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On b-edge consecutive edge labeling of some regular trees

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Abstract

Let $G = (V, E)$ be a finite (non-empty), simple, connected and undirected graph, where V and E are the sets of vertices and edges of G. An edge magic total labeling is a bijection α from $V \cup E$ to the integers $1, 2, \ldots, n + e$, with the property that for every $xy \in E$, $\alpha(x) + \alpha(y) + \alpha(xy) = k$, for some constant k. Such a labeling is called a b-edge consecutive edge magic total if $\alpha(E) = \{b + 1, b + 2, \ldots, b + e\}$. In this paper, we proved that several classes of regular trees, such as regular caterpillars, regular firecrackers, regular caterpillar-like trees, regular path-like trees, and regular banana trees, have a b-edge consecutive edge magic labeling for some $0 < b < |V|$.

Keywords: banana tree, caterpillar, consecutive edge magic labeling, edge magic labeling, Firecracker
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1. Introduction

All graphs which are considered in this paper are finite, simple, connected and undirected. Let $G = (V, E)$ be a graph with vertex set V and edge set E. The labeling $\alpha : V \cup E \rightarrow \{1, 2, ..., |V| + |E|\}$ of G is called *edge magic* total if every edge xy has the same weight $w(x) = \alpha(x) + \alpha(y) + \alpha(xy) = k$, and G is called an *edge magic total graph* if an edge magic total labeling of G exists. If $\alpha(V) = \{1, ..., n\}$ then α is called a *super edge magic total labeling*. Magic labeling introduced by Sedláček [8] in 1963, and until now, the research is grown and there are many results in magic labeling, especially in edge magic labeling. There are some results on some classes of trees, such as banana trees [5]. The super edge magic strength of caterpillars, firecrackers and

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banana trees were studied by Swaminathan and Jeyanthi [12]. For further results in graph labeling, including the (super) edge magic total labeling, we can see [4].

A bijection $f : V(G) \cup E(G) \to \{1, 2, ..., |V| + |E|\}$ is called a b-edge consecutive edge magic total labeling of $G = G(V, E)$ if f is an edge magic total labeling and $f(E) = \{b + 1, ..., b + e\}, 0 \leq b \leq n$. A graph G that has b-edge consecutive edge magic total labeling is called a *b-edge consecutive edge magic total graph*. For simplicity, for the rest of the paper, we will use b-ECEMTL for an abbreviation of b-edge consecutive edge magic total labeling. Since if $b = 0$ the 0-ECEMTL will be a well known super edge magic total labeling, which are already studied by many researchers, and in the case $b = n$ the labeling can be found by dual of super edge magic total labeling (if any), then in this paper, we only consider the case of $0 < b < n$. The most famous conjecture on edge magic labeling area is from Enomoto *et al.* [3], which is "every tree is super edge magic graph." This conjecture might also be true for b-ECEMTL. On the direction of showing the conjecture is true, in this paper, we study several classes of regular trees, such as regular caterpillars, regular firecrackers, regular caterpillar-like trees, regular path-like trees and regular banana trees. Thus, this paper is a mini survey on b edge consecutive edge labeling on several regular trees. Moreover, we add new result on regular banana trees.

2. Known Results

Sugeng and Miller introduced the concept of b-ECEMTL in 2008. This paper was inspired by the concept of the edge consecutive vertex magic total labeling and vertex consecutive vertex magic total labeling by Balbuena *et al.* [1]. Sugeng and Miller [9] proved several results as follows. The first theorem said that we always can find a graph that has b-ECEMTL for every $b, 0 < b < n$.

Theorem 2.1. *[9] There exists a b-ECEMT graph for every b, $0 < b < n$.*

Theorem 2.2. *[9] If a connected graph G has a b-ECEMTL, where $b \in \{1, ..., n-1\}$, then G is a tree.*

Theorem 2.3. *[9] Every caterpillar has a b-ECEMTL, where*

$$b = \begin{cases} \lceil \frac{r+1}{2} \rceil + \sum_{i\ even} n_i - 2, & \text{if } i \text{ is odd,} \\ \lceil \frac{r+1}{2} \rceil + \sum_{i\ even,\ i<r} n_i - 2 + (n_r - 1), & \text{if } i \text{ is even.} \end{cases}$$

2.1. Regular Caterpillar and Regular Firecracker

A caterpillar is a graph derived from a path by hanging any number of leaves from the vertices of the path. If the number of leaves of every center the same, then we called it a regular caterpillar. We call the path which its vertices are the centers of the caterpillar as a backbone path of the caterpillar. A firecracker is a graph obtained from the concatenation of stars by linking one leaf from each. We call the linking leaf as a backbone path of the firecracker. If a firecracker is obtained from the concatenation of isomorphic stars, we get a regular firecracker. A caterpillar can be obtained from firecracker by moving the edges linking one leaf from each star S_i to linking each center of S_i and vice versa. Theorem 2.3 gives the result that every caterpillar has a b-ECEMTL for some $b \in (0, |V|)$. The similar result has done by Kang *et al.* [6] that caterpillar has a b-ECMTL for some specific b. They also proved that if G is a tree with the bipartite set $V(G) = V_1 \cup V_2$ and having a b-edge consecutive magic labeling then $b \in \{0, |V_1|, |V_2|, |V|\}$. However, in their paper they also included the value $b = 0$ and $b = |V|$, which we do not consider in this paper. The preliminary results in this subsection already presented in [10] and [11]. It is known that every caterpillar has a b-ECEMTL (Theorem 2.3 and in [6]). However, in the following theorem, we give an alternative proof for the regular caterpillar.

Theorem 2.4. *Every r-regular caterpillar has a b-ECEMTL, for $0 < b < n$.*

Proof. Let G be a regular caterpillar with c_i as its center vertices, for $i = 1, 2, ..., k$ and r as the number of leaves of every center. Let v_i^j be the j-th leaf of the center c_i, $i = 1, ..., k$ and $j = 1, ..., r$. Since caterpillar is a bipartite graph, then we can divide the set of its vertices as two disjoint sets of vertices, say V_1 and V_2. Arrange the vertices such that if we draw the caterpillar, then the edges do not intersect each other. As an example we can put the center c_1 in the set V_1 and label it with 1, and put the leaf vertices of the center v_1^j, $j = 1, ..., r$ in the set V_2 and label it with $|E| + b + 1, ..., |E| + b + r$. The next step, put the leaves of center c_2, v_2^j, $j = 1, ..., r$, in the set V_1 and label it with $2, ..., 2 + r - 1$, then put the center vertex c_2 in the set V_2 and label it with $|E| + b + r + 1$. This process can continue until all vertices have its label.

The weight $f(u) + f(v)$ for every edge uv in the caterpillar will form consecutive integers. By completing the edge label with $b + 1, b + 2, ..., b + |E|$, following the edge weight starts from the edge with the biggest label, then we can see that f is a b-ECEMTL, with $b = |V_1|$.

□

Theorem 2.5. *Every r-regular firecrackers has a b-ECEMTL, where $0 < b < n$*

Proof. The firecracker G is a concatenation of stars S_i, $i = 1, ..., k$. Let c_i be center of each S_i, v_i^j be the j-th leaf of S_i, $i = 1, ..., k$ and $j = 1, ..., r$. Let P_s be the backbone path of the firecracker, with $V(P_s) = \{v_1^*, ..., v_k^*\}$. Note that the firecracker is regular, then the number of leaves is the same. Label the firecracker using Algorithm 1.

Algorithm 1.

1. Move the edges $v_i^* v_{i+1}^*$ to $c_i c_{i+1}$, for all $i = 1, ..., k$, to obtain a caterpillar.
2. Label the caterpillar with b-ECEMTL given in the proof of Theorem 3.1 in such a way that all leaves of c_i have the smallest label if i is odd and have the biggest label if i is even, .
3. Move back the edges $c_i c_{i+1}$ to $v_i^* v_{i+1}^*$, , for all $i = 1, ..., k$, to return the graph to the firecracker form.

The moving process of the edges $c_i c_{i+1}$ to $v_i^* v_{i+1}^*$, for all $i = 1, ..., k$, in step (iii) guarantee the b-ECEMTL for the firecracker. □

2.2. Path-like and Caterpillar-like trees

Let P_n be a path with n vertices. Embed the path in the two dimensional grid where the vertex is located in the intersection point of the grid. An elementary transformation of the path is a process by replacing the edge xy by a new edge $x^* y^*$, such that the edge weight set does not change. A tree T of order n is called path-like tree when it can be obtained from embedding a path in the two-dimensional grid and using set of elementary transformations. The structure of the path-like tree was studied by Muntaner-Batle and Rius-Font [7]. Later, Sugeng and Silaban in [10] use generalisation of the path-like tree on a backbone path of caterpillar to obtain a super edge magic total labeling on new subclass of trees that they called caterpillar-like trees. This idea can be use for the regular caterpillar to obtain a b-ECEMTL for regular caterpillar-like trees. Theorem 2.6 is published in the proceedings[10]. The super edge magic strength of caterpillar and firecracker was studied by Swaminathan and Jeyanthi [12].

Figure 1 gives the example of b-ECEMTL for the regular caterpillar-like tree.

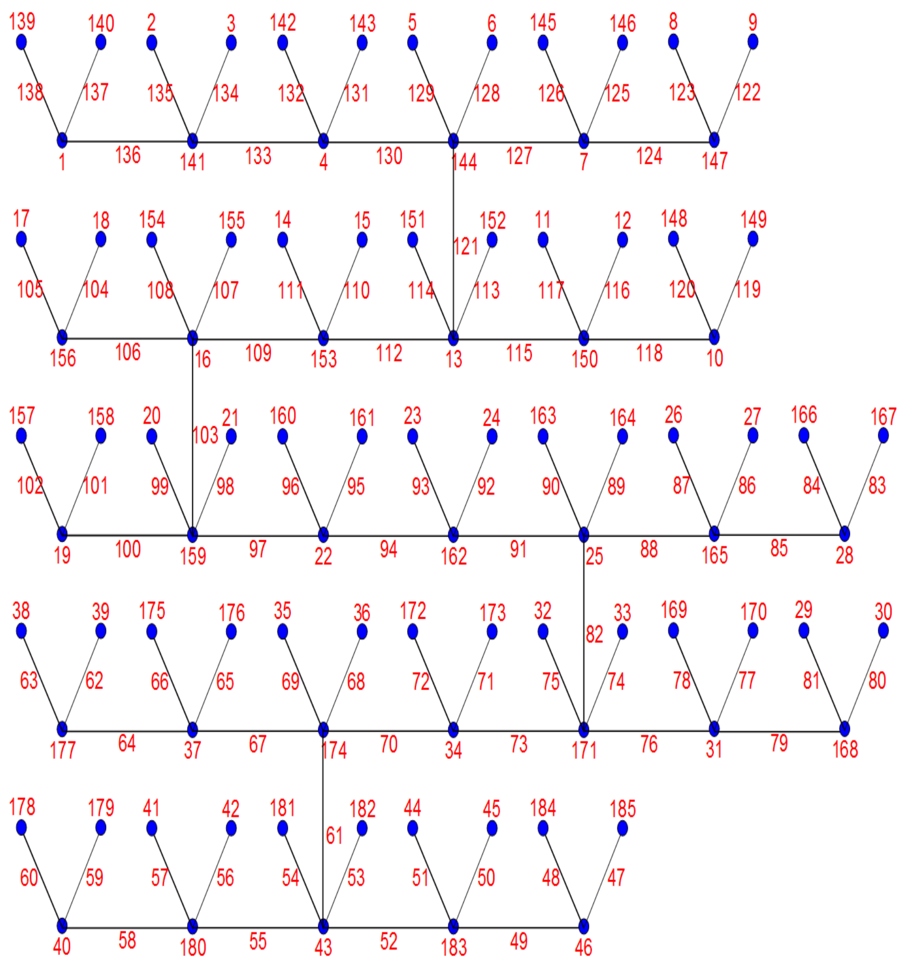

Figure 1. 47-ECEMTL of caterpillar-like, with magic constant 278.

Theorem 2.6. *Every regular caterpillar-like tree has a b-ECEMTL, where $0 < b < n$.*

Proof. Label the regular caterpillar-like tree using the label in Algorithm 2.

Algorithm 2.

1. Label the regular caterpillar with b-ECEMTL given in the proof of Theorem 3.
2. Remove all the labeled leaves from the vertices of the backbone path.
3. Embed the backbone path of the labeled caterpillar in the two-dimensional grid.
4. Do some elementary transformation on the backbone path by replacing the edge by a new edge
5. Put back all labeled leaves to the associated vertices of the path-like tree.

The elementary transformation in step (iv) keeps the b-ECEMTL property of the new graph. □

Corollary 2.1. *All regular path-like trees have a b-ECEMTL, where $0 < b < n$.*

2.3. Regular Banana Trees

In this subsection, we give new result on regular banana tree. A regular (k, r)-banana tree is a graph obtained by connecting one leaf of each of k copies of a star S_r graph with a single root vertex that is distinct from all the stars [2].

Theorem 2.7. *For $r \geq 3$, every regular banana tree $B(r, r)$ has an r^2-ECEMTL.*

Proof. Let a be the root vertex of the regular banana tree $B(r, r)$. Let $c_1, ..., c_r$ be the center of the star and v_i^j be the j-th vertex of i-th star, $i, j = 1, ..., r$. Let v_i^1 be a vertex which is adjacent to the root, for $i = 1, ..., r$.

Label the regular banana tree using the label in Algorithm 3.

Algorithm 3.

1. Set $b = r^2$
2. Label the root vertex with $f(a) = b + |E| + r + 1$
3. Label the center of the stars with $f(c_i) = b + |E| + i, \quad i = 1, ..., r$
4. Label the vertex v_i^1 with $f(v_i^1) = (i - 1)(r + 1) + 1$
5. Label the leaves from the first branch with $f(v_1^j) = j$ for $j = 2, ..., r$.
6. Label the leaves from the other branches as follows

$$f(v_i^j) = \begin{cases} (i-1)r + j & \text{for } j = 2, .., i-1, \\ (i-1)r + j + 1 & \text{for } j = i, ..., r. \end{cases}$$

The weight $f(u) + f(v)$ for every edge uv will form consecutive integer from $|E| + r^2 + 2$ to $|E| + 2r^2 + r + 1$. Then complete the label of edges with element of $\{r^2 + 1, ..., r^2 + |E|\}$ to obtain edge magic total labeling. □

3. Summary

In this paper, we give the construction of b-ECEMTL for several regular trees: regular caterpillars, regular firecrackers, regular caterpillar-like trees, regular path-like trees and regular banana trees. The research can continue to find the construction for general regular tree and non regular tree. We conclude this paper by giving a conjecture:

Conjecture 1. *All trees have a b-ECEMTL.*

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