic graphs having locating-dominating number two. Some authors also have proven the locating-dominating number of certain classes of graphs. Interested readers are referred to a number of relevant literature that are mentioned in the bibliography section, including [2, 4, 6, 7].

Some authors also have determined the locating-dominating number of graphs obtained from a product graphs. Canoy and Malacas [3] provided the bounds for the locating-dominating number of corona product graphs. They also investigated a locating-dominating set of the composition product graphs. Moreover, they determined an exact value of the locating-dominating number of composition product graphs between G and H where G is a connected totally point determining graph and H is a non-trivial connected graph.

We are interested to apply the locating-dominating concept to a product graphs. In this paper, we consider the *comb product* of connected graphs G and H and both graphs have order at least two. This product graphs is constructed as follows.

1. Given two connected graphs G and H.
2. Choose a vertex in a graph H, say it o.
3. Make $|V(G)|$ copies of H.
4. Identified the i-th vertex of G to the vertex o in the i-th copy of H

By the construction above, we can say that $V(G \rhd_o H) = \{(x, u) | x \in V(G), u \in V(H)\}$ and $(x, u)(y, v) \in E(G \rhd_o H)$ if ($x = y$ and $uv \in E(H)$) or ($xy \in E(G)$ and $u = v = o$). In chemistry [1], some classes of chemical graphs can be considered as the comb product graphs. This product graphs has been widely investigated in many areas, including metric distance problems [5, 13, 14] and graph labeling problems [11, 12].

For the purpose to determine the locating-dominating number of $G \rhd_o H$, we will use some definitions. For $o \in V(H)$ and $x \in V(G)$, we define $G_o = \{(x, o) | x \in V(G)\}$ and $H_x = \{(x, u) | u \in V(H)\}$. We also define $H_x^- = H_x \setminus \{(x, o)\}$. Note that, since the order of H is at least 2, it follows H_x^- is a non-empty set. Furthermore, if $z \in H_x^-$, then $N_{G \rhd_o H}(z) \subseteq H_x$. For $S \subseteq V(G)$, we also use the notation $G[S]$ which is a maximal subgraph of G induced by all vertices of S.

2. Main Results

From now on, every connected graphs G and H stated here are not trivial graph. In order to determine $\lambda(G \rhd_o H)$, we consider H_x for every $x \in V(G)$. We also define W as a locating-dominating set of $G \rhd_o H$ and $W_x = W \cap H_x$. In Lemma 2.1, we show that H_x contributes at least $\lambda(H) - 1$ vertices in W.

Lemma 2.1. *For every vertex $x \in V(G)$, $W \cap H_x \neq \emptyset$. Moreover, $|W \cap H_x| \geq \lambda(H) - 1$.*

Proof. For $x \in V(G)$, if $W \cap H_x = \emptyset$, then there exists a vertex $z \in H_x^-$ such that $N_{G \rhd_o H}(z) \cap W = \emptyset$, a contradiction.

Now, suppose that we have a vertex $x \in V(G)$ such that $|W_x| \leq \lambda(H) - 2$ where $W_x = W \cap H_x$. So, two different vertices in H_x^- are not in W_x, let them be a and b. These two vertices satisfy $N_{G \rhd_o H}(a) \cap H_x = \emptyset$, $N_{G \rhd_o H}(b) \cap H_x = \emptyset$, or $N_{G \rhd_o H}(a) \cap W_x = N_{G \rhd_o H}(b) \cap W_x$. Therefore, we obtain $N_{G \rhd_o H}(a) \cap W = N_{G \rhd_o H}(a) \cap W_x = \emptyset$, $N_{G \rhd_o H}(b) \cap W = N_{G \rhd_o H}(b) \cap W_x = \emptyset$, or $N_{G \rhd_o H}(a) \cap W = N_{G \rhd_o H}(a) \cap W_x = N_{G \rhd_o H}(b) \cap W_x = N_{G \rhd_o H}(b) \cap W$, a contradiction. □

From the proof of Lemma 2.1 above, for $x \in V(G)$, if $z \in H_x^-$, then $N_{G \rhd_o H}(z) \subseteq H_x$. The only vertex in H_x which is adjacent to a vertex outside H_x is (x, o). So, we have a direct consequences in corollary below.

Corollary 2.1. *If $|W_x| = \lambda(H) - 1$, then $(x, o) \notin W_x$. Furthermore, $(W_x \cup \{(x, o)\})$ is a locating-dominating set of graph $(G \rhd_o H)[H_x]$.*

By Lemma 2.1 above, the lower bound of $\lambda(G \rhd_o H)$ is obtained, that is $\lambda(G \rhd_o H) \geq |V(G)| \cdot (\lambda(H) - 1)$. Note that, if $\lambda(G \rhd_o H) = |V(G)| \cdot (\lambda(H) - 1)$ and W is a locating-dominating set of $G \rhd_o H$ where $|W| = |V(G)| \cdot (\lambda(H) - 1)$, then by Corollary 2.1, all vertices in G_o are not in W. Since for every $x \in V(G)$, H_x contributes $\lambda(H) - 1$ vertices in W, it may be happen that there exists a vertex $z \in H_x^-$ such that $N_{G \rhd_o H}(z) \cap W = N_{G \rhd_o H}((x, o)) \cap W$ or $N_{G \rhd_o H}((x, o)) \cap W = \emptyset$. So, we must add more vertices to W such that a new set is a locating-dominating set of $G \rhd_o H$.

Lemma 2.2. *If a vertex $x \in V(G)$ satisfies $|W_x| = \lambda(H) - 1$, then $N_{G \rhd_o H}((x, o)) \cap G_o \cap W \neq \emptyset$.*

Proof. Since $|W_x| = \lambda(H) - 1$, then W_x is not a locating-dominating set of $(G \rhd_o H)[H_x]$ and by Corollary 2.1, $(x, o) \notin W_x$. Therefore, there exists a vertex $a \in H_x^-$ such that $N_{G \rhd_o H}(a) \cap W_x = N_{G \rhd_o H}((x, o)) \cap W_x$ or $N_{G \rhd_o H}((x, o)) \cap W_x = \emptyset$. Since W is a locating-dominating set and the only vertex of H_x which is adjacent to vertex in $V(G \rhd_o H) \setminus H_x$ is (x, o), there must be a vertex $y \in W$ which is adjacent to (x, o). Note that, y is an element of G_o. □

Now, in Lemma 2.3 below, we consider that the set H_x can contribute $\lambda(H)$ vertices in a locating-dominating set of $G \rhd_o H$.

Lemma 2.3. *Let B be a locating-dominating set of H with $\lambda(H)$ vertices. For $x \in V(G)$, let $B_x = \{(x, v) | x \in V(G), v \in B\}$. Then $D = \bigcup_{x \in V(G)} B_x$ is a locating-dominating set of $G \rhd_o H$.*

Proof. Let us consider $a, b \in V(G \rhd_o H) \setminus D$ where $a \neq b$. If both vertices $a, b \in H_x$ for $x \in V(G)$, then it is clear that $\emptyset \neq N_{G \rhd_o H}(a) \cap B_x \neq N_{G \rhd_o H}(b) \cap B_x \neq \emptyset$ which implies $\emptyset \neq N_{G \rhd_o H}(a) \cap D \neq N_{G \rhd_o H}(b) \cap D \neq \emptyset$.

Now, we assume that $a \in H_x$ and $b \in H_y$ with $x, y \in V(G)$ and $x \neq y$. Then there exist two different vertices $u \in B_x$ and $v \in B_y$ such that $ua, vb \in E(G \rhd_o H)$ but $ub, va \notin E(G \rhd_o H)$. Therefore, $\emptyset \neq N_{G \rhd_o H}(a) \cap D \neq N_{G \rhd_o H}(b) \cap D \neq \emptyset$. □

According to Lemmas 2.1 and 2.3 above, we obtain some direct corollaries below.

Corollary 2.2. *Let G and H be a connected graphs of order at least 2. Then $|V(G)| \cdot (\lambda(H)-1) \leq \lambda(G \rhd_o H) \leq |V(G)| \cdot \lambda(H)$.*

Corollary 2.3. *Let W be a locating-dominating set of $G \rhd_o H$ where $|W| = \lambda(G \rhd_o H)$. For $x \in V(G)$, let $W_x = W \cap H_x$. Then either $|W_x| = \lambda(H) - 1$ or $|W_x| = \lambda(H)$.*

Let $o \in V(H)$ be an identifying vertex. Let W be a locating-dominating set of $G \rhd_o H$ where $|W| = \lambda(G \rhd_o H)$. By Corollary 2.3, for $x \in V(G)$, the set $W_x = W \cap H_x$ satisfies $|W_x| = \lambda(H) - 1$ or $|W_x| = \lambda(H)$. So, we define

$$T^+ = \{x \in V(G) || W_x| = \lambda(H)\} \tag{1}$$

and

$$T^- = \{x \in V(G) || W_x| = \lambda(H) - 1\}. \tag{2}$$

Note that $T^+ \cap T^- = \emptyset$ and $T^+ \cup T^- = V(G)$. Therefore, we obtain the lemma below.

Lemma 2.4. *Let W be a locating-dominating set of $G \rhd_o H$ where $|W| = \lambda(G \rhd_o H)$. Then*

$$|W| = (|T^+| \cdot \lambda(H)) + (|T^-| \cdot \lambda(H) - 1)$$

Considering Corollary 2.1, Lemma 2.2, and Corollary 2.3 above, we will characterize graph H based on its identifying vertex. Let $o \in V(H)$ be an identifying vertex. We say that a graph H is of:

- Type $\mathcal{A}_o$ if there exists a locating-dominating set D of $H \setminus \{o\}$ with $\lambda(H) - 1$ vertices and there exists $v \in V(H) \setminus \{o\}$ such that $\emptyset \neq N_H(o) \cap D = N_H(v) \cap D \neq \emptyset$.
- Type $\mathcal{B}_o$ if every locating-dominating set D of $H \setminus \{o\}$ with $\lambda(H) - 1$ vertices, satisfies $N_H(o) \cap D = \emptyset$.
- Type $\mathcal{C}_o$ if H is neither of type $\mathcal{A}_o$ nor $\mathcal{B}_o$.

By characterization above, we can say that every locating-dominating set D of $H \setminus \{o\}$ of type of $\mathcal{C}_o$ consists of at least $\lambda(H)$ vertices. Note that, the type of H is based on the identifying vertex o chosen. For example, let H with the identifying vertex $o \in V(H)$ be of type $\mathcal{A}_o$. If we choose another identifying vertex $a \in V(H) \setminus \{o\}$, the type of H may be $\mathcal{A}_a$, $\mathcal{B}_a$, or $\mathcal{C}_a$.

Now, we will provide the lower bound of $\lambda(G \rhd_o H)$ for type of $\mathcal{A}_o$ and $\mathcal{B}_o$ of H.

Lemma 2.5. *Let G and H be connected graphs of order at least 2. Let $o \in V(H)$.*

1. *If H is of type $\mathcal{A}_o$, then $\lambda(G \rhd_o H) \geq \gamma(G) + |V(G)| \cdot (\lambda(H) - 1)$.*
2. *If H is of type $\mathcal{B}_o$, then $\lambda(G \rhd_o H) \geq \lambda(G) + |V(G)| \cdot (\lambda(H) - 1)$.*

Proof. We recall the sets T^+ and T^- defining on (1) and (2), respectively.

Let $X = G_o \cap W$. By Corollary 2.1, for $x \in V(G)$, if $|W_x| = \lambda(H) - 1$, then $(x, o) \notin W_x$. Thus, $(T^- \cap W) = \emptyset$ and X should be a subset of T^+. Moreover, Lemma 2.2 provides that for every $x \in T^-$, $N_{G \rhd_o H}((x, o)) \cap X \neq \emptyset$. Then $N_{G \rhd_o H}((x, o)) \cap T^+ \neq \emptyset$.

1. If H is of type $\mathcal{A}_o$, then T^+ should dominate vertices in G_o, which implies $|T^+| \geq \gamma(G)$. Then by Lemma 2.4, we obtain

$$\begin{aligned}|W| &= |T^+| \cdot \lambda(H) + |T^-| \cdot (\lambda(H) - 1)\\ &= |T^+| \cdot \lambda(H) + (|V(G)| - |T^+|) \cdot (\lambda(H) - 1)\\ &= |T^+| + |V(G)| \cdot (\lambda(H) - 1)\\ &\geq \gamma(G) + |V(G)| \cdot (\lambda(H) - 1).\end{aligned}$$

2. If H is of type $\mathcal{B}_o$, then T^+ should locate and dominate vertices in G_o, which implies $|T^+| \geq \lambda(G)$. Then by Lemma 2.4, we obtain

$$\begin{aligned}|W| &= |T^+| \cdot \lambda(H) + |T^-| \cdot (\lambda(H) - 1)\\ &= |T^+| \cdot \lambda(H) + (|V(G)| - |T^+|) \cdot (\lambda(H) - 1)\\ &= |T^+| + |V(G)| \cdot (\lambda(H) - 1)\\ &\geq \lambda(G) + |V(G)| \cdot (\lambda(H) - 1).\end{aligned}$$

□

Now, we are ready to determine the locating-dominating number of $G \rhd_o H$ for connected graphs G and H of order at least 2, with an identifying vertex $o \in V(H)$.

Theorem 2.1. *Let G and H be a non-trivial connected graphs. Let $o \in V(H)$. Then*

$$\lambda(G \rhd_o H) = \begin{cases} \gamma(G) + |V(G)| \cdot (\lambda(H) - 1), & \text{if } H \text{ is of type } \mathcal{A}_o,\\ \lambda(G) + |V(G)| \cdot (\lambda(H) - 1), & \text{if } H \text{ is of type } \mathcal{B}_o,\\ |V(G)| \cdot \lambda(H), & \text{if } H \text{ is of type } \mathcal{C}_o. \end{cases}$$

Proof. We distinguish two cases.
Case 1. H is of type $\mathcal{A}_o$ or of type $\mathcal{B}_o$.
By Lemma 2.5,

- if H is of type $\mathcal{A}_o$, then we only need to show that $\lambda(G \rhd_o H) \leq \gamma(G) + |V(G)| \cdot (\lambda(H) - 1)$;
- if H is of type $\mathcal{B}_o$, then we only need to show that $\lambda(G \rhd_o H) \leq \lambda(G) + |V(G)| \cdot (\lambda(H) - 1)$.

Now, let us consider a locating-dominating set D of $H \setminus \{o\}$ with $\lambda(H) - 1$ vertices where

- if H is of type $\mathcal{A}_o$, then there exists $v \in V(H) \setminus \{o\}$ such that $\emptyset \neq N_H(o) \cap D = N_H(v) \cap D \neq \emptyset$;
- if H is of type $\mathcal{B}_o$, then $N_H(o) \cap D = \emptyset$.

For $x \in V(G)$, we define $D_x = \{(x, u) | u \in D\}$. Let $X \subseteq V(G)$ be a dominating set of G with $\gamma(G)$ vertices if H is of type $\mathcal{A}_o$ and be a locating-dominating set of G with $\lambda(G)$ vertices if H is of type $\mathcal{B}_o$. We also define $X_o = \{(a, o) | a \in X\}$. Let $S = X_o \cup \bigcup_{x \in V(G)} D_x$. We will show that S is a locating-dominating set of $G \rhd_o H$.

Let a and b be two distinct vertices in $V(G \rhd_o H) \setminus S$.

- $a, b \in H_x$ for $x \in V(G)$

  If $a, b \in H_x \setminus \{(x, o)\}$ for $x \in V(G)$, then it is clear that $\emptyset \neq N_{G \triangleright_o H}(a) \cap D_x \neq N_{G \triangleright_o H}(b) \cap D_x \neq \emptyset$. If $a = (x, o)$, then note that a is the only vertex in H_x which is adjacent to a vertex in X_o. Therefore, we obtain $\emptyset \neq N_{G \triangleright_o H}(a) \cap S \neq N_{G \triangleright_o H}(b) \cap S \neq \emptyset$.

- $a \in H_x$ and $b \in H_y$ for $x, y \in V(G)$ and $x \neq y$

  We distinguish two cases.

  1. $a \in H_x \setminus \{(x, o)\}$ and $b \in H_y \setminus \{(y, o)\}$
     Then there exists $u \in D_x$ and $v \in D_y$ such that $au, bv \in E(G \triangleright_o H)$ but $av, bu \notin E(G \triangleright_o H)$.
  2. $a = (x, o)$ or $b = (y, o)$
     If H is of type $\mathcal{A}_o$, then there exist $u \in D_x$ and $v \in D_y$ such that $au, bv \in E(G \triangleright_o H)$ but $av, bu \notin E(G \triangleright_o H)$. Now, we assume H is of type $\mathcal{B}_o$. Let $a = (x, o)$. Then there exists a vertex $z \in X_o$ such that $az \in E(G \triangleright_o H)$ but $bz \notin E(G \triangleright_o H)$.

  According two cases above, we obtain $\emptyset \neq N_{G \triangleright_o H}(x) \cap S \neq N_{G \triangleright_o H}(y) \cap S \neq \emptyset$.

Case 2. H is of type $\mathcal{C}_o$.

By Corollary 2.2, we only need to show that $\lambda(G \triangleright_o H) \geq |V(G)| \cdot \lambda(H)$. We recall the sets T^+ and T^- defining on (1) and (2), respectively. Let D be a locating-dominating set of $H \setminus \{o\}$. Note that $|D| \geq \lambda(H)$. Let W be a locating-dominating set of $G \triangleright_o H$ and for $x \in V(G)$, $W_x = W \cap H_x$. Since H is of type $\mathcal{C}_o$, by considering Corollary 2.3, then $|W_x| \geq |D| = \lambda(H)$ for every $x \in V(G)$. So, we can say $|T^-| = 0$. By Lemma 2.4, we have $|W| \geq |V(G)| \cdot \lambda(H)$. □

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