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The complete short proof of the Berge conjecture

Ikorong Annouk

University Pierre Et Marie Curie (Paris Vi), France

ikorong@ccr.jussieu.fr

Abstract

We say that a graph B is *berge* (see [1]) if every graph $B' \in \{B, \bar{B}\}$ does not contain an induced cycle of odd length ≥ 5 [$\bar{B}$ is the complementary graph of B]. A graph G is *perfect* if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$, where $\chi(G')$ is the *chromatic number* of G' and $\omega(G')$ is the *clique number* of G'. The Berge conjecture states that a graph H is *perfect* if and only if H is *berge*. Indeed, the *Berge problem* (or the *difficult part of the Berge conjecture*) consists to show that $\chi(B) = \omega(B)$ for every *berge* graph B. In this paper, we give the direct short proof of the *Berge conjecture* by reducing the *Berge problem* into a simple equation of three unknowns and by using trivial complex calculus coupled with elementary computation and a trivial reformulation of that *problem* via the reasoning by reduction to absurd [we recall that the *Berge conjecture* was first proved by Chudnovsky, Robertson, Seymour and Thomas in a paper of at least 143 pages long (see [3]). That being said, the new proof given in this paper is far more easy and more short than the first given in [3]]. Our work in this paper is original and is completely different from all strong investigations made by Chudnovsky, Robertson, Seymour and Thomas in their manuscript of at least 143 pages long.

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Preliminary and Definitions
Recall that in a graph $G = [V(G), E(G), \chi(G), \omega(G), \alpha(G), \bar{G}]$, $V(G)$ is the set of vertices, $E(G)$ is the set of edges, $\chi(G)$ is the *chromatic number* i.e. the smallest number of colors needed to color all vertices of G such that two adjacent vertices do not receive the same color, $\omega(G)$ is the *clique number* of G i.e. the size of a largest *clique* of G. Recall that a graph F is a *subgraph* of G, if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. We say that a graph F is an *induced subgraph* of G by Z, if F is a subgraph of G such that $V(F) = Z$, $Z \subseteq V(G)$, and for any pair of vertices x and y of F (note that x and y are in $V(F) = Z$), xy is an edge of F if and only if xy is an edge of G. For $X \subseteq V(G)$, $G \setminus X$ denotes the *subgraph* of G *induced* by $V(G) \setminus X$. A *clique* of G is a subgraph of G that is *complete*; such a subgraph is necessarily an induced subgraph (recall that a graph K is complete if every pair of vertices of K is an edge of K), $\alpha(G)$ is the *stability number* of G i.e. the size of a largest *stable set* of G. Recall that a *stable set* of a graph G is a set of vertices of G that induces a subgraph with no edges, and $\bar{G}$ is the *complementary* graph of G recall $\bar{G}$ is the complementary graph of G, if $V(G) = V(\bar{G})$ and two vertices are adjacent in G if and only if they are not adjacent in $\bar{G}$. We say that a graph B is *berge*, if every $B' \in \{B, \bar{B}\}$ does not contain an induced cycle of odd length ≥ 5. A graph G is *perfect* if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$. The Berge conjecture states that a graph H is *perfect* if and only if H is *berge*. Indeed the *difficult part* of the Berge conjecture consists to show that $\chi(B) = \omega(B)$ for every *berge* graph B. Briefly, the *difficult part* of the Berge conjecture will be called the *Berge problem*. It is easy to see:

Assertion 0.0.*Let G be a graph and F be a subgraph of G. Then*

$$\omega(G) \leq \chi(G) \textit{ and } \chi(F) \leq \chi(G). \quad \square$$

It is known and it is not very difficult to prove that:
Assertion 0.1. *The Berge conjecture is true for every graph G such that* $0 \leq \chi(G) \leq 2$. □
Assertion 0.2. *The Berge problem is true for every graph G such that* $0 \leq \chi(G) \leq 2$.
Proof. Immediate and is an obvious consequence of Assertion 0.1.□

Assertion 0.3. *For every berge graph B such that* $0 \leq \chi(B) \leq 2$*, we have* $\omega(B) = \chi(B)$.
Proof. Immediate and is a trivial consequence of Assertion 0.2.□

That being so, this paper is divided into five simple sections. In Section **1** , we introduce a graph parameter denoted by β [β is called the *berge index*], and, using this graph parameter, we give the original reformulation of the *Berge problem*. In Section **2**, we use the original reformulation of the *Berge problem* to introduce *uniform* graphs and *relative* subgraphs [uniform graphs and relative subgraphs are crucial for the proof of the result which implies the Berge conjecture], and we give some elementary properties of these graphs; in Section **2**, we also define another graph parameter denoted by b [the graph parameter b is called the *berge caliber*, and is related to the berge index defined in Section **1**], and using the graph parameter b, we prove a trivial proposition which is equivalent to the *Berge problem*. We will use this proposition in Section **4** for the proof of the result which implies the Berge conjecture. In Section **3**, we prove elementary properties linked to isomorphisms, trivial complex calculus, elementary computation, and we reduce the *Berge problem* into a simple equation of three unknowns; we will use them in Section **4** [In Section **3**, we will let one Proposition unproved and we will prove this Proposition in Section **3'** (Epilogue)]. In Section **4**, using elementary properties proved in Section **3** and a trivial proposition of Section **2**, we prove a Theorem which immediately implies the Berge problem, and, at the same time, we also prove the Berge conjecture. In Section **3'** (Epilogue), we end this paper by proving the only Proposition we let unproved in Section **3**. In this paper, every graph is finite, is simple and undirected.

1. The Berge index of a graph and the original reformulation of the Berge problem

In this section, we introduce some important definitions that are not standard. In particular, we define a graph parameter called the *berge index* [and denoted by β], and we use it to give the original reformulation of the Berge problem.

Definition 1.0 (*true pal*). We say that a graph G is a *true pal* of a graph F, if F is a subgraph of G and $\chi(F) = \chi(G)$. *trpl(F)* denotes the set of all true pals of F; **so** $G \in trpl(F)$ **means** G is a true pal of F .

Definition 1.1 (*complete* $\omega(Q)$*-partite* graph and Ω). We recall that a graph Q is a *complete* $\omega(Q)$*-partite* graph, if there exists a partition $\Xi(Q) = \{Y_1, ..., Y_{\omega(Q)}\}$ of $V(Q)$ into $\omega(Q)$ stable set(s), such that $x \in Y_j \in \Xi(Q)$, $y \in Y_k \in \Xi(Q)$ and $j \neq k$, $\Rightarrow$ x and y are adjacent in Q. Ω denotes the set of all *complete* $\omega(Q)$*-partite graphs*; **so** $Q \in \Omega$ **means** Q is a complete $\omega(Q)$-partite graph. For example, if G is a complete $\omega(G)$-partite graph with $\omega(G) \in \{0, 1, 2, 3, 4, ..., etc..\}$, then $G \in \Omega$. *More generally*, G is a complete $\omega(G)$-partite graph with $\omega(G) \geq 0$, *if and only if* $G \in \Omega$. It is immediate that $\chi(Q) = \omega(Q)$ for all $Q \in \Omega$ (it is also immediate that for every $Q \in \Omega$, the partition $\Xi(Q) = \{Y_1, ..., Y_{\omega(Q)}\}$ of $V(Q)$ into $\omega(Q)$ stable set(s) is *canonical*).

Now, using the previous definitions, then the following Assertion becomes immediate.

Assertion 1.2 . *Let G be a graph. Then, there exists a graph $P \in \Omega$ such that P is a true pal of G [i.e. there exists $P \in \Omega$ such that $P \in trpl(G)$].*

Proof. Indeed let G be a graph and let $\Xi(G) = \{Y_1, ..., Y_{\chi(G)}\}$ be a partition of $V(G)$ into $\chi(G)$ stable set(s) (it is immediate that such a partition $\Xi(G)$ exists). Now let Q be a graph defined as follows: (i) $V(Q) = V(G)$, (ii) $\Xi(Q) = \{Y_1, ..., Y_{\chi(G)}\} = \Xi(G)$ is a partition of $V(Q)$ into $\chi(G)$ stable set(s) such that $x \in Y_j \in \Xi(Q)$, $y \in Y_k \in \Xi(Q)$ and $j \neq k$, $\Rightarrow$ x and y are adjacent in Q. Clearly $Q \in \Omega$, $\chi(Q) = \omega(Q) = \chi(G)$, and G is visibly a subgraph of Q; observe that Q is a true pal of G such that $Q \in \Omega$ (because G is a subgraph of Q and $\chi(Q) = \chi(G)$ and $Q \in \Omega$). Now put $Q = P$; Assertion 1.2 follows.□

Using Assertion 1.2, let us define.

Definition 1.3 (*parent*). We say that a graph P is a *parent* of a graph F, if $P \in \Omega \bigcap trpl(F)$. In other words, a graph P is a *parent* of F, if P is a complete $\omega(P)$-partite graph and P is also a true pal of F [note that such a P clearly exists, via Assertion 1.2]. *parent(F)* denotes the set of all parents of F; **so** $P \in parent(F)$ **means** P is a parent of F.

The following assertion is an immediate consequence of Definition 1.3 and Assertion 1.2.

Assertion 1.4. *Let G be a graph. Then, there exists a graph P which is a parent of G [i.e. there exists a graph P such that $P \in parent(G)$].*
Proof. Immediate [use Definition 1.3 and Assertion 1.2]. □

Using the definition of a parent [use Definition 1.3], the definition of a true pal [use Definition 1.0], the definition of a *berge graph* [use Preliminary and definitions] and the definition of Ω [use Definition 1.1], then the following two Assertions are immediate.

Assertion 1.5. *Let F be a graph and let $P \in parent(F)$; then $\chi(F) = \chi(P) = \omega(P)$.* □

Assertion 1.6. *Let $G \in \Omega$. Then, $\omega(G) = \chi(G)$ and G is berge.* □

Assertion 1.6 says that the set Ω is an obvious example of berge graphs. Curiously the set Ω will be fundamental for the proof of the Berge conjecture.

Now, we define the *berge index* and a *representative*.

Definition 1.7 *(The berge index and a representative).* If $G \in \Omega$, then the *berge index* of G is denoted by $\beta(G)$ where $\beta(G) = \min_{F \in \mathcal{B}(G)} \omega(F)$ and where $\mathcal{B}(G) = \{F; G \in parent(F)$ and $F\ is\ berge\}$ [$\mathcal{B}(G)$ is the set of graphs F such that G is a parent of F and F is berge]; and a *representative* of G is any graph $S \in \mathcal{B}(G)$ such that $\omega(S) = \beta(G)$.
If $G \notin \Omega$, then the *berge index* of G is denoted by $\beta(G)$ where $\beta(G) = \min_{P \in parent(G)} \beta(P)$ and where $parent(G)$ is the set of all parents of G [use Definition 1.3]; and a *representative* of G is any graph S such that $\omega(S) = \beta(G)$ and S is an element of any set $\mathcal{B}(P)$ such that $P \in parent(G)$ and $\beta(P) = \beta(G)$ [notice that $\mathcal{B}(P) = \{F; P \in parent(F)$ and $F\ is\ berge\}$] [we will prove that the *berge index* $\beta(G)$ exists and is well defined for every graph G and we will also prove that for every graph G, there exists at least one *representative* of G].

Using Definitions 1.7, let us Remark.

Remark 1.7′. *Let G be a graph. Then $\beta(G)$ exists, is well defined and there exists S such that S is a representative of G [such a S is berge and we have $\chi(S) = \chi(G)$].*

Proof.

(i) If $G \in \Omega$, then $\beta(G)$ exists, is well defined and there exists a graph S such that S is a representative of G [such a S is berge and we have $\chi(S) = \chi(G)$]. (Indeed let $\mathcal{B}(G)$ [recall that $\mathcal{B}(G) = \{F; G \in parent(F)$ and $F\ is\ berge\}$], since $G \in \Omega$, then G is berge [use

Assertion 1.6]; so $G \in \mathcal{B}(G)$ and clearly

$$\min_{F\in\mathcal{B}(G)} \omega(F) \ exists\, and\, there\, exists\, S \in \mathcal{B}(G)\, such\, that\, \omega(S) = \min_{F\in\mathcal{B}(G)} \omega(F). \tag{1.1}$$

Using *(1.1)* and the meaning of ($\beta(G), \mathcal{B}(G)$), then we immediately deduce that

$$\beta(G) \text{ exists, is well defined, } S \text{ is a representative of } G \text{ and } S \text{ is berge and } \chi(S) = \chi(G). \tag{1.2}$$

(ii) If $G \notin \Omega$, then $\beta(G)$ exists, is well defined and there exists a graph S such that S is a representative of G [such a S is berge and we have $\chi(S) = \chi(G)$]. (Indeed let P be a parent of G [such a P exists by using Assertion 1.4]; since in particular $P \in \Omega$ [because P is a parent of G] , then using **(i)**, we immediately deduce that $\beta(P)$ exists and therefore

$$\min_{P\in parent(G)} \beta(P) \text{ also exists.} \tag{1.3}$$

(1.3) clearly says that

$$\beta(G) = \min_{P\in parent(G)} \beta(P) \text{ exists and is well defined.} \tag{1.4}$$

Now let $Q' \in parent(G)$ such that $\beta(Q') = \beta(G)$ [such a Q' exists, via (1.4)] and let $S \in \mathcal{B}(Q')$ such that $\omega(S) = \beta(G)$ [we recall that $\mathcal{B}(Q') = \{F; Q' \in parent(F)$ and $F\ is\ berge\}$]; then using the previous, it becomes immediate to deduce that [2]

$$S \text{ is berge and is a representative of } G \text{ and } \chi(S) = \chi(Q') = \chi(G). \tag{1.5}$$

So

$$\beta(G) \text{ exists, is well defined and } S \text{ is a representative of } G \text{ and } S \text{ is berge and}$$

$$\chi(S) = \chi(G) \tag{1.6}$$

[use (1.4) and (1.5)]). Remark $1.7'$ follows (use **(i)** and **(ii)**). □

Remark $1.7''$. *Let* (K, G, B, P), *where* K *is a complete graph,* $G \in \Omega$, B *is berge and* $P \in parent(B)$. *We have the following elementary properties.*
($1.7''.1$). *If* $\omega(G) \leq 1$, *then* $\omega(G) = \chi(G) = \beta(G)$.
($1.7''.2$). $\omega(K) = \chi(K) = \beta(K)$.
($1.7''.3$). $\omega(G) \geq \beta(G)$.
($1.7''.4$). $\beta(P) \leq \omega(B)$. (*Proof.* Property ($1.7''.1$) is immediate.
($1.7''.2$). Indeed let $\mathcal{B}(K) = \{F; K \in parent(F)$ and $F\ is\ berge\}$, since K is a complete graph, clearly K is berge and $\mathcal{B}(K) = \{K\}$; observe that $K \in \Omega$ and so $\beta(K) = \min_{F\in\mathcal{B}(K)} \omega(F)$ [use the definition of the parameter β and observe that $K \in \Omega$], and using the preceding, we easily deduce that $\beta(K) = \omega(K) = \chi(K)$. Property ($1.7''.2$) follows.
($1.7''.3$). Indeed let $\mathcal{B}(G) = \{F; G \in parent(F)$ and $F\ is\ berge\}$; since G is a complete graph,

clearly $\beta(G) = \min_{F \in \mathcal{B}(G)} \omega(F)$ [use the definition of the parameter β and observe that $G \in \Omega$]; observe that G is berge [use Assertion 1.6], so $G \in \mathcal{B}(G)$ and the previous equality implies that $\omega(G) \geq \beta(G)$.Property $(1.7''.3)$ follows.
$(1.7''.4)$.Indeed let $\mathcal{B}(P) = \{F; P \in parent(F)$ and $F\ is\ berge\}$; clearly $B \in \mathcal{B}(P)$; observe that $P \in \Omega$, so $\beta(P) = \min_{F \in \mathcal{B}(P)} \omega(F)$ [use the definition of the parameter β and observe that $P \in \Omega$], and using the preceding, we easily deduce that $\beta(P) \leq \omega(B)$.Property $(1.7''.4)$ follows. and Remark $1.7''$ immediately follows.□)

Now the following Theorem is the original reformulation of the Berge problem.

Theorem 1.8. (The original reformulation of the Berge problem). *The following are equivalent.*
(1) The Berge problem is true [i.e. For every berge graph B, we have $\chi(B) = \omega(B)$]
(2) For every graph F, we have $\chi(F) = \beta(F)$.
(3) For every $G \in \Omega$, we have $\omega(G) = \beta(G)$.
Proof. $(1) \Rightarrow (2)$. Indeed let F be graph and let S be a representative of F, in particular S is berge [use Remark $1.7'$] and clearly $\chi(S) = \omega(S)$; now observing that $\omega(S) = \beta(F)$ [since S is a representative of F], then the previous two equalities imply that $\chi(S) = \beta(F)$. Observe that $\chi(S) = \chi(F)$ [by observing that S is a representative of F and by using Remark $1.7'$] and the last two equalities immediately become $\chi(F) = \beta(F)$.
$(2) \Rightarrow (3)$]. Let $G \in \Omega$; in particular G is a graph and so $\chi(G) = \beta(G)$. Note $\chi(G) = \omega(G)$ (since $G \in \Omega$) and the previous two equalities imply that $\omega(G) = \beta(G)$.
$(3) \Rightarrow (1)$]. Let B be a berge and let $P \in parent(B)$; property $(1.7''.4)$ of Remark $1.7''$ implies that

$$\beta(P) \leq \omega(B). \tag{1.7}$$

Note

$$\beta(P) = \omega(P) \tag{1.8}$$

[since $P \in \Omega$] and clearly

$$\omega(P) \leq \omega(B) \tag{1.9}$$

[use (1.7) and (1.8)]. It is immediate that $\chi(B) = \chi(P) = \omega(P)$ [since $P \in parent(B)$] and inequality *(1.9)* becomes $\chi(B) \leq \omega(B)$. Observe that $\chi(B) \geq \omega(B)$ and the previous two inequalities imply that $\chi(B) = \omega(B)$. □

We will use Theorem 1.8 in Section **2** to define uniform graphs which are crucial for the proof of the Berge conjecture. Now using the definition of the Berge problem and the definition of the berge index β and Assertion 0.2, then the following assertion becomes trivial and we leave it to the reader.
Assertion 1.9. *We have the following three properties.*
(1). The Berge problem is true for every graph G' such that $0 \leq \chi(G') \leq 2$ [i.e. For every berge graph B such that $0 \leq \chi(B) \leq 2$, we have $\chi(B) = \omega(B)$].
(2) For every graph F such that $0 \leq \chi(F) \leq 2$, we have $\chi(F) = \beta(F)$.
(3) For every $G \in \Omega$ such that $0 \leq \chi(G) \leq 2$, we have $\omega(G) = \beta(G)$. □

2. Uniform graphs, relative subgraphs, the berge caliber and some consequences

In this section, we use the original reformulation of the *Berge problem* given by Theorem 1.8 to introduce *uniform* graphs and *relative* subgraphs [uniform graphs and relative subgraphs are crucial for the proof of the result which implies the Berge conjecture], and we give some elementary properties of these graphs. In this section, we also define another graph parameter denoted by b [the graph parameter b is called the *berge caliber*, and is related to the berge index defined in Section **1**], and using the graph parameter b, we prove simple propositions that we will use in Section **4** to give the short proof of the *Berge problem* [in particular, we prove a trivial proposition which is equivalent to the *Berge problem*]. In this section, the definition of a *berge graph* [use *Preliminary and definitions*], the definition of a *true pal* [use Definition 1.0], the denotation of Ω [use Definition 1.1], the definition of a *parent* [use Definition 1.3], and the definition of the *berge index* β [use Definitions 1.7], are fundamental and crucial. Now let us remark.

Remark 2.0. *Let B be berge and let P be a parent of B; then $\beta(P) \leq \omega(B)$.*
Proof. Immediate and is an obvious consequence of property $(1.7''.4)$ of Remark $1.7''$.□

Remark 2.1. *Let K be a complete graph; then $\beta(K) = \omega(K) = \chi(K)$.*
Proof. Immediate and is an obvious consequence of property $(1.7''.2)$ of Remark $1.7''$. □

From Theorem 1.8, we are going to define a new class of graphs in Ω [called *uniform graphs*]; we will also define *relative subgraphs*, and we will present some properties related to these graphs. These properties are elementary and curiously, are crucial for the result which immediately implies the Berge problem and the Berge conjecture. Before, let us define.

Definition 2.2 [*optimal coloration* and $\ominus(G)$]. An *optimal coloration* of a graph G is a partition $\Xi(G) = \{Y_1, ..., Y_{\chi(G)}\}$ of $V(G)$ into $\chi(G)$ stable set(s) [where $\chi(G)$ is the chromatic number of G]; $\ominus(G)$ denotes the set of all optimal colorations of G; so, $\Xi(G) \in \ominus(G)$ means $\Xi(G)$ is an optimal coloration of G.

Definition 2.3 (the *canonical coloration*). Let G be a graph and let $\Xi(G) \in \ominus(G)$ [use Definition 2.2]. We say that $\Xi(G)$ is the *canonical coloration* of G, if and only if, $\ominus(G) = \{\Xi(G)\}$ [observe that such a canonical coloration does not always exist].

Using the denotation of $\ominus(G)$ [use Definition 2.2], then the following Assertion is immediate.

Assertion 2.4. *Let $G \in \Omega$ and let $\Xi(G) \in \ominus(G)$. Then $\Xi(G)$ is the canonical coloration of G [i.e. $\ominus(G) = \{\Xi(G)\}$, by Definition 2.3].*
Proof. Immediate, by observing that $G \in \Omega$. □

So, let $G \in \Omega$ and let $\Xi(G) \in \ominus(G)$; then Assertion 2.4 clearly says that $\Xi(G)$ is the canonical coloration of G [indeed, we have no choice, since $\ominus(G) = \{\Xi(G)\}$].

Definition 2.5 *(Uniform graph)* Let $G \in \Omega$ and let $\Xi(G)$ be the canonical coloration of G [observe that $\Xi(G)$ exists, via Assertion 2.4]; we say that G is *uniform*, if for every $Y \in \Xi(G)$, we have $card(Y) = \alpha(G)$, where $card(Y)$ is the cardinality of Y and where $\alpha(G)$ is the *stability number* of G.

Definition 2.5 gets sense, since $G \in \Omega$ and so $\Xi(G)$ is canonical [via Assertion 2.4]. Using the definition of a uniform graph [use Definition 2.5], then the following Assertion is immediate.

Assertion 2.6. *Let $G \in \Omega$ and let $\Xi(G)$ be the canonical coloration of G [observe that $\Xi(G)$ exists, via Assertion 2.4]. We have the following trivial properties.*
(2.6.0). If $0 \leq \omega(G) \leq 1$, then G is uniform.
(2.6.1). If $0 \leq \alpha(G) \leq 1$, then G is uniform.
(2.6.2). If G is a complete graph, then G is uniform.
(2.6.3). If $\alpha(G) \geq 2$ and if for every $Y \in \Xi(G)$ we have $card(Y) = \alpha(G)$, then G is uniform and is not a complete graph.
Proof. Properties (2.6.0) and (2.6.1) are trivial [it suffices to observe that $G \in \Omega$ and to use Definition 2.5]; property (2.6.2) is an immediate consequence of property (2.6.1); and property (2.6.3) is trivial (indeed, observe [by the hypotheses] that $G \in \Omega$ and use Definition 2.5).□

Uniform graphs have nice properties when we study isomorphism of graphs.
Recall 2.7. Recall that two graphs are *isomorphic* if there exists a one to one correspondence between their vertex set that preserves adjacency.

Assertion 2.8. *Let $G \in \Omega$; then there exists a uniform graph U which is isomorphic to a parent of G [use Definition 1.3 for the meaning of parent].*
Proof. If $0 \leq \omega(G) \leq 1$, clearly G is uniform [use property (2.6.0) of Assertion 2.6]; now put $U = G$, clearly U is a uniform graph which is a parent of G. Now, if $\omega(G) \geq 2$, let $\Xi(G)$ be the canonical coloration of G [observe that $\Xi(G)$ exists, by remarking that $G \in \Omega$ and by using Assertion 2.4]; since it is immediate that $\chi(G) = \omega(G)$, clearly $\Xi(G)$ is of the form $\Xi(G) = \{Y_1, ..., Y_{\omega(G)}\}$. Now let Q be a graph defined as follows: (i) $\Xi(Q) = \{Z_1, ..., Z_{\omega(G)}\}$ is a partition of $V(Q)$ into $\omega(G)$ stable sets such that, $x \in Z_j \in \Xi(Q)$, $y \in Z_k \in \Xi(Q)$ and $j \neq k$, $\Rightarrow$ x and y are adjacent in Q; (ii) For every $j = 1, 2, ..., \omega(G)$ and for every $Z_j \in \Xi(Q) = \{Z_1, ..., Z_{\omega(G)}\}$, $card(Z_j) = \alpha(G)$. Clearly $Q \in \Omega$, $card(V(Q)) = \omega(G)\alpha(G)$, Q is uniform, $\chi(Q) = \omega(Q) = \omega(G) = \chi(G)$, and visibly, G is isomorphic to a subgraph of Q; observe that Q is isomorphic to a true pal of G and $Q \in \Omega$ [since G is isomorphic to a subgraph of Q and $\chi(Q) = \omega(Q) = \omega(G) = \chi(G)$ and $Q \in \Omega$] and Q is uniform. Using the previous and the definition of a parent, then we immediately deduce that Q is a uniform graph which is isomorphic to a parent of G. Now put $Q = U$; clearly U is a uniform graph which is isomorphic to a parent of G. Assertion 2.8 follows.□

Now the following assertion is only an immediate consequence of Assertion 2.8.
Assertion 2.9. *Let H be a graph [H is not necessarily in Ω]; then there exists a uniform graph U which is isomorphic to a parent of H.*
Proof. Let P be a parent of H [such a P exists via Assertion 1.4] and let U be a uniform graph such that U is isomorphic to a parent of P [such a U exists, by observing that $P \in \Omega$ and by

using Assertion 2.8]; clearly U is a uniform graph and is isomorphic to a parent of H [since U is a uniform graph which is isomorphic to a parent of P and P is a parent of H] .□

Now we define *relative* subgraphs.

Definition 2.10 *(relative subgraph).* Let G and F be uniform. Now let $\Xi(G)$ be the canonical coloration of G and let $\Xi(F)$ be the canonical coloration of F [observe that the couple $(\Xi(G), \Xi(F))$ exists, by remarking that $G \in \Omega$ and $F \in \Omega$, and by using Assertion 2.4]. We say that F is a *relative subgraph* of G, if $\Xi(F) \subseteq \Xi(G)$ (it is immediate that the previous gets sense, since in particular $(G, F) \in \Omega \times \Omega$ [because G and F are uniform], and so $\Xi(G)$ and $\Xi(F)$ are canonical [by using Assertion 2.4 and Definition 2.3]. It is also immediate that relative subgraphs are defined for uniform graphs, and only for uniform graphs).

Using the definition of a relative subgraph [use Definition 2.10] and the definition of a uniform graph [use Definition 2.5], then the following assertion is immediate and will help us later.

Assertion 2.11. *Let (P, U) be a couple of uniform graphs such that $\omega(P) \geq 1$ and $\omega(U) \geq 1$. Now let $\Xi(P)$ be the canonical coloration of P and let $\Xi(U)$ be the canonical coloration of U [observe that the couple $(\Xi(P), \Xi(U))$ exists, by remarking that $P \in \Omega$ and $U \in \Omega$, and by using Assertion 2.4]. Then we have the following trivial properties.*
(2.11.0). If U is a relative subgraph of P, then $\alpha(U) = \alpha(P)$ and $\omega(U) \leq \omega(P)$.
(2.11.1). If U is a relative subgraph of P and if $\omega(U) = \omega(P)$, then $U = P$.
(2.11.2). If U is a relative subgraph of P and if $\omega(U) < \omega(P)$, then there exists $Y \in \Xi(P)$ such that U is a relative subgraph of $P \setminus Y$.
(2.11.3). If $\alpha(U) = \alpha(P)$ and $\omega(U) = \omega(P)$, then U and P are isomorphic.
(2.11.4). If $\omega(P) \geq 2$, then, for every $Y \in \Xi(P)$, $P \setminus Y$ is a relative subgraph of P and $P \setminus Y$ is uniform and $\omega(P \setminus Y) = \omega(P) - 1$ and $\alpha(P \setminus Y) = \alpha(P)$.

Proof. Properties (2.11.0) and (2.11.1) and (2.11.2) are immediate [it suffices to use the definition of a uniform graph and the definition of a relative subgraph]. Properties (2.11.3) and (2.11.4) are trivial consequences of the definition of uniform graphs and relative subgraphs. □

We will see in Section **4** that uniform graphs and relative subgraphs play a major role in the proof of Theorem which immediately implies the Berge problem. Now we introduce again definitions that are not standard; in particular, we introduce a graph parameter denoted by b and called the *berge caliber* [the *berge caliber* b is related to the *berge index* β introduced in Definitions 1.7 (Section **1**)], and using the parameter b on uniform graphs, we prove an elementary proposition which will help us in Section **4** to prove the Berge problem and the Berge conjecture. Before, let us define.

Definition 2.12 (*Fundamental*) We say that a graph G is *bergerian*, if G is uniform and if $\omega(G) = \beta(G)$ [use Definitions 1.7 for the meaning of $\beta(G)$ and Definition 2.5 for the meaning of uniform].

The following two assertions are obvious consequences of Remark 2.1 and Definition 2.12.

Assertion 2.13. *Let K be a complete graph; then K is bergerian.*
Proof. Immediate, and is a consequence of Remark 2.1 and Definition 2.12 and property (2.6.2) of Assertion 2.6.□

Assertion 2.14. *The set of all complete graphs is an obvious example of bergerian graphs.*
Proof. Immediate, and is a trivial consequence of Assertion 2.13.□

Definitions 2.15 (*bergerian subgraph* and *maximal bergerian subgraph*). Let G be uniform. We say that a graph F is a *bergerian subgraph* of G, if F is *bergerian* and is a relative subgraph of G [use Definition 2.10 for the meaning of a relative subgraph and Definition 2.12 for the meaning of bergerian]. We say that F is a *maximal bergerian* subgraph of G [we recall that G is uniform], if F is a *bergerian subgraph* of G and $\omega(F)$ is **maximum** for this property [it is immediate that such a F exists and is well defined].

Now we define the *berge caliber*.
Definition 2.16 (*berge caliber*). Let G be uniform, and let F be a **maximal bergerian** subgraph of G [use Definitions 2.15], then the *berge caliber* of G is denoted by $b(G)$, where $b(G) = \omega(F)$.

The following remark clearly shows that for every uniform graph G, $b(G)$ exists and is well defined.

Remark 2.17.*For every uniform graph G, the berge caliber $b(G)$ exists and is well defined.*
Proof. Let G be uniform and let F be a *maximal bergerian subgraph* of G [use Definitions 2.15]; observing [by definition of a *maximal bergerian subgraph* of G] that F is a *bergerian subgraph* of G and $\omega(F)$ is **maximum** for this property, clearly $\omega(F)$ is unique and therefore $b(G)$ is also unique, since $b(G) = \omega(F)$. So $b(G)$ exists and is well defined. □

It is immediate that the berge caliber [i.e. the graph parameter b] is defined for *uniform graphs* and only for *uniform graphs*. We will see in Section **4** that the berge caliber plays a crucial role in the proof of Theorem which immediately implies the Berge problem. Now, using the definition of a uniform graph, the definition of a relative subgraph and the definition of the berge caliber, then the following assertion is immediate.

Assertion 2.18. *Let G be uniform and let $b(G)$ be the berge caliber of G. Consider $\beta(G)$ [$\beta(G)$ is the berge index of G (use Definitions 1.7)]. We have the following six properties.*
*(*2.18.0*)* $\omega(G) \geq b(G)$.
*(*2.18.1*)* *G is bergerian* $\Leftrightarrow \beta(G) = \omega(G) = b(G) \Leftrightarrow \beta(G) = \omega(G) \Leftrightarrow b(G) = \omega(G)$.
*(*2.18.2*)* *G is not bergerian* $\Leftrightarrow \omega(G) > b(G) \Leftrightarrow \omega(G) \neq \beta(G)$.
*(*2.18.3*)* *If* $\omega(G) \in \{0, 1, 2\}$, *then* $b(G) = \omega(G) = \beta(G)$ *[i.e. G is bergerian].*
*(*2.18.4*)* *If* $\omega(G) \geq j$ *[where* $j \in \{0, 1, 2\}$*], then* $b(G) \geq j$.
*(*2.18.5*)* *For every relative subgraph R of G, we have* $b(R) \leq b(G)$.

Proof. Property (2.18.0) is immediate [use the definition of $b(G)$]; properties (2.18.1) and (2.18.2) are trivial [use the definitions of $b(G)$ and $\beta(G)$]. Property (2.18.3) is easy (indeed, let G be uniform such that $\omega(G) = j$ where $j \in \{0, 1, 2\}$, clearly $\chi(G) = j$ where $j \in \{0, 1, 2\}$; observe that $\omega(G) = \beta(G)$ [use the previous and property (3) of Assertion 1.9]. Now using the previous equality and property (2.18.1), then it becomes trivial to deduce that $b(G) = \omega(G) = \beta(G)$ and G is bergerian). Property (2.18.4) is an immediate consequence of property (2.18.3) and property (2.18.5) immediately results via the definition of a relative subgraph [use Definition 2.10] and the definition of the parameter b [use Definition 2.16]. □

The previous definitions and simple properties made, now the following trivial proposition is crucial for the proof of the Berge problem and the Berge conjecture.

Proposition 2.19 (*The trivial reformulation of the Berge problem*). *The following are equivalent.*
(i) For every uniform graph U, we have $\omega(U) = b(U)$.
(ii) The Berge problem is true [i.e. For every berge graph B', we have $\chi(B') = \omega(B')$].
Proof. $(i) \Rightarrow (ii)$]. Indeed, observe [by the hypotheses] that for every uniform graph U, we have $\omega(U) = b(U)$; now using the previous and property (2.18.1) of Assertion 2.18, then it becomes trivial to deduce that

$$\text{for every uniform graph } U, \text{ we have } \omega(U) = b(U) = \beta(U). \tag{2.1}$$

Now let B be berge and let P be uniform such that P is isomorphic to a parent of B [such a P clearly exists, via Assertion 2.9], clearly

$$\beta(P) \leq \omega(B) \tag{2.2}$$

[by observing that in particular P is isomorphic to a parent of B and by using Remark 2.0]. It trivial that

$$\omega(B) \leq \chi(B). \tag{2.3}$$

Since in particular P is isomorphic to a parent of B, clearly P is isomorphic to a true pal of B and so

$$\chi(P) = \chi(B). \tag{2.4}$$

Clearly $\omega(P) = \chi(P)$ [since $P \in \Omega$], and using the previous equality, then it becomes trivial to deduce that equality (2.4) clearly says that

$$\omega(P) = \chi(B). \tag{2.5}$$

Recalling that P is uniform and using (2.1), then it becomes trivial to deduce that

$$\omega(P) = b(P) = \beta(P). \tag{2.6}$$

Now using (2.2) and (2.3) and (2.5) and (2.6), then it becomes trivial to deduce that

$$\beta(P) \leq \omega(B) \leq \chi(B) \leq \omega(P) \leq b(P) \leq \beta(P). \tag{2.7}$$

(2.7) immediately implies that

$$\beta(P) = \omega(B) = \chi(B) = \omega(P) = b(P). \tag{2.8}$$

Clearly $\omega(B) = \chi(B)$ [use (2.8)], and the previous equality clearly says that the Berge problem is true for B; using the previous and observing that the berge graph B was arbitrary chosen, then it becomes trivial to deduce that every berge graph B' satisfies $\omega(B') = \chi(B')$; so the Berge problem is true and therefore $(i) \Rightarrow (ii)$.
$(ii) \Rightarrow (i)$. Immediate (indeed, if the Berge problem is true [i.e. if for every berge graph B', we have $\chi(B') = \omega(B')$], then, using Theorem 1.8, we immediately deduce that

$$for\ every\ G \in \Omega,\ we\ have\ \omega(G) = \beta(G). \tag{2.9}$$

Now let U be uniform; observing that $U \in \Omega$ and using (2.9), then we immediately deduce that

$$\text{for every uniform graph } U, \text{ we have } \omega(U) = \beta(U). \tag{2.10}$$

Now using (2.10) and property (2.18.1) of Proposition (2.18), then it becomes trivial to deduce that for every uniform graph U, we have $\omega(U) = b(U)$. So $(ii) \Rightarrow (i)$) Proposition 2.19 follows. □

In Section **4**, Proposition 2.19 coupled with trivial complex calculus and elementary computation we help us to give the short proof of the Berge problem and the Berge conjecture.

3. Isomorphism of uniform graphs, properties linked to trivial complex calculus on uniform graphs, elementary computation, and the reduction of the Berge problem into a simple equation of three unknowns

In this section, we prove elementary properties linked to trivial complex calculus on uniform graphs and elementary computation; and we reduce the *Berge problem* into a simple equation of three unknowns. We will use them in Section **4**, where we will give the short complete simple proof of the Berge problem and the Berge conjecture [In this Section, we let one Proposition unproved and we will prove this Proposition in Section **3'** (Epilogue)]. In this section, the definition of a *uniform* graph [use Definition 2.5], the definition of a *relative subgraph* [use Definition 2.10], the definition of the *berge caliber* $b(G)$ of a uniform graph G [use Definition 2.16], are crucial. Before, let us remark that uniform graphs and relative subgraphs have interesting properties when we study isomorphism of graphs. Indeed, using the definition of uniform graphs and the definition of relative subgraphs and the definition of isomorphism [use Remark 2.7], then the following Remark becomes immediate:

Remark 3.0. *Let U and U' be uniform graphs such that $\omega(U) \geq 2$ and $\omega(U') \geq 1$. Now let $\Xi(U)$ be the canonical coloration of U [observe that $\Xi(U)$ exists, by remarking that $U \in \Omega$ (since U is uniform) and by using Assertion 2.4], and let $(Y, Y') \in \Xi(U) \times \Xi(U)$; look at the couple ($U \setminus Y$, $U \setminus Y'$) [recall that $U \setminus Z$ is the induced subgraph of U by $V(U) \setminus Z$]. Then we have the following four simple properties.*

(i) U and U' are isomporphic, if and only if, $\omega(U) = \omega(U')$ and $\alpha(U) = \alpha(U')$.

(ii) *U' is isomorphic to a relative subgraph of U if and only if $\omega(U) \geq \omega(U')$ and $\alpha(U) = \alpha(U')$.*
(iii) *$U \setminus Y$ is uniform and $\omega(U \setminus Y) = \omega(U) - 1$ and $\alpha(U \setminus Y) = \alpha(U)$ and $\omega(U \setminus Y) \geq 1$ and $U \setminus Y$ is a relative subgraph of U.*
(iv) *$U \setminus Y$ is isomorphic to $U \setminus Y'$.*

Proof. Property (i) is immediate [indeed, observe that U and U' are uniform graphs and use the definition of uniform graphs]. Property (ii) is trivial [indeed, observe that U and U' are uniform graphs and use the definition of uniform graphs and the definition of relative subgraphs]. Property (iii) is immediate [indeed since $Y \in \Xi(U)$, then it becomes trivial to see that $U \setminus Y$ is a relative subgraph of U and $U \setminus Y$ is uniform and $\omega(U \setminus Y) = \omega(U) - 1$ and $\alpha(U \setminus Y) = \alpha(U)$ and $\omega(U \setminus Y) \geq 1$]. Property (iv) is very easy [indeed, it is trivial that $U \setminus Y$ and $U \setminus Y'$ are uniform such that $\omega(U \setminus Y) \geq 1$ and $\omega(U \setminus Y) = \omega(U \setminus Y')$ and $\alpha(U \setminus Y) = \alpha(U \setminus Y')$; so $U \setminus Y$ is isomorphic to $U \setminus Y'$, by using the previous and property (2.11.3) of Assertion 2.11]. □

Proposition 3.1. *Let U be uniform such that $\omega(U) \geq 2$ and let $\Xi(U)$ be the canonical coloration of U [observe that $\Xi(U)$ exists, by remarking that $U \in \Omega$ (since U is uniform) and by using Assertion 2.4]; now let $Y \in \Xi(U)$ and look at the couple $(\, b(U), b(U \setminus Y)\,)$, where $b(U)$ is the berge caliber of U and $b(U \setminus Y)$ the berge caliber of $U \setminus Y$ [recall that $U \setminus Y$ is the induced subgraph of U by $V(U) \setminus Y$]. If $\omega(U) > b(U)$, then $b(U \setminus Y) = b(U)$.*

Proof. Indeed, let F' be a maximal bergerian subgraph of U; since $\omega(U) > b(U)$ [by the hypotheses], then, using the previous inequality coupled with the definition of F' [note that $\omega(F') = b(U)$, via the definition of F'], it becomes trivial to deduce that

$$there\ exists\ Y' \in \Xi(U)\ such\ that\ Y' \notin \Xi(F'), \tag{3.0}$$

where $\Xi(F')$ is the canonical coloration of F'. (3.0) immediately implies that

$$F'\ is\ also\ a\ maximal\ bergerian\ subgraph\ of\ U \setminus Y'. \tag{3.1}$$

It is trivial that (3.1) clearly says that

$$\omega(F') = b(U \setminus Y'). \tag{3.2}$$

Recalling that F' is a maximal bergerian subgraph of U [and so $\omega(F') = b(U)$] and using equality (3.2), then it becomes trivial to deduce that

$$b(U) = b(U \setminus Y'). \tag{3.3}$$

Now let $Y \in \Xi(U)$; observing that $U \setminus Y$ is isomorphic to $U \setminus Y'$ [use property (iv) of Remark 3.0], and using equality (3.3), then it becomes trivial to deduce that $b(U) = b(U \setminus Y)$. Proposition 3.1 follows. □

Proposition 3.1 will help us to simplify complex calculus on uniform graphs.

Recalls and Definitions 3.2 **(Fundamental).** (*Real numbers*, *complex numbers*, $\mathcal{C}$, $\mathcal{L}_U$ and *tackle*). Recall that $\mathcal{R}$ is the set all *real numbers* and θ is a *complex number* if $\theta = x + iy$, where $x \in \mathcal{R}$, $y \in \mathcal{R}$ and $i^2 = -1$; $\mathcal{C}$ is the set of all *complex numbers*. That being said, clearly

$$\mathcal{C}^2 = \{(x, y);\ x \in \mathcal{C}\ and\ y \in \mathcal{C}\},\ \ \mathcal{C}^2 \times \mathcal{R} = \{(x, y, k); (x, y) \in \mathcal{C}^2\ and\ k \in \mathcal{R}\}$$

and

$$\mathcal{C}^3 = \{(x, y, z); (x, y) \in \mathcal{C}^2\ and\ z \in \mathcal{C}\}.$$

Now let n be an integer ≥ 1 and let U be uniform such that $\omega(U) = n$; look at the berge caliber $b(U)$ [use Definition 2.16 for $b(U)$] and denote $\mathcal{L}_U = 2b(U)+2$; we say that $(\phi(U), \nu(U), \epsilon(U)) \in \mathcal{C}^3$ *tackles* $(1, 3i\mathcal{L}_U^{-1})$ *around* $6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$, if there exists $(x, y, k) \in \mathcal{C}^2 \times \mathcal{R}$ such that

$$x + 3iy\mathcal{L}_U^{-1} + k(6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i) + \phi(U) = 0$$

$$\text{where } x + y - \nu(U) = 0 \text{ and } x - y - \epsilon(U) = 0.$$

We will see that the definition of tackle introduced above helps to reduce the Berge problem into an elementary equation of three unknowns. Before, let us define:

Definitions 3.3 **(Fundamental).** Let n be an integer ≥ 1 and let U be a uniform graph such that $\omega(U) = n$; look at $\mathcal{L}_U$ [use Recalls and Definitions 3.2]. Now let $(\phi(U), \nu(U), \epsilon(U))$, where

$$\phi(U) = \sum_{j=1}^{4} \phi(U.j) \tag{3.4}$$

and where

$$\begin{aligned} \phi(U.1) &= \frac{1}{1331} \big(126301i\mathcal{L}_U^{-3} + 2273418i\mathcal{L}_U^{-4} - 23837\mathcal{L}_U - 357566\mathcal{L}_U^{-1} + \\ &\quad 2010800\mathcal{L}_U^{-2} + 2399719\mathcal{L}_U^{-4} - 757806\mathcal{L}_U^{-3}\big), \end{aligned} \tag{3.5}$$

$$\begin{aligned} \phi(U.2) &= \big((2n+1)^2 - 1 - \mathcal{L}_U^2 - 2\mathcal{L}_U)(16\mathcal{L}_U^{-3} + 50\mathcal{L}_U^{-2} + 11i\mathcal{L}_U^{-3} - 13i + 5i\mathcal{L}_U\big) + \\ &\quad \frac{(\mathcal{L}_U - 2n)}{2}(8\mathcal{L}_U - 8i - 16), \end{aligned} \tag{3.6}$$

$$\phi(U.3) = ((2n+1)^2 - 1 - \mathcal{L}_U^2 + 2\mathcal{L}_U)(34i\mathcal{L}_U^{-2} - 70i\mathcal{L}_U^{-3} - 11i + 5 + 6i\mathcal{L}_U) + \frac{289080}{1331}, \tag{3.7}$$

and

$$\phi(U.4) = ((2n+1)^2 - 1 - \mathcal{L}_U^2 + 7\mathcal{L}_U)(11i\mathcal{L}_U + 7 - 50\mathcal{L}_U^{-1} + 23i\mathcal{L}_U^{-3} - 54\mathcal{L}_U^{-3}) - \frac{1003112i}{1331}; \tag{3.8}$$

$$\nu(U) = (\,i\mathcal{L}_U - 4in - 4i + 1\,)^2 - 1 + \mathcal{L}_U^2, \tag{3.9}$$

and

$$\epsilon(U) = 4((2n+1)^2 - 1 - \mathcal{L}_U^2 + \mathcal{L}_U) + (\,i\mathcal{L}_U - 2in + 1\,)(4i\mathcal{L}_U + 4 - 8i). \tag{3.10}$$

It is immediate that for every integer $n \geq 1$ and for every uniform graph U such that $\omega(U) = n$, $(\phi(U), \nu(U), \epsilon(U))$ is well defined and gets sense. Now using the notion of *tackle* (use Recalls and Definition 3.2), then the following Theorem immediately implies the Berge problem.

Theorem B. *Let n be an integer ≥ 1 and let (U, Q) be a couple of uniform graphs such that $\omega(Q) = n + 1$ and U is a relative subgraph of Q with $\omega(U) = n$; consider $(b(U), \mathcal{L}_U, b(Q))$ [where $b(U)$ is the berge caliber of U, $\mathcal{L}_U$ is introduced in Recalls and Definitions 3.2 and $b(Q)$ is the berge caliber of Q]. Now look at $(\phi(U), \nu(U), \epsilon(U))$ introduced in Definitions 3.3. Then at least one of the following three properties is satisfied.*
(A_1). *U is a complete graph.*
(A_2). $\omega(Q) = b(Q) = b(U) + 1$.
(A_3). *$(\phi(U), \nu(U), \epsilon(U))$ tackles $(1, 3i\mathcal{L}_U^{-1})$ around $6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$ (use Recalls and Definitions 3.2 for the meaning of tackle and $\mathcal{L}_U$).*

We will simply prove Theorem **B** in Section **4**. But before, let us remark.

Remark 3.4 *(fundamental)* *Let (n, U) where n is an integer ≥ 1 and U is a uniform graph such that $\omega(U) = n$; consider $\mathcal{L}_U$ (use Recalls and Definitions 3.2) and look at $(\phi(U), \nu(U), \epsilon(U))$ introduced in Definitions 3.3. If $\mathcal{L}_U = 2n + 2$, then $(\phi(U), \nu(U), \epsilon(U))$ tackles $(1, 3i\mathcal{L}_U^{-1})$ around $6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$ (use Recalls and Definitions 3.2 for the meaning of tackle).*

Proof. Indeed, observing (via the hypotheses) that $\mathcal{L}_U = 2n + 2$ and using the preceding equality, we easily deduce that

$$\mathcal{L}_U - 2n = 2;\ i\mathcal{L}_U - 4in - 4i + 1 = 1 - i\mathcal{L}_U;\ 2n + 1 = \mathcal{L}_U - 1;\ \textit{and}\ i\mathcal{L}_U - 2in + 1 = 2i + 1. \quad (3.11)$$

Now look at $(\phi(U), \nu(U), \epsilon(U))$ introduced in Definitions 3.3 and let $(\phi(U.2), \phi(U.3), \phi(U.4))$ explicited in Definitions 3.3. Clearly

$$\phi(U.2) = -4\mathcal{L}_U(16\mathcal{L}_U^{-3} + 50\mathcal{L}_U^{-2} + 11i\mathcal{L}_U^{-3} - 13i + 5i\mathcal{L}_U) + 8\mathcal{L}_U - 8i - 16 \quad (3.12)$$

(use (3.6) of Definitions 3.3 and the third equality of (3.11) (observe [via the third equality of (3.11)] that $(\mathcal{L}_U - 1)^2 = \mathcal{L}_U^2 - 2\mathcal{L}_U + 1$); and the first equality of *(3.11)* (observe [via the first equality of (3.11)] that $\frac{(\mathcal{L}_U - 2n)}{2} = 1$)), and

$$\phi(U.3) = \frac{289080}{1331} \quad (3.13)$$

(use (3.7) of Definitions 3.3 and the third equality of (3.11) (observe [via the third equality of (3.11)] that $(\mathcal{L}_U - 1)^2 = \mathcal{L}_U^2 - 2\mathcal{L}_U + 1$)), and

$$\phi(U.4) = 5\mathcal{L}_U(11i\mathcal{L}_U + 7 - 50\mathcal{L}_U^{-1} + 23i\mathcal{L}_U^{-3} - 54\mathcal{L}_U^{-3}) - \frac{1003112i}{1331} \quad (3.14)$$

(use (3.8) of Definitions 3.3 and the third equality of (3.11) (observe [via the third equality of (3.11)] that $(\mathcal{L}_U - 1)^2 = \mathcal{L}_U^2 - 2\mathcal{L}_U + 1$)). That being said let $(\nu(U), \epsilon(U))$ explicited in Definitions

3.3; clearly

$$\nu(U) = (\,1 - i\mathcal{L}_U\,)^2 - 1 + \mathcal{L}_U^2 \tag{3.15}$$

(use (3.9) of Definitions 3.3 and the second equality of (3.11)), and clearly

$$\epsilon(U) = 4(\,(\mathcal{L}_U - 1)^2 - 1 - \mathcal{L}_U^2 + \mathcal{L}_U\,) + (\,2i+1\,)(\,4i\mathcal{L}_U + 4 - 8i\,) \tag{3.16}$$

(use (3.10) of Definitions 3.3 and the last two equalities of (3.11)). That being so, let (x, y, k) such that

$$x = 10 - 6\mathcal{L}_U + i\mathcal{L}_U;\ y = -10 + 6\mathcal{L}_U - 3i\mathcal{L}_U;\ and\, k = -\frac{35}{11} - \frac{618\mathcal{L}_U^{-1}}{121} + \frac{7494\mathcal{L}_U^{-2}}{121} + \frac{126301\mathcal{L}_U^{-4}}{1331}, \tag{3.17}$$

and let $(\,\phi(U), \nu(U), \epsilon(U), x, y, k\,)$ where (x, y, k) is explicted in (3.17) and where $(\phi(U), \nu(U), \epsilon(U))$ is introduced in Definitions 3.3. Now consider $(\,\nu(U), \epsilon(U), x, y\,)$; using (3.15) and (3.16) and the first two equalities of (3.17), we easily check (by elementary computation and the fact that $i^2 = -1$) that

$$x + y - \nu(U) = 0\, and\, x - y - \epsilon(U) = 0. \tag{3.18}$$

That being so, look again at $(\phi(U), \nu(U), \epsilon(U), x, y, k\,)$ and let $(\,\phi(U), x, y, k\,)$; using the three equalities of (3.17), it becomes very easy to check (by elementary computation and the fact that $i^2 = -1$) that

$$x + 3iy\mathcal{L}_U^{-1} + k(6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i) + \phi(U) = 0;\ k \in \mathcal{R}\, and\, \phi(U) = \sum_{j=1}^{4} \phi(U.j). \tag{3.19}$$

(use the three equalities of (3.17) [for x and y and k]; and (3.5) [for $\phi(U.1)$]; and $(3.j + 10)$ [for $\phi(U.j)$ where $2 \leq j \leq 4$]; and *(3.4)* [for $\phi(U)$]). Clearly $(\,\phi(U), \nu(U), \epsilon(U)\,)$ tackles $(1, 3i\mathcal{L}_U^{-1})$ around $6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$ (use *(3.18)* and *(3.19)* and the notion of tackle introduced in Recalls and Definitions 3.2 [observe that $(x, y) \in \mathcal{C}^2$ and $k \in \mathcal{R}$; so $(x, y, k) \in \mathcal{C}^2 \times \mathcal{R}$]). Remark 3.4 follows.

Remark 3.5 (fundamental): **reduction of the Berge problem into a trivial equation of three unknowns**. *Let (n, U) where n is an integer ≥ 1 and U is a uniform graph such that $\omega(U) = n$; consider $\mathcal{L}_U$ (use Recalls and Definitions 3.2) and look at $(\,\phi(U), \nu(U), \epsilon(U)\,)$ introduced in Recalls and Definitions 3.2. If $\mathcal{L}_U = 2n$, then $(\,\phi(U), \nu(U), \epsilon(U)\,)$ does not tackle $(1, 3i\mathcal{L}_U^{-1})$ around $6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$ (use Recalls and Definition 3.2 for the meaning of tackle).*
Proof. Indeed, observing (via the hypotheses) that $\mathcal{L}_U = 2n$ and using the preceding equality, we easily deduce that

$$\mathcal{L}_U - 2n = 0;\ i\mathcal{L}_U - 4in - 4i + 1 = 1 - 4i - i\mathcal{L}_U;\ 2n + 1 = \mathcal{L}_U + 1;\ and\ i\mathcal{L}_U - 2in + 1 = 1. \tag{3.20}$$

Now look at $(\,\phi(U), \nu(U), \epsilon(U)\,)$ introduced in Definitions 3.3 and let $(\phi(U.2), \phi(U.3), \phi(U.4))$ explicited in Definitions 3.3. Clearly

$$\phi(U.2) = 0 \tag{3.21}$$

(use (3.6) of Definitions 3.3 and the third equality of (3.20) (observe [via the third equality of (3.20)] that $(\mathcal{L}_U + 1)^2 = \mathcal{L}_U^2 + 2\mathcal{L}_U + 1$); and the first equality of (3.20) (observe [via the first equality of (3.20)] that $\frac{(\mathcal{L}_U - 2n)}{2} = 0$)), and

$$\phi(U.3) = 4\mathcal{L}_U(34i\mathcal{L}_U^{-2} - 70i\mathcal{L}_U^{-3} - 11i + 5 + 6i\mathcal{L}_U) + \frac{289080}{1331} \tag{3.22}$$

(use (3.7) of Definitions 3.3 and the third equality of (3.20) (observe [via the third equality of (3.20)] that $(\mathcal{L}_U + 1)^2 = \mathcal{L}_U^2 + 2\mathcal{L}_U + 1$)), and

$$\phi(U.4) = 9\mathcal{L}_U(11i\mathcal{L}_U + 7 - 50\mathcal{L}_U^{-1} + 23i\mathcal{L}_U^{-3} - 54\mathcal{L}_U^{-3}) - \frac{1003112i}{1331} \tag{3.23}$$

(use (3.8) of Definitions 3.3 and the third equality of (3.20) (observe [via the third equality of (3.20)] that $(\mathcal{L}_U + 1)^2 = \mathcal{L}_U^2 + 2\mathcal{L}_U + 1$)). That being said let $(\nu(U), \epsilon(U))$ explicited in Definitions 3.3; clearly

$$\nu(U) = (\,1 - 4i - i\mathcal{L}_U\,)^2 - 1 + \mathcal{L}_U^2 \tag{3.24}$$

(use (3.9) of Definitions 3.3 and the second equality of (3.20)) and clearly

$$\epsilon(U) = 4(\;(\mathcal{L}_U + 1)^2 - 1 - \mathcal{L}_U^2 + \mathcal{L}_U\;) + \;4i\mathcal{L}_U + 4 - 8i \tag{3.25}$$

(use (3.10) of Definitions 2.3 and the last two equalities of (3.20)). That being so, let (x, y) such that

$$x = 2\mathcal{L}_U + i\mathcal{L}_U - 6 - 8i \;and\; y = -10 - 10\mathcal{L}_U - 3i\mathcal{L}_U, \tag{3.26}$$

and let $(\,\phi(U), \nu(U), \epsilon(U), x, y\,)$ where (x, y) is explicted in (3.26) and where $(\,\phi(U), \nu(U), \epsilon(U))$ is introduced in Definitions 3.3 (use (3.4) for $\phi(U)$; and (3.9) for $\nu(U)$; and (3.10) for $\epsilon(U)$; and (3.26) for (x, y)). Using (3.24) and (3.25) and the two equalities of (3.26), then we easily check (by elementary computation and the fact that $i^2 = -1$) that

$$x + y - \nu(U) = 0 \,and\, x - y - \epsilon(U) = 0. \tag{3.27}$$

That being said, we have this fact.
Fact.0.

$$there\,exists\,not\,k \in \mathcal{R}\,such\,that\,x + 3iy\mathcal{L}_U^{-1} + k(6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i) + \phi(U) = 0.$$

Otherwise (we reason by reduction to absurd)

$$Let\,k \in \mathcal{R}\,such\,that\,x + 3iy\mathcal{L}_U^{-1} + k(6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i) + \phi(U) = 0. \tag{3.28}$$

It is immediate to see that (3.28) says that

$$x + 3iy\mathcal{L}_U^{-1} + \phi(U) = -k(6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i);\; k \in \mathcal{R}\,and\,\phi(U) = \sum_{j=1}^{4} \phi(U.j) \tag{3.29}$$

(use (3.4) for $\phi(U)$). Now let $(\phi(U), \nu(U), \epsilon(U), x, y)$ explicited above; consider $(\phi(U), x, y)$ and look at (3.29); using elementary computation and elementary divisibility coupled with the fact that $i^2 = -1$ and $k \in \mathcal{R}$, it becomes very easy to check that (3.29) immediately implies that

$$k = k_{n.1} = k_{n.2}, \tag{3.30}$$

where

$$k_{n.1} = \frac{-23837\mathcal{L}_U - 357566\mathcal{L}_U^{-1} + 2010800\mathcal{L}_U^{-2} + 2399719\mathcal{L}_U^{-4} - 757806\mathcal{L}_U^{-3} + 289080}{1331(19 - 6\mathcal{L}_U)} + \frac{85\mathcal{L}_U - 447 - 486\mathcal{L}_U^{-3}}{19 - 6\mathcal{L}_U}. \tag{3.31}$$

and

$$k_{n.2} = \frac{2273418\mathcal{L}_U^{-4} + 126301\mathcal{L}_U^{-3} - 1003112}{1331(\mathcal{L}_U - 11\mathcal{L}_U^2 + 18)} + \frac{123\mathcal{L}_U^2 + 106\mathcal{L}_U^{-1} - 38 - 43\mathcal{L}_U - 73\mathcal{L}_U^{-2}}{\mathcal{L}_U - 11\mathcal{L}_U^2 + 18} \tag{3.32}$$

(via (3.29), use the two equalities of (3.26) [for x and y]; and (3.5) [for $\phi(U.1)$]; and $(3.j+19)$ [for $\phi(U.j)$ where $2 \leq j \leq 4$]; and (3.4) [for $\phi(U)$]). That being so, using (3.30) and (3.31) and (3.32), then we immediately deduce (via elementary computation and the fact that $k_{n.1} = k_{n.2}$) that

$$10996722\mathcal{L}_U^{-2} - 11643588\mathcal{L}_U^{-3} - 7115526 + 7762392\mathcal{L}_U^{-1} = 0. \tag{3.33}$$

Equality (3.33) is clearly impossible (since $\mathcal{L}_U \geq 4$ [indeed recall that $\mathcal{L}_U = 2b(U) + 2$ (use Recalls and Definitions 3.2), and since $n \geq 1$, clearly $b(U) \geq 1$ (recall that $n = \omega(U)$ where $n \geq 1$ and use property (2.18.4) of Assertion 2.18) and so $\mathcal{L}_U \geq 4$] and therefore

$$10996722\mathcal{L}_U^{-2} - 11643588\mathcal{L}_U^{-3} - 7115526 + 7762392\mathcal{L}_U^{-1} < 0).$$

So assuming that

$$there\,exists\;k \in \mathcal{R}\,such\,that\,x + 3iy\mathcal{L}_U^{-1} + k(6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i) + \phi(U) = 0$$

gives rise to a serious contradiction; therefore

$$there\,exists\,not\,k \in \mathcal{R}\,such\,that\,x + 3iy\mathcal{L}_U^{-1} + k(6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i) + \phi(U) = 0.$$

Fact.0. follows.

Clearly $(\phi(U), \nu(U), \epsilon(U))$ does not tackle $(1, 3i\mathcal{L}_U^{-1})$ around $6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$ (use (3.27) and **Fact.0** and the notion of tackle introduced in Recalls and Definitions 3.2). Remark 3.5 follows.

Remark.3.5 reduces the Berge problem into a simple equation of three unknowns. Indeed Remark 3.5 clearly says that, if $\mathcal{L}_U = 2n$ [U is a uniform graph with $\omega(U) = n$], we will have a simple equation of three unknowns which implies that $(\phi(U), \nu(U), \epsilon(U))$ does not tackle $(1, 3i\mathcal{L}_U^{-1})$ around $6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$. We will use Remark 3.5 in Section **4** to immediately deduce the Berge problem and the Berge conjecture. Now using Definitions 3.3, then we have the following elementary Proposition.

Proposition 3.6. *Let (n, U) where n is an integer ≥ 2 and U is a uniform graph such that $\omega(U) = n$; consider $\mathcal{L}_U$ (use Recalls and Definitions 3.2) and look at $(\phi(U), \nu(U), \epsilon(U))$ introduced in Definitions 3.3. Now let $\Xi(U)$ be the canonical coloration of U (use Definition 2.3), and let $Y \in \Xi(U)$; look at $(U \setminus Y, b(U \setminus Y), \mathcal{L}_{U\setminus Y})$ [where $U \setminus Y$ is the induced subgraph of U by $V(U) \setminus Y$; $b(U \setminus Y)$ is the berge caliber of $U \setminus Y$ and $\mathcal{L}_{U\setminus Y} = 2b(U \setminus Y) + 2$]. Via $(\phi(U), \nu(U), \epsilon(U))$ introduced in Definitions 3.3, consider $(\phi(U \setminus Y), \nu(U \setminus Y), \epsilon(U \setminus Y))$ (this consideration gets sense, since U is uniform such that $\omega(U) = n$ with $n \geq 2$ and therefore $U \setminus Y$ is uniform such that $\omega(U \setminus Y) = n - 1$ [use property (2.11.4) of Assertion 2.11], with $n - 1 \geq 1$). If the berge caliber $b(U)$ is of the form $b(U) \leq n - 1$, then $\mathcal{L}_U \leq 2n$ and $(\phi(U \setminus Y) - \phi(U), \nu(U \setminus Y) - \nu(U), \epsilon(U \setminus Y) - \epsilon(U))$ tackles $(1, 3i\mathcal{L}_U^{-1})$ around $6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$.*

Proposition 3.6 is not very difficult to prove and we will prove Proposition 3.6 in Section **3'** (Epilogue). The previous simple Proposition made, we are now ready to give the short complete simple proof of the Berge problem and the Berge conjecture.

4. The proof of the Berge problem and the Berge conjecture

In this Section, using the trivial reformulation of the Berge problem given by Proposition 2.19 and simple properties of Section **3**, we give the short complete simple proof of the Berge problem and the Berge conjecture. In this section, the definition of a uniform graph (use Definition 2.5), the definition of a relative subgraph (use Definition 2.10), the definition of berge caliber (use Definition 2.16), the definition of tackle (use Recalls and Definitions 3.2) and $(\phi(U), \nu(U), \epsilon(U))$ introduced in Definitions 3.3, are *fundamental and crucial*. Now the following Theorem immediately implies the Berge problem and the Berge conjecture.

Theorem 4.1. *Let n be an integer ≥ 1 and let (U, Q) be a couple of uniform graphs such that $\omega(Q) = n + 1$ and U is a relative subgraph of Q with $\omega(U) = n$; consider $(b(U), \mathcal{L}_U, b(Q))$ [where $b(U)$ is the berge caliber of U, $\mathcal{L}_U$ is introduced in Recalls and Definitions 3.2 and $b(Q)$ is the berge caliber of Q]. Now look at $(\phi(U), \nu(U), \epsilon(U))$ introduced in Definitions 3.3. Then at least one of the following three properties is satisfied.*
(A_1). U is a complete graph.
(A_2). $\omega(Q) = b(Q) = b(U) + 1$.
(A_3). $(\phi(U), \nu(U), \epsilon(U))$ tackles $(1, 3i\mathcal{L}_U^{-1})$ around $6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$ (use Recalls and Definitions 3.2 for the meaning of tackle]).

Observe that Theorem 4.1 is exactly Theorem.**B** stated after Definitions 3.3 (Section **3**). We are going to prove simply Theorem 4.1. But before, let us remark.

Remark 4.1'. *Let n be an integer ≥ 2 and let U be uniform such that $\omega(U) = n$; consider the canonical coloration $\Xi(U)$ of U [observe that $\Xi(U)$ exists, by remarking that $U \in \Omega$ (since U is*

uniform) and by using Assertion 2.4*]. Now let* $Y \in \Xi(U)$ *and look at the couple* $(\ b(U), b(U \setminus Y)\ )$*, where* $b(U)$ *is the berge caliber of* U *and* $b(U \setminus Y)$ *the berge caliber of* $U \setminus Y$ *[recall that* $U \setminus Y$ *is the induced subgraph of* U *by* $V(U) \setminus Y$*]. If* U *is bergerian, then* $U \setminus Y$ *is also bergerian .*
Proof. Observe that

$$b(U) = \omega(U) \tag{4.0}$$

[note that U is bergerian and use property $(2.18.1)$ of Assertion 2.18], and notice that

$$U \setminus Y\, is\, uniform\, such\, that\, \omega(U \setminus Y) = \omega(U) - 1 = n - 1\, and\ n - 1 \geq 1 \tag{4.1}$$

[note that U is uniform such that $n = \omega(U) \geq 2$, $Y \in \Xi(U)$, and use property (iii) of Remark 3.0]. Clearly

$$b(U \setminus Y) = \omega(U \setminus Y) = \omega(U) - 1 \tag{4.2}$$

[use *(4.0)* and *(4.1)* and the fact that U is bergerian]. So $U \setminus Y$ is bergerian [note that $U \setminus Y$ is uniform such that $\omega(U \setminus Y) = n - 1 \geq 1$ (use *(4.1)*) and use (4.2) coupled with property $(2.18.1)$ of Assertion 2.18]. Remark $4.1'$ follows. □

Remark $4.1''$. *Let* n *be an integer* ≥ 2 *and let* Q *be uniform such that* $\omega(Q) = n$*. Now let* U *be a relative subgraph of* Q*. If* Q *is bergerian, then* U *is also bergerian .*
Proof. (We reason by reduction to absurd). Indeed let Q be a counter-example such that $card(V(Q))$ is minimun; clearly

$$U \neq Q\, and\, \omega(Q) \geq 3 \tag{4.2'}$$

[otherwise $U = Q$ or $\omega(Q) = 2$, and it becomes immediate to deduce that U is bergerian; a contradiction]. Now using $(4.2')$ and the definition of relative subgraph, then we immediately deduce that

$$\text{there exists}\, Y \in \Xi(Q)\, \text{such that}\, U\, \text{is a relative subgraph}\, Q \setminus\ Y\, \text{and}$$

$$card(\ V(Q \setminus Y)\ ) < card(V(Q)). \tag{4.2''}$$

Clearly U is bergerian [use $(4.2'')$ and the minimality of $card(V(Q))$ and Remark $4.1'$]; a contradiction. Remark $4.1''$ follows. □
Remark 4.2. *Let* n *be an integer* ≥ 1 *and let* (U, Q) *be a couple of uniform graphs such that* $\omega(Q) = n + 1$ *and* U *is a relative subgraph of* Q *with* $\omega(U) = n$*; consider* $(b(U), b(Q))$ *(where* $b(U)$ *is the berge caliber of* U *and* $b(Q)$ *is the berge caliber of* Q*). We have the following trivial properties.*
$(4.2.0.)$ *If* U *is a complete graph, then Theorem* 4.1 *is satisfied.*
$(4.2.1.)$ *If* $\omega(Q) = b(Q)$*, then Theorem* 4.1 *is satisfied.*
$(4.2.2.)$ *If* $b(U) = n$*, then Theorem* 4.1 *is satisfied.*
$(4.2.3.)$ *If* $n = 1$*, then Theorem* 4.1 *is satisfied.*
$(4.2.4.)$ *If* $n = 2$*, then Theorem* 4.1 *is satisfied.*

Proof. Property (4.2.0) is immediate. Property (4.2.1) is very easy (indeed if

$$\omega(Q) = b(Q), \tag{4.3}$$

using equality (4.3) and property (2.18.1) of Assertion 2.18, we immediately deduce that

$$Q\, is\, bergerian. \tag{4.4}$$

Observing that U is a relative subgraph of Q with $\omega(U) + 1 = \omega(Q)$, then we immediately deduce that

$$\text{there exists } Y \in \Xi(Q) \text{ such that } U = Q \setminus Y \text{ where } \Xi(Q) \text{ is the canonical coloration of } Q \tag{4.5}$$

[$Q \setminus Y$ is the induced subgraph of Q by $V(Q) \setminus Y$]. Clearly

$$U\, is\, bergerian \tag{4.6}$$

[use (4.4) and (4.5) and Remark $4.1'$]. Observe that

$$\omega(U) = b(U) \tag{4.7}$$

[use (4.6) and property (2.18.1) of Assertion 2.18]. So

$$\omega(Q) = b(Q) = b(U) + 1$$

[use (4.3) and (4.7) and the fact that $\omega(Q) = \omega(U) + 1 = n + 1$]. (4.7) clearly says that property (A_2) of Theorem 4.1 is satisfied; therefore Theorem 4.1 is also satisfied. Property (4.2.1) follows). Property (4.2.2) is immediate (indeed if $b(U) = n$, observing that $n \geq 1$ and U is uniform such that $\omega(U) = n$, then using the previous and the denotation of $\mathcal{L}_U$ given in Recall and Definition 3.2, we clearly deduce that $\mathcal{L}_U = 2n + 2$; now using the previous equality and Remark 3.4, we immediately deduce that

$$(\,\phi(U), \nu(U), \epsilon(U)\,)\, tackles\, (1, 3i\mathcal{L}_U^{-1})\, around\, 6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i. \tag{4.8}$$

(4.8) clearly says that property (A_3) of Theorem 4.1 is satisfied; therefore Theorem 4.1 is also satisfied. Property (4.2.2) follows).
Property (4.2.3.) is also immediate (indeed, observing (by the hypotheses) that $n = 1$, clearly

$$2 = n + 1 = \omega(Q). \tag{4.9}$$

Since Q is uniform, then, using (4.9) and property (2.18.3) of Assertion 2.18, it becomes very easy to deduce that

$$n + 1 = \omega(Q) = b(Q) = \beta(Q). \tag{4.10}$$

Clearly

$$\omega(Q) = b(Q) \tag{4.11}$$

[use (4.10)]. So Theorem 4.1 is satisfied [use (4.11) and property $(4.2.1)$]. Property $(4.2.3)$ follows). Property $(4.2.4)$ is trivial (indeed, observing (by the hypotheses) that $n = 2$, clearly

$$2 = n = \omega(U). \tag{4.12}$$

Since U is uniform, then, using (4.12) and property (2.18.3) of Assertion 2.18, it becomes very easy to deduce that

$$n = \omega(U) = b(U) = \beta(U). \tag{4.13}$$

Clearly

$$b(U) = n \tag{4.14}$$

[use (4.13)]. So Theorem 4.1 is satisfied [use (4.14) and property $(4.2.2)$]. Property $(4.2.4)$ follows.)□

Remark 4.3. *Suppose that Theorem* 4.1 *is false. Then there exists an integer* $n \geq 1$ *which is a* **minimum** *counter-example to Theorem* 4.1.
Proof. Immediate. □

Using Remark 4.3, then the following last definition comes.
Definition 4.4 **(Fundamental)**. Suppose that Theorem 4.1 is false and let n be a **minimum** counter-example to Theorem 4.1 (such a n exists, by using Remark 4.3). We say that (U, Q) is a *remarkable couple of uniform graphs*, if (U, Q) is a couple of uniform graphs such that $\omega(Q) = n+1$ and U is a relative subgraph of Q with $\omega(U) = n$ [$n \geq 1$ and n is a **minimum** counter-example to Theorem 4.1] and (U, Q) is a counter-example to Theorem 4.1 [it is immediate (via Remark 4.3) that such a couple of uniform graphs (U, Q) exists, if Theorem 4.1 is false]. Briefly, (U, Q) is a remarkable couple of uniform graphs if (U, Q) is a counter-example to Theorem 4.1 with $\omega(U)$ **minimun** (recall that $\omega(U) = n$).
The previous simple remarks and the previous definition made, we now prove simply Theorem 4.1.

Proof of Theorem 4.1. Otherwise (we reason by reduction to absurd), let n be a minimum counter-example to Theorem 4.1 (such a n exists, by using Remark 4.3), and let (U, Q) be a **remarkable couple of uniform graphs** (such a couple (U, Q) exists, by using Remark 4.3 and Definition 4.4); **fix once and for all the couple** (U, Q) (**the couple** (U, Q) **is fixed once and for all, so** (U, Q) **does not move anymore**), and look at ($\phi(U), \nu(U), \epsilon(U)$) introduced in Definitions 3.3. We observe the following.
Observation.$4.1.i.$ $n + 1 = \omega(Q)$*;* $n = \omega(U)$ *and* ($\phi(U), \nu(U), \epsilon(U)$) *does not tackle* $(1, 3i\mathcal{L}_U^{-1})$ *around* $6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$ *[* $\mathcal{L}_U = 2b(U) + 2$ *[use Recalls and Definitions* 3.2*] and* $b(U)$ *is the berge caliber of* U*].*

Immediate, since (U, Q) is a remarkable couple of uniform graphs [use Remark 4.3 and Definition 4.4 for the meaning of a remarkable couple of uniform graphs].
Observation.$4.1.ii.$ *U is not a complete graph and property (A_2) of Theorem* 4.1 *is not satifised.*

Immediate, since (U, Q) is a remarkable couple of uniform graphs.

Observation.$4.1.iii$. *Let* n *[we recall* n *is a minumun counter-example to Theorem* 4.1*]. Then* $n \geq 3$.

Otherwise

$$1 \leq n \leq 2 \tag{4.15}$$

[notice that $n \geq 1$, by the hypotheses]. Clearly Theorem 4.1 is satisfied (use inequality (4.15) and property $(4.2.3)$ of Remark 4.2 if $n = 1$; use inequality (4.15) and property $(4.2.4)$ of Remark 4.2 if $n = 2$), and we have a contradiction since in particular n is a counter-example to Theorem 4.1.

Observation.$4.1.iv$. *Consider* n *[we recall* n *is a minumun counter-example to Theorem* 4.1*] and look at the couple* (U, Q) *[we recall that* (U, Q) *is a couple of remarkable uniform graphs and is fixed once and for all]. Now let* $\Xi(U)$ *be the canonical coloration of* U *[observe that* $\Xi(U)$ *exists, by remarking that* $U \in \Omega$ *(since* U *is uniform) and by using Assertion* 2.4*], and let* $Y \in \Xi(U)$*; look at the couple* $(U \setminus Y, Q \setminus Y)$ *(note that* $G \setminus Y$ *is the induced subgraph of* G *by* $V(G) \setminus Y$*). Then* $n + 1 = \omega(Q) \geq 4$*;* $n = \omega(U) \geq 3$*;* $\omega(Q) = \omega(U) + 1$*;* $Q \setminus Y$ *and* $U \setminus Y$ *are relative subgraphs of* Q*;* $U \setminus Y$ *is a relative subgraph of* $Q \setminus Y$*;* $\omega(Q \setminus Y) = \omega(Q) - 1 = n$*;* $\omega(U \setminus Y) = n - 1 \geq 2$*; and* $\Xi(U) \subseteq \Xi(Q)$.

Clearly

$$n + 1 = \omega(Q) \geq 4 \, and \, n = \omega(U) \geq 3 \, and \, \omega(Q) = \omega(U) + 1 \tag{4.16}$$

[use Observation.$4.1.iii$ and the first two equalities of Observation.$4.1.i$], and clearly

$$Q \setminus Y \, and \, U \setminus Y \, are \, relative \, subgraphs \, of \, Q; \;\; U \setminus Y \, is \, a \, relative \, subgraph \, of \, Q \setminus Y;$$

$$\omega(Q \setminus Y) = \omega(Q) - 1 = n; \, \omega(U \setminus Y) = n - 1 \geq 2; \, and \, \Xi(U) \subseteq \Xi(Q)$$

[Indeed recalling that $Y \in \Xi(U)$ and observing that U is a relative subgraph of Q such that $\omega(Q) = \omega(U) + 1$ (because (U, Q) is a couple of remarkable uniform graphs), then using the preceding and *(4.16)* via the meaning of a relative subgraph (use Definition 2.10), it becomes immediate to deduce that $Q \setminus Y$ and $U \setminus Y$ are relative subgraphs of Q and $U \setminus Y$ is a relative subgraph of $Q \setminus Y$ and $\omega(Q \setminus Y) = \omega(Q) - 1 = n$ and $\omega(U \setminus Y) = \omega(U) - 1 = n - 1$ and $n - 1 \geq 2$ and $\Xi(U) \subseteq \Xi(Q)$]. Observation.$4.1.iv$ follows.

Observation $4.1.v$. *Let* n *[we recall* n *is a minumun counter-example to Theorem* 4.1*] and look at the couple* (U, Q) *[we recall that* (U, Q) *is a couple of remarkable uniform graphs and is fixed once and for all]. Now let* $\Xi(Q)$ *be the canonical coloration of* Q *[observe that* $\Xi(Q)$ *exists, by remarking that* $Q \in \Omega$ *(since* Q *is uniform) and by using Assertion* 2.4*]. Then there exists* $Z \in \Xi(Q)$ *such that* $Z \notin \Xi(U)$ *and* U *is isomorphic to* $Q \setminus Z$ *(note that* $G \setminus Y$ *is the induced subgraph of* G *by* $V(G) \setminus Y$ *and* $\Xi(U)$ *is the canonical coloration of* U *(observe that* $\Xi(U)$ *exists, by remarking that* $U \in \Omega$ *(since* U *is uniform) and by using Assertion* 2.4*).*

Indeed observing that U is a relative subgraph of Q such that $\omega(Q) = \omega(U) + 1$ (because (U, Q) is a couple of remarkable uniform graphs), then using the preceding and the meaning of a relative subgraph [use Definition 2.10], we immediately deduce that

$$\text{there exists } Z \in \Xi(Q) \text{ such that } Z \notin \Xi(U). \tag{4.17}$$

Clearly

$$\omega(Q \setminus Z) = \omega(Q) - 1 \text{ and } \omega(Q \setminus Z) \geq 3 \text{ and } \omega(Q \setminus Z) = \omega(U) \tag{4.18}$$

[use *(4.17)* and property (2.11.4) of Assertion 2.11 coupled with Observation.$4.1.iii$ and the second equality of Observation.$4.1.i$] and clearly

$$Q \setminus Z \text{ is a relative of subgraph of } Q \tag{4.19}$$

[recall that $Z \in \Xi(Q)$ and use property (2.11.4) of Assertion 2.11 via the meaning of a relative subgraph (use Definition 2.10)]. Observe that

$$Q \setminus Z \text{ is uniform and } \omega(Q \setminus Z) \geq 3 \text{ and } \omega(Q \setminus Z) = \omega(U) \tag{4.20}$$

[use (4.19) and the the meaning of a relative subgraph (use Definition 2.10) and the last two equalities of (4.18)]. Since in particular (U, Q) is a couple of uniform graphs such that U is a relative subgraph of Q with $\omega(Q) = \omega(U) + 1$, then using the previous and the couple ((4.19), (4.20)), it becomes immediate to deduce that

$$\alpha(Q \setminus Z) = \alpha(Q) = \alpha(U);\ (U, Q \setminus Z) \text{ is a couple of uniform graphs and}$$

$$\omega(Q \setminus Z) = \omega(U) \geq 2. \tag{4.21}$$

So

$$U \text{ is isormorphic to } Q \setminus Z \tag{4.22}$$

[use (4.21) and property (i) of Remark 3.0]. Observation.$4.1.v$ follows [use (4.17) and (4.22)].

Observation $4.1.vi$. *Let n [we recall n is a minumun counter-example to Theorem 4.1] and look at the couple (U, Q) [we recall that (U, Q) is a couple of remarkable uniform graphs and is fixed once and for all]. Now let $\Xi(U)$ be the canonical coloration of U [observe that $\Xi(U)$ exists, by remarking that $U \in \Omega$ (since U is uniform) and by using Assertion 2.4]; look at $Y \in \Xi(U)$ and let the couple $(U \setminus Y, Q \setminus Y)$ (note that $G \setminus Y$ is the induced subgraph of G by $V(G) \setminus Y$). Then $U \setminus Y$ is uniform; $\omega(U \setminus Y) = \omega(U) - 1 = n - 1$; $\omega(Q \setminus Y) = \omega(Q) - 1 = n$; $b(U) \leq n - 1$; $b(U) = b(U \setminus Y)$; and $b(Q) \leq n - 1$ [notice that $b(G)$ is the berge caliber of G].*

Firstly, we prove that $U \setminus Y$ is uniform and $\omega(U \setminus Y) = \omega(U) - 1 = n - 1$ and $\omega(Q \setminus Y) = \omega(Q) - 1 = n$. **Fact.0:** $U \setminus Y$ is uniform and $\omega(U \setminus Y) = \omega(U) - 1 = n - 1$ and $\omega(Q \setminus Y) = \omega(Q) - 1 = n$. Indeed, using Observation.$4.1.iv$, then it becomes immediate to deduce that $U \setminus Y$ is a relative subgraph of $Q \setminus Y$ such that $\omega(U \setminus Y) = \omega(U) - 1 = n - 1 \geq 2$ and $\omega(Q \setminus Y) = \omega(Q) - 1 = n$; in particular $U \setminus Y$ is uniform [because $U \setminus Y$ is a relative subgraph of $Q \setminus Y$].

Secondly, we prove that $b(U) \leq n - 1$. **Fact.1:** $b(U) \leq n - 1$. (Otherwise $b(U) > n - 1$ and the previous inequality clearly implies that

$$b(U). \geq n \tag{4.23}$$

Remark that

$$b(U) \leq n, \tag{4.24}$$

since $n = \omega(U)$ [use Observation.$4.1.i$] and $b(U) \leq \omega(U)$ [use the meaning of $b(U)$ via Definition 2.16 and use property $(2.18.0)$ of Assertion 2.18]. Now using inequalities (4.23) and (4.24), then it becomes trivial to deduce that $b(U) = n$; so Theorem 4.1 is satisfied [use the previous equality and property (4.2.2) of Remark 4.2]; a contradiction [since in particular (U, Q) is a couple of remarkable uniform graphs]).

Thirdly, we prove that $b(U) = b(U \setminus Y)$. **Fact.2:** $b(U) = b(U \setminus Y)$. (indeed, observing [by **Fact.1**] that $b(U) \leq n - 1$ and noticing [by using Observation.$4.1.iii$] that $n \geq 3$, then using the previous two inequalities [notice that $n = \omega(U)$ via Observation $4.1.i$], it becomes trivial to deduce that all the hypotheses of Proposition 3.1 are satisfied, therefore, the conclusion of Proposition 3.1 is satisfied; consequently $b(U) = b(U \setminus Y)$).

Now, to prove Observation.$4.1.vi$, it suffices to show that $b(Q) \leq n-1$. **Fact.3:** $b(Q) \leq n-1$. (indeed, observe that

$$b(Q) \leq n \tag{4.25}$$

otherwise $b(Q) > n$ and the previous inequality clearly implies that

$$b(Q) \geq n + 1. \tag{4.26}$$

Remark that

$$b(Q) \leq n + 1, \tag{4.27}$$

since $n + 1 = \omega(Q)$ [use Observation.$4.1.i$] and $b(Q) \leq \omega(Q)$ [use the meaning of $b(Q)$ via Definition 2.16 and use property $(2.18.0)$ of Assertion 2.18]. Now using the previous equality and inequalities (4.26) and (4.27), then it becomes trivial to deduce that

$$b(Q) = n + 1 = \omega(Q). \tag{4.28}$$

Clearly

$$Q \text{ is bergerian} \tag{4.29}$$

[use (4.28) and property $(2.18.1)$ of Assertion 2.18]. Now recalling that

$$U \text{ is relative subgraph of } Q \tag{4.30}$$

[since (U, Q) is a couple of remarkable uniform graphs], then, using (4.29) and (4.30) and Remark $4.1''$, we immediately deduce that

$$U \text{ is bergerian.} \tag{4.31}$$

So $b(U) = \omega(U)$ [use (4.31) and property $(2.18.1)$ of Assertion 2.18]; a contradiction, since $b(U) \leq n-1$ [by **Fact.1**] and $\omega(U) = n$ [by using Observation.$4.1.i$]. So $b(Q) \leq n$]]. Now notice that

$$b(Q) \leq n - 1 \tag{4.32}$$

otherwise $b(Q) > n - 1$; now using the previous inequality coupled with inequality (4.25), then we immediately deduce that

$$b(Q) = n. \tag{4.33}$$

$$\text{now let } Z \in \Xi(Q) \text{ such that } Z \notin \Xi(U) \tag{4.34}$$

[such a Z exists, via Observation.$4.1.v$]; clearly

$$U \text{ is isomorphic to } Q \setminus Z \tag{4.35}$$

[use (4.34) and Observation.$4.1.v$]. Observe that

$$b(Q) < \omega(Q) \tag{4.36}$$

[use (4.25) and observe (by using Observation.$4.1.i$) that $n + 1 = \omega(Q)$]. Now let (4.35) and look at $Q \setminus Z$ [recall that in particular $Z \in \Xi(Q)$]; then using (4.36) and the preceding [observe that $\omega(Q) \geq 4$, by using Observation.$4.1.iv$], it becomes trivial to deduce that all the hypotheses of Proposition 3.1 are satisfied, therefore, the conclusion of Proposition 3.1 is satisfied; consequently

$$b(Q) = b(Q \setminus Z). \tag{4.37}$$

Clearly

$$b(Q) = b(U) \tag{4.38}$$

[use (4.35) and (4.37)] and so

$$b(U) = n \tag{4.39}$$

[use (4.38) and (4.33)]. Equality (4.39) clearly contradicts **Fact.1**. So $b(Q) \leq n - 1$.]]). Observation $4.1.vi$ follows.

Observation $4.1.vii$**.** Let n and look at the couple (U, Q) [we recall that (U, Q) is a couple of remarkable uniform graphs and is fixed once and for all]. Now let $\Xi(U)$ be the canonical coloration of U, and let $Y \in \Xi(U)$; look at $U \setminus Y$ [$U \setminus Y$ is the induced subgraph of U by $V(U) \setminus Y$]. Now via $(\ \phi(U), \nu(U), \epsilon(U)\ )$ introduced in Definitions 3.3, consider $(\ \phi(U \setminus Y), \nu(U \setminus Y), \epsilon(U \setminus Y)\ )$ (this consideration gets sense, since (U, Q) is couple of remarkable uniform graphs such that $\omega(U) = n$ with $n \geq 3$ [use Observation.$4.1.iii$] and therefore $U \setminus Y$ is uniform such that $\omega(U \setminus Y) = n - 1$ [use property (2.11.4) of Assertion 2.11], with $n - 1 \geq 2$). Then $(\ \phi(U \setminus Y) - \phi(U), \nu(U \setminus Y) - \nu(U), \epsilon(U \setminus Y) - \epsilon(U)\ )$ tackles $(1, 3i\mathcal{L}_U^{-1})$ around $6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$.

Indeed oberving that

$$b(U) \leq n - 1 \tag{4.40}$$

[use Observation.$4.1.vi$] and remaking that

$$n = \omega(U)\, and\, n \geq 3 \tag{4.41}$$

[use Observation.$4.1.i$ and Observation.$4.1.iii$], then using (4.40) and (4.41) and the the fact that (U, Q) is a couple of remarkable uniform graphs, it becomes trivial to deduce that (n, U) satifies all the hypotheses of Proposition 3.6; therefore (n, U) satisfies the conclusion of Proposition 3.6.

So $\mathcal{L}_U \leq 2n$ and $(\,\phi(U \setminus Y) - \phi(U), \nu(U \setminus Y) - \nu(U), \epsilon(U \setminus Y) - \epsilon(U)\,)$ tackles $(1, 3i\mathcal{L}_U^{-1})$ around $6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$. Observation.$4.1.vii$ follows.

Observation $4.1.viii$**.** Let n [we recall n is a minumun counter-example to Theorem 4.1] and look at the couple (U, Q) [we recall that (U, Q) is a couple of remarkable uniform graphs and is fixed once and for all]. Let $\Xi(U)$ be the canonical coloration of U and let $Y \in \Xi(U)$; look at $(U \setminus Y, Q \setminus Y)$ [$U \setminus Y$ is the induced subgraph of U by $V(U) \setminus Y$; $Q \setminus Y$ is the induced subgraph of Q by $V(Q) \setminus Y$ (observe that $Q \setminus Y$ gets sence, since $\Xi(U) \subseteq \Xi(Q)$ by using Observation.$4.1.iv$)]. Now look at $(\,\phi(U), \nu(U), \epsilon(U)\,)$ introduced in Definitions 3.3, and via $(\,\phi(U), \nu(U), \epsilon(U)\,)$, consider $(\,\phi(U \setminus Y), \nu(U \setminus Y), \epsilon(U \setminus Y)\,)$ (this consideration gets sense, since (U, Q) is a couple of remarkable uniform graphs such that $\omega(U) = n$ with $n \geq 3$ [use Observation.$4.1.iii$] and therefore $U \setminus Y$ is uniform such that $\omega(U \setminus Y) = n - 1$ [use Observation.$4.1.vi$], with $n - 1 \geq 2$) . Then $(\,\phi(U \setminus Y), \nu(U \setminus Y), \epsilon(U \setminus Y)\,)$ tackles $(1, 3i\mathcal{L}_U^{-1})$ around $6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$.

Indeed, look at n [recall n is a minimum counter-example to Theorem 4.1], and via n, consider $n-1$ (this consideration gets sense, since $n \geq 3$ [use Observation.$4.1.iii$], and therefore $n-1 \geq 2$).

$$\text{Then by the minimality of } n, \quad n - 1 \text{ is not a counter-example to Theorem 4.1.} \tag{4.42}$$

Observing that

$$U \setminus Y \text{ is a relative subgraph of } Q \setminus Y \text{ and } \omega(U \setminus Y) = n-1 \text{ and } \omega(Q \setminus Y) = n \; and \; n-1 \geq 2 \tag{4.43}$$

[use Observation.$4.1.iv$]; then using *(4.42)* and (4.43), it becomes trivial to deduce that

$$\text{the couple of uniform graphs } (U \setminus Y,\, Q \setminus Y) \text{ satisfies Theorem 4.1} \tag{4.44}$$

[notice that $U \setminus Y$ and $Q \setminus Y$ are uniform graphs, since $U \setminus Y$ is a relative subgraph of $Q \setminus Y$ (use *(4.43)* and the meaning of a relative subgraph given in Definition 2.10)]. Now using *(4.44)*, it immediately follows that

$$\text{at least one of properties } (A_1) \text{ and } (A_3) \text{ of Theorem 4.1 is satisfied by } U \setminus Y \tag{4.45}$$

$$\text{or property} (A_2) \text{ of Theorem 4.1 is satisfied by } (U \setminus Y,\, Q \setminus Y). \tag{4.45$'$}$$

That being said, we have the following three Claims.
Claim 1. Property (A_1) of Theorem 4.1 is not satisfied by $U \setminus Y$. Otherwise

$$U \setminus Y \text{ is a complete graph} \tag{4.46}$$

[since Property (A_1) of Theorem 4.1 is satisfied by $U \setminus Y$]. Observing that

$$\omega(U \setminus Y) = n - 1 \text{ where } n - 1 \geq 2 \tag{4.46$'$}$$

[use (4.43)], then using (4.46) and $(4.46')$ and the fact that U is uniform, it becomes immediate to deduce that

$$U \text{ is a complete graph.} \tag{4.47}$$

(4.47) contradicts Observation.$4.1.ii$. Claim.1 follows.
Claim 2. Property (A_2) of Theorem 4.1 is not satisfied by $(U \setminus Y, Q \setminus Y)$. Otherwise

$$\omega(Q \setminus Y) = b(Q \setminus Y) = b(U \setminus Y) + 1 \tag{4.48}$$

[since Property (A_2) of Theorem 4.1 is satisfied by $(U \setminus Y, Q \setminus Y)$]. Observing that

$$\omega(Q \setminus Y) = n \tag{4.49}$$

[use Observation.$4.1.vi$], then using (4.49), it becomes immediate to deduce that (4.48) clearly says that

$$n = b(Q \setminus Y) = b(U \setminus Y) + 1. \tag{4.50}$$

So

$$n = b(Q \setminus Y) \tag{4.51}$$

[use (4.50)]. Now observing [by using Observation.$4.1.vi$] that

$$b(Q) \leq n - 1 \tag{4.51'}$$

and noticing [by using Observation.$4.1.iv$] that

$$Q \setminus Y \text{ is a relative subgraph, } Q \tag{4.52}$$

clearly

$$b(Q \setminus Y) \leq b(Q) \tag{4.52'}$$

[use (4.52) and property $(2.18.5)$ of Assertion 2.18]. So

$$b(Q \setminus Y) \leq n - 1 \tag{4.52''}$$

[use $(4.52')$ and $(4.51')$], and inequality $(4.52'')$ contradicts equality (4.51). **Claim.2** follows.
Claim 3. Property (A_3) of Theorem 4.1 is satisfied by $U \setminus Y$. This Claim is immediate [use Claim 1 and Claim 2 and (4.45) and $(4.45')$].

Having made the previous three Claims, then we immediately deduce that

$$(\, \phi(U \setminus Y), \nu(U \setminus Y), \epsilon(U \setminus Y)\,) \text{ tackles } (1, 3i\mathcal{L}_{U\setminus Y}^{-1}) \text{ around}$$

$$6\mathcal{L}_{U\setminus Y} - 19 - i\mathcal{L}_{U\setminus Y} + 11i\mathcal{L}_{U\setminus Y}^2 - 18i \tag{4.53}$$

[use Claim 3]. Now let (4.53) and look at $\mathcal{L}_{U\setminus Y}$; clearly

$$\mathcal{L}_{U\setminus Y} = 2b(U \setminus Y) + 2 \tag{4.54}$$

[use the definition of $\mathcal{L}_{U\setminus Y}$ via the definition of $\mathcal{L}_U$ given in Recalls and Definitions 3.2 and notice that $U \setminus Y$ is uniform (via Observation.$4.1.vi$)]. Now observing [via Observation.$4.1.vi$] that

$$b(U) = b(U \setminus Y), \tag{4.55}$$

clearly

$$\mathcal{L}_{U\setminus Y} = 2b(U) + 2 = \mathcal{L}_U \tag{4.56}$$

[use (4.55) and (4.54) and the definition of $\mathcal{L}_U$ given in Recalls and Definitions 3.2]. So

$$(\ \phi(U \setminus Y), \nu(U \setminus Y), \epsilon(U \setminus Y)\ ) \text{ tackles } (1, 3i\mathcal{L}_U^{-1}) \text{ around } 6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$$

[use (4.56) and (4.53)]. Observation.$4.1.viii$ follows.

Observation $4.1.ix$**.** Let the couple (U, Q) [we recall that (U, Q) is a couple of remarkable uniform graphs and is fixed once and for all] and look at $(\ \phi(U), \nu(U), \epsilon(U)\ )$ introduced in Definitions 3.3. Then $(\ \phi(U), \nu(U), \epsilon(U)\ )$ tackles $(1, 3i\mathcal{L}_U^{-1})$ around $6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$.

Indeed using Observation.$4.1.vii$ and the definition of *tackle* (use Recalls and Definitions 3.2), then we immediately deduce that there exists $(\ x, y, k\ ) \in \mathcal{C}^2 \times \mathcal{R}$ such that

$$x + 3iy\mathcal{L}_U^{-1} + k(6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i) + \phi(U \setminus Y) - \phi(U) = 0 \tag{4.57}$$

where

$$x + y - (\nu(U \setminus Y) - \nu(U)) = 0 \, and \, x - y - (\epsilon(U \setminus Y) - \epsilon(U)) = 0. \tag{4.58}$$

That being so, using Observation.$4.1.viii$ and the definition of *tackle* (use Recalls and Definitions 3.2), then we immediately deduce that there exists $(x', y', k') \in \mathcal{C}^2 \times \mathcal{R}$ such that

$$x' + 3iy'\mathcal{L}_U^{-1} + k'(6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i) + \phi(U \setminus Y) = 0 \tag{4.59}$$

where

$$x' + y' - \nu(U \setminus Y) = 0 \ and \ x' - y' - \epsilon(U \setminus Y) = 0. \tag{4.60}$$

Now using equality of (4.59), then we immediately deduce that equality of (4.57) clearly says that

$$x - x' + 3i(y - y')\mathcal{L}_U^{-1} + (k - k')(6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i) - \phi(U) = 0. \tag{4.61}$$

It is trivial to see that equality *(4.61)* clearly says that

$$x' - x + 3i(y' - y)\mathcal{L}_U^{-1} + (k' - k)(6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i) + \phi(U) = 0. \tag{4.62}$$

Using the two equalities of (4.60), then we immediately deduce that the two equalities of (4.58) clearly say that

$$(x - x') + (y - y') + \nu(U) = 0 \ \ and \ \ (x - x') - (y - y') + \epsilon(U) = 0. \tag{4.63}$$

It is trivial to see that (4.63) clearly says that

$$(x' - x) + (y' - y) - \nu(U) = 0 \ \ and \ \ (x' - x) - (y' - y) - \epsilon(U) = 0. \tag{4.64}$$

Now observing that $(\ x, y, k\ ) \in \mathcal{C}^2 \times \mathcal{R}$ (use (4.57)) and since $(\ x', y', k'\ ) \in \mathcal{C}^2 \times \mathcal{R}$ (use (4.59)), then it becomes trivial to deduce that (4.62) and (4.64) clearly say that

$$\text{there exists } (x'', y'', k'') \in \mathcal{C}^2 \times \mathcal{R} \text{ such that } x'' = x' - x \text{ and } y'' = y' - y \text{ and } k'' = k' - k \tag{4.65}$$

where

$$x'' + y'' - \nu(U) = 0,\ x'' - y'' - \epsilon(U) = 0 \text{ and}$$
$$x'' + 3iy''\mathcal{L}_U^{-1} + k''(6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i) + \phi(U) = 0. \tag{4.66}$$

Clearly

$$(\,\phi(U), \nu(U), \epsilon(U)\,) \text{ tackles } (1, 3i\mathcal{L}_U^{-1}) \text{ around } 6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$$

(use (4.65) and (4.66) and the definition of tackle introduced in Recalls and Definitions 3.2). Observation.$4.1.ix$ follows.

These simple observations made, then it becomes trivial to see that Observation.$4.1.ix$ clearly contradicts Observation.$4.1.i$. Theorem 4.1 follows.□

Now the following theorem is a consequence of Theorem 4.1.

Theorem 4.5 *Let n be an integer ≥ 1 and let U be uniform such that $\omega(U) = n$; look at $b(U)$ ($b(U)$ is the berge caliber of U). Then $b(U) = n$.*
Proof. Otherwise, let n be a minimum counter-example [such a n clearly exists] and let U be uniform such that

$$b(U) \neq n \text{ and } n = \omega(U), \tag{4.67}$$

fix once and for all U [U is fixed once and for all, so U does not move anymore]. Now let $\Xi(U)$ be the canonical coloration of U [observe that $\Xi(U)$ exists, by remarking that $U \in \Omega$ (since U is uniform) and by using Assertion 2.4], and let $Y \in \Xi(U)$; look at the couple $(b(U),\ b(U \setminus Y))$, where $b(U)$ is the berge caliber of U and $b(U \setminus Y)$ is the berge caliber of $U \setminus Y$ (note that $U \setminus Y$ is the induced subgraph of U by $V(U) \setminus Y$). We observe the following.

Observation $4.5.1$**.** $n > 2$ and $U \setminus Y$ is uniform and $\omega(U \setminus Y) = \omega(U) - 1 = n - 1$ and $b(U) \leq n - 1$ and $b(U) = b(U \setminus Y)$ and $b(U) = n - 1$.
Firstly, we prove that $n > 2$. **Claim.0:** $n > 2$. Otherwise clearly

$$1 \leq n \leq 2, \text{ where } n = \omega(U) \text{ and } \chi(U) = n \text{ since } U \in \Omega. \tag{4.68}$$

Noticing that $U \in \Omega$ [since U is uniform], then, using (4.68) and property (3) of Assertion 1.9, it becomes trivial to deduce that

$$n = \beta(U), \text{ where } n = \omega(U). \tag{4.69}$$

Using (4.69) and property (2.18.1) of Assertion 2.18, then we immediately deduce that

$$n = \beta(U) = b(U). \tag{4.70}$$

Clearly $b(U) = n$ [use (4.70)], and the previous equality contradicts (4.67).
Secondly, we prove that $U \setminus Y$ is uniform and $\omega(U \setminus Y) = \omega(U) - 1 = n - 1$. **Claim 1:** $U \setminus Y$ is uniform and $\omega(U \setminus Y) = \omega(U) - 1 = n - 1$. Indeed, observing [by Claim 0] that $n > 2$ and noticing [by using (4.67)] that $n = \omega(U)$, then, using the previous and property (iii) of Remark

3.0, it becomes trivial to deduce that $U \setminus Y$ is uniform such that $\omega(U \setminus Y) = \omega(U) - 1 = n - 1$. Now, to prove Observation 4.5.1, it suffices to show that $b(U) \leq n - 1$ and $b(U) = b(U \setminus Y)$ and $b(U) = n - 1$. **Claim 2:** $b(U) \leq n - 1$ and $b(U) = b(U \setminus Y)$ and $b(U) = n - 1$. Clearly

$$b(U) \leq n - 1 \tag{4.71}$$

(Otherwise $b(U) > n - 1$ and the previous inequality clearly implies that

$$b(U) \geq n. \tag{4.71'}$$

Remark that

$$b(U) \leq n, \tag{4.72}$$

since $n = \omega(U)$ [use (4.67)] and $b(U) \leq \omega(U)$ [use the definition of $b(U)$]. Now using inequalities $(4.71')$ and (4.72), then it becomes trivial to deduce that $b(U) = n$; the previous equality contradicts (4.67). So $b(U) \leq n - 1$); and clearly

$$b(U) = b(U \setminus Y) \tag{4.73}$$

(indeed, observing [by (4.71)] that $b(U) \leq n - 1$ and noticing [by Claim 0] that $n > 2$, then using the previous two inequalities [recall that $n = \omega(U)$], it becomes trivial to deduce that all the hypotheses of Proposition 3.1 are satisfied, therefore, the conclusion of Proposition 3.1 is satisfied; consequently $b(U) = b(U \setminus Y)$); and clearly $b(U) = n - 1$ (indeed, look at n [recall n is a minimum counter-example], and via n, consider $n - 1$ [this consideration gets sense, since $n > 2$ (by Claim 0), and therefore $n - 1 > 1$]. Observing [by Claim 1] that $U \setminus Y$ is uniform such that $\omega(U \setminus Y) = n - 1$ and $n > 2$, then, by the minimality of n, $n - 1$ is not a counter-example to Theorem 4.5; so

$$b(U \setminus Y) = n - 1, \tag{4.74}$$

since [by Claim 1] $U \setminus Y$ is uniform such that $\omega(U \setminus Y) = n - 1$. Clearly $b(U) = n - 1$ [use equalities (4.73) and (4.74)]). Observation.4.5.1 follows.

Observation 4.5.2. Let the couple (n, U) [we recall that n is a minimum counter-example and U is uniform such $\omega(U) \neq b(U)$ and $\omega(U) = n$ (use (4.67)) and U is fixed once and for all]. Now consider $(\,\phi(U), \nu(U), \epsilon(U)\,)$ introduced in Definitions 3.3 and look at $\mathcal{L}_U$ [use Recalls and Definitions 3.2]. Then $(\,\phi(U), \nu(U), \epsilon(U)\,)$ does not tackle $(1, 3i\mathcal{L}_U^{-1})$ around $6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$.

Indeed look at U; observing [by using Observation.4.5.1] that $b(U) = n - 1$, clearly

$$\mathcal{L}_U = 2n. \tag{4.75}$$

Now observing that U is uniform and $n > 2$ [use Observation.4.5.1], then using the preceding and (4.75), it becomes trivial to see that (n, U) satisfies all the hypotheses of Remark 3.5, therefore (n, U) satisfies the conclusion of Remark 3.5; consequently $(\,\phi(U), \nu(U), \epsilon(U)\,)$ does not tackle

$(1, 3i\mathcal{L}_U^{-1})$ around $6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$. Observation.4.5.2 follows.

Observation 4.5.3. Let the couple (n, U) [we recall that n is a minimum counter-example and U is uniform such $\omega(U) \neq b(U)$ and $\omega(U) = n$ (use (4.67)) and U is fixed once and for all]. Then U is not a complete graph.

Otherwise

$$b(U) = n \tag{4.76}$$

[use Remark 2.1 and recall that $n = \omega(U)$ (by (4.67))] and equality (4.76) contradicts *(4.67)*. Observation.4.5.3 follows.

These simple observations made, let Q be uniform such that $\omega(Q) = n + 1$ and U is a relative subgraph of Q [observe that such a Q clearly exists, since $n > 2$ (use Observation.4.5.1)] and look at the couple (U, Q); recalling that in particular U is uniform such $\omega(U) = n$ [note that $n > 2$ (use Observation.4.5.1)], then, using Theorem 4.1, it becomes trivial to deduce that

$$U \text{ is a complete graph or } b(Q) = \omega(Q) = b(U) + 1 \tag{4.77}$$

or

$$(\,\phi(U), \nu(U), \epsilon(U)\,) \text{ tackles } (1, 3i\mathcal{L}_U^{-1}) \text{ around } 6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i. \tag{4.78}$$

Recalling that

$$b(U) \neq n \text{ and } \omega(U) = n \text{ and } \omega(Q) = n + 1 \tag{4.79}$$

[use (4.67) and the definition of Q], and using Observation.4.5.3 coupled with (4.79), then it becomes trivial to deduce that

$$\text{property (4.77) is not satisfied;} \tag{4.80}$$

so

$$\text{property (4.78) is satisfied} \tag{4.81}$$

[use (4.80) and (4.77) and (4.78)]. Clearly

$$(\,\phi(U), \nu(U), \epsilon(U)\,) \text{ tackles } (1, 3i\mathcal{L}_U^{-1}) \text{ around } 6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$$

[use (4.81) and (4.78)] and the previous clearly contradicts Observation.4.5.2. Theorem 4.5 follows.□

Theorem 4.5 immediately implies the Berge problem and the Berge conjecture.

Theorem 4.6. (The proof of the Berge problem). The Berge problem is true [i.e. For every berge graph B, we have $\chi(B) = \omega(B)$].

Proof. Observing [by Theorem 4.5] that for every integer $n \geq 1$ and for every uniform graph U such that $\omega(U) = n$, we have $b(U) = n$, then using the previous and Proposition 2.19, it becomes immediate to deduce that the Berge problem is true.□

Theorem 4.7 *(The Proof of the Berge Conjecture).* Let F be a graph. Then, F is perfect $\Leftrightarrow$ F

is berge.
Proof. ($\Rightarrow$) is very easy and was rapidly proved by Berge (see [2]).
($\Leftarrow$) Otherwise [we reason by reduction to absurd]. Let F be a counter-example such that - $card(V(F))$ is minimum. Then, we have the following three simple claims.
Claim 4.7.0. $card(V(F)) > 0$.

Otherwise, clearly $V(F) = \{\emptyset\}$, and clearly F is perfect. A contradiction.
Claim 4.7.1. Let $x \in V(F)$ [such a x exists, by using Claim 4.7.0], and let $F' = F \setminus \{x\}$ [recall F' is the induced subgraph of F by $V(F) \setminus \{x\}$]. Then F' is perfect.

Indeed, recalling that F is berge, clearly F' is also berge [since F' is an induced subgraph of F]. Since $card(V(F')) = card(V(F)) - 1$, clearly F' is berge and is such that $card(V(F')) = card(V(F)) - 1$; then, by the minimality of $card(V(F))$, F' is not a counter-example. So, F' is perfect. Claim.4.7.1 follows.
*Claim.*4.7.2. $\chi(F) > \omega(F)$.

Otherwise, clearly

$$\chi(F) = \omega(F). \tag{4.82}$$

Now let $x \in V(F)$ [such a x exists, by using Claim 4.7.0], and let $F' = F \setminus \{x\}$; observing that F' is perfect [use Claim 4.7.1], then, using equality (4.82), we immediately deduce that F is also perfect. A contradiction [since F is a counter-example]. Claim 4.7.2 follows.

These three simple claims made, consider F; recalling that F is berge, then Theorem 4.6 immediately implies that $\chi(F) = \omega(F)$, and this contradicts claim 4.7.2. Theorem 4.7 follows. □

3'. Epilogue
Our article clearly shows that elementary complex calculus coupled with elementary arithmetic calculus and trivial computation help to give a simple analytic proof of the Berge problem and the Berge conjecture. Now we end this article by proving the only Proposition that we let unproved in Section **3**.

Proposition 3.6. *Let (n, U) where n is an integer ≥ 2 and U is a uniform graph such that $\omega(U) = n$; consider $\mathcal{L}_U$ (use Recalls and Definitions 3.2) and look at $(\phi(U), \nu(U), \epsilon(U))$ introduced in Definitions 3.3. Now let $\Xi(U)$ be the canonical coloration of U (use Definition 2.3), and let $Y \in \Xi(U)$; look at $(U \setminus Y, b(U \setminus Y), \mathcal{L}_{U\setminus Y})$ [where $U \setminus Y$ is the induced subgraph of U by $V(U) \setminus Y$; $b(U \setminus Y)$ is the berge caliber of $U \setminus Y$ and $\mathcal{L}_{U\setminus Y} = 2b(U \setminus Y) + 2$]. Via $(\phi(U), \nu(U), \epsilon(U))$ introduced in Definitions 3.3, consider $(\phi(U \setminus Y), \nu(U \setminus Y), \epsilon(U \setminus Y))$ (this consideration gets sense, since U is uniform such that $\omega(U) = n$ with $n \geq 2$ and therefore $U \setminus Y$ is uniform such that $\omega(U \setminus Y) = n - 1$ [use property (2.11.4) of Assertion 2.11], with $n - 1 \geq 1$). If the berge caliber $b(U)$ is of the form $b(U) \leq n - 1$, then $\mathcal{L}_U \leq 2n$ and $(\phi(U \setminus Y) - \phi(U), \nu(U \setminus Y) - \nu(U), \epsilon(U \setminus Y) - \epsilon(U))$ tackles $(1, 3i\mathcal{L}_U^{-1})$ around $6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$.*
To prove Proposition 3.6, we need four elementary remarks.

Remark 3.7. *Let n be an integer ≥ 2 and let U be uniform such that $\omega(U) = n$; look at $\mathcal{L}_U$ (use Recalls and Definitions* 3.2*). Now let $\Xi(U)$ be the canonical coloration of U (use Definition* 2.3*), and let $Y \in \Xi(U)$; consider $(U \setminus Y, \mathcal{L}_{U\setminus Y})$ [where $U \setminus Y$ is the induced subgraph of U by $V(U) \setminus Y$ and $\mathcal{L}_{U\setminus Y} = 2b(U \setminus Y) + 2$]. If $\mathcal{L}_U \leq 2n$, then $\mathcal{L}_U = \mathcal{L}_{U\setminus Y}$.*

Proof. Indeed observing (via the hypotheses) that

$$\mathcal{L}_U \leq 2n \text{ where } n = \omega(U) \text{ and } n \geq 2 \text{ (} U \text{ is uniform)}, \tag{3.34}$$

and recalling [via Recalls and Definitions 3.2] that

$$\mathcal{L}_U = 2b(U) + 2, \tag{3.35}$$

then using (3.34) and (3.35), it becomes trivial to deduce that

$$\omega(U) > b(U). \tag{3.36}$$

Now via (3.34) and inequality (3.36), then it becomes trivial to deduce that U satisfies all the hypotheses of Proposition 3.1; therefore U satisfies the conclusion of Proposition 3.1 and so

$$b(U) = b(U \setminus Y). \tag{3.37}$$

Clearly $\mathcal{L}_U = \mathcal{L}_{U\setminus Y}$ (use equalities (3.37) and (3.35) by recalling that $\mathcal{L}_{U\setminus Y} = 2b(U \setminus Y) + 2$). Remark 3.7 follows.□

Remark 3.8.*Let n be an integer ≥ 2 and let U be uniform such that $\omega(U) = n$; consider $\mathcal{L}_U$ (use Recalls and Definitions* 3.2*). Now let $\Xi(U)$ be the canonical coloration of U (use Definition* 2.3*), and let $Y \in \Xi(U)$; consider $(U \setminus Y, \mathcal{L}_{U\setminus Y})$ [where $U \setminus Y$ is the induced subgraph of U by $V(U) \setminus Y$ and $\mathcal{L}_{U\setminus Y} = 2b(U\setminus Y)+2$]. Look at $(\phi(U.1), \phi(U.2), \phi(U.3), \phi(U.4))$ introduced in Definitions* 3.3*, and via $(\phi(U.1), \phi(U.2), \phi(U.3), \phi(U.4))$, consider $(\phi(U \setminus Y.1), \phi(U \setminus Y.2), \phi(U \setminus Y.3), \phi(U \setminus Y.4))$ (this consideration gets sense, since U is uniform such that $\omega(U) = n$ with $n \geq 2$ and therefore $U \setminus Y$ is uniform such that $\omega(U \setminus Y) = n - 1$ [use property* (2.11.4) *of Assertion* 2.11*], with $n - 1 \geq 1$). If $\mathcal{L}_U \leq 2n$, then we have the following four properties.*
(3.8.1) $\phi(U \setminus Y.1) - \phi(U.1) = 0$.
(3.8.2) $\phi(U \setminus Y.2) - \phi(U.2) = -8n(16\mathcal{L}_U^{-3} + 50\mathcal{L}_U^{-2} + 11i\mathcal{L}_U^{-3} - 13i + 5i\mathcal{L}_U) + 8\mathcal{L}_U - 8i - 16$.
(3.8.3) $\phi(U \setminus Y.3) - \phi(U.3) = -8n(34i\mathcal{L}_U^{-2} - 70i\mathcal{L}_U^{-3} - 11i + 5 + 6i\mathcal{L}_U)$.
(3.8.4) $\phi(U \setminus Y.4) - \phi(U.4) = -8n(11i\mathcal{L}_U + 7 - 50\mathcal{L}_U^{-1} + 23i\mathcal{L}_U^{-3} - 54\mathcal{L}_U^{-3})$.

Proof. (3.8.1) Indeed let $\phi(U \setminus Y.1)$; clearly

$$\phi(U\setminus Y.1) = \frac{126301i\mathcal{L}_{U\setminus Y}^{-3} + 2273418i\mathcal{L}_{U\setminus Y}^{-4} - 23837\mathcal{L}_{U\setminus Y} - 357566\mathcal{L}_{U\setminus Y}^{-1}}{1331} + \phi(U\setminus Y.1)' \tag{3.38}$$

where

$$\phi(U \setminus Y.1)' = \frac{2010800\mathcal{L}_{U\setminus Y}^{-2} + 2399719\mathcal{L}_{U\setminus Y}^{-4} - 757806\mathcal{L}_{U\setminus Y}^{-3}}{1331} \tag{3.38'}$$

(use (3.5) of Definitions 3.3). So

$$\begin{aligned}\phi(U \setminus Y.1) &= \tfrac{1}{1331}(126301i\mathcal{L}_U^{-3} + 2273418i\mathcal{L}_U^{-4} - 23837\mathcal{L}_U - 357566\mathcal{L}_U^{-1} \\ &\quad +2010800\mathcal{L}_U^{-2} + 2399719\mathcal{L}_U^{-4} - 757806\mathcal{L}_U^{-3})\end{aligned} \tag{3.39}$$

(use (3.38) and (3.38′) and notice [by observing that $\mathcal{L}_U \leq 2n$ and by using Remark 3.7] that $\mathcal{L}_U = \mathcal{L}_{U\setminus Y}$). Clearly

$$\phi(U \setminus Y.1) = \phi(U.1) \tag{3.40}$$

(use (3.5) of Definitions 3.3 and (3.39)) and so $\phi(U \setminus Y.1) - \phi(U.1) = 0$ (use (3.40)). Property (3.8.1) immediately follows.
(3.8.2) Indeed let $\phi(U \setminus Y.2)$; clearly

$$\phi(U\setminus Y.2) = ((2(n-1)+1)^2-1-\mathcal{L}_{U\setminus Y}^2-2\mathcal{L}_{U\setminus Y})(16\mathcal{L}_{U\setminus Y}^{-3}+50\mathcal{L}_{U\setminus Y}^{-2}+11i\mathcal{L}_{U\setminus Y}^{-3}-13i+5i\mathcal{L}_{U\setminus Y})+\gamma_n \tag{3.41}$$

where

$$\gamma_n = \frac{(\mathcal{L}_{U\setminus Y} - 2(n-1))}{2}(8\mathcal{L}_{U\setminus Y} - 8i - 16) \tag{3.41'}$$

(use (3.6) of Definitions 3.3 and observe that $\omega(U \setminus Y) = \omega(U) - 1 = n - 1$). So

$$\begin{aligned}\phi(U \setminus Y.2) &= ((2(n-1)+1)^2 - 1 - \mathcal{L}_U^2 - 2\mathcal{L}_U)(16\mathcal{L}_U^{-3} + 50\mathcal{L}_U^{-2} + 11i\mathcal{L}_U^{-3} - 13i + 5i\mathcal{L}_U) \\ &\quad + \frac{(\mathcal{L}_U - 2(n-1))}{2}(8\mathcal{L}_U - 8i - 16)\end{aligned} \tag{3.42}$$

(use (3.41) and (3.41′) and notice [by observing that $\mathcal{L}_U \leq 2n$ and by using Remark 3.7] that $\mathcal{L}_U = \mathcal{L}_{U\setminus Y}$). Clearly

$$\begin{aligned}\phi(U \setminus Y.2) &= ((2n+1)^2 - 1 - \mathcal{L}_U^2 - 2\mathcal{L}_U)(16\mathcal{L}_U^{-3} + 50\mathcal{L}_U^{-2} + 11i\mathcal{L}_U^{-3} - 13i + 5i\mathcal{L}_U) \\ &\quad + \frac{(\mathcal{L}_U - 2n)}{2}(8\mathcal{L}_U - 8i - 16) + \phi(U \setminus Y.2)'\end{aligned} \tag{3.43}$$

where

$$\phi(U \setminus Y.2)' = -8n(16\mathcal{L}_U^{-3} + 50\mathcal{L}_U^{-2} + 11i\mathcal{L}_U^{-3} - 13i + 5i\mathcal{L}_U) + 8\mathcal{L}_U - 8i - 16 \tag{3.43'}$$

(use the first member of (3.42) and observe [by elementary computation and by using the first member of (3.42)] that

$$((2(n-1)+1)^2 - 1 - \mathcal{L}_U^2 - 2\mathcal{L}_U) = ((2n+1)^2 - 1 - \mathcal{L}_U^2 - 2\mathcal{L}_U) - 8n,$$

and remark [by elementary computation and by using the second member of (3.42)] that

$$\frac{(\mathcal{L}_U - 2(n-1))}{2} = \frac{(\mathcal{L}_U - 2n)}{2} + 1)$$

. So

$$\phi(U \setminus Y.2) = \phi(U.2) - 8n(16\mathcal{L}_U^{-3} + 50\mathcal{L}_U^{-2} + 11i\mathcal{L}_U^{-3} - 13i + 5i\mathcal{L}_U) + 8\mathcal{L}_U - 8i - 16 \quad (3.44)$$

(use (3.6) of Definitions 3.3 and (3.43) and (3.43′)). Clearly

$$\phi(U \setminus Y.2) - \phi(U.2) = -8n(16\mathcal{L}_U^{-3} + 50\mathcal{L}_U^{-2} + 11i\mathcal{L}_U^{-3} - 13i + 5i\mathcal{L}_U) + 8\mathcal{L}_U - 8i - 16$$

(use (3.44)). Property (3.8.2) immediately follows.
(3.8.3) Indeed let $\phi(U \setminus Y.3)$; clearly

$$\phi(U\setminus Y.3) = ((2(n-1)+1)^2 - 1 - \mathcal{L}_{U\setminus Y}^2 + 2\mathcal{L}_{U\setminus Y})(34i\mathcal{L}_{U\setminus Y}^{-2} - 70i\mathcal{L}_{U\setminus Y}^{-3} - 11i + 5 + 6i\mathcal{L}_{U\setminus Y}) + \frac{289080}{1331} \quad (3.45)$$

(use (3.7) of Definitions 3.3 and observe that $\omega(U \setminus Y) = \omega(U) - 1 = n - 1$). So

$$\phi(U\setminus Y.3) = ((2(n-1)+1)^2 - 1 - \mathcal{L}_U^2 + 2\mathcal{L}_U)(34i\mathcal{L}_U^{-2} - 70i\mathcal{L}_U^{-3} - 11i + 5 + 6i\mathcal{L}_U) + \frac{289080}{1331} \quad (3.46)$$

(use (3.45) and notice [by observing that $\mathcal{L}_U \leq 2n$ and by using Remark 3.7] that $\mathcal{L}_U = \mathcal{L}_{U\setminus Y}$). Clearly

$$\phi(U\setminus Y.3) = ((2n+1)^2 - 1 - \mathcal{L}_U^2 + 2\mathcal{L}_U)(34i\mathcal{L}_U^{-2} - 70i\mathcal{L}_U^{-3} - 11i + 5 + 6i\mathcal{L}_U) + \frac{289080}{1331} + \phi(U\setminus Y.3)' \quad (3.47)$$

where

$$\phi(U \setminus Y.3)' = -8n(34i\mathcal{L}_U^{-2} - 70i\mathcal{L}_U^{-3} - 11i + 5 + 6i\mathcal{L}_U) \quad (3.48)$$

(use (3.46) and observe [by elementary computation] that $((2(n-1)+1)^2 - 1 - \mathcal{L}_U^2 + 2\mathcal{L}_U) = ((2n+1)^2 - 1 - \mathcal{L}_U^2 + 2\mathcal{L}_U) - 8n$). So

$$\phi(U \setminus Y.3) = \phi(U.3) - 8n(34i\mathcal{L}_U^{-2} - 70i\mathcal{L}_U^{-3} - 11i + 5 + 6i\mathcal{L}_U) \quad (3.49)$$

(use (3.47) and (3.48) and (3.7) of Definitions 3.3) and clearly

$$\phi(U \setminus Y.3) - \phi(U.3) = -8n(34i\mathcal{L}_U^{-2} - 70i\mathcal{L}_U^{-3} - 11i + 5 + 6i\mathcal{L}_U)$$

(use (3.49)). Property (3.8.3) immediately follows.
(3.8.4) Indeed let $\phi(U \setminus Y.4)$; clearly

$$\begin{aligned}\phi(U \setminus Y.4) \;=\; & ((2(n-1)+1)^2 - 1 - \mathcal{L}_{U\setminus Y}^2 + 7\mathcal{L}_{U\setminus Y}) \\ & (11i\mathcal{L}_{U\setminus Y} + 7 - 50\mathcal{L}_{U\setminus Y}^{-1} + 23i\mathcal{L}_{U\setminus Y}^{-3} - 54\mathcal{L}_{U\setminus Y}^{-3}) \\ & -\frac{1003112i}{1331} \end{aligned} \quad (3.50)$$

(use (3.8) of Definitions 3.3 and observe that $\omega(U \setminus Y) = \omega(U) - 1 = n - 1$). So

$$\begin{aligned}\phi(U \setminus Y.4) \;=\; & ((2(n-1)+1)^2 - 1 - \mathcal{L}_U^2 + 7\mathcal{L}_U)(11i\mathcal{L}_U + 7 - 50\mathcal{L}_U^{-1} + 23i\mathcal{L}_U^{-3} - 54\mathcal{L}_U^{-3}) \\ & -\frac{1003112i}{1331}\end{aligned} \quad (3.51)$$

(use (3.50) and notice [by observing that $\mathcal{L}_U \leq 2n$ and by using Remark 3.7] that $\mathcal{L}_U = \mathcal{L}_{U \setminus Y}$). Clearly

$$\phi(U \setminus Y.4) = ((2n+1)^2 - 1 - \mathcal{L}_U^2 + 7\mathcal{L}_U)(11i\mathcal{L}_U + 7 - 50\mathcal{L}_U^{-1} + 23i\mathcal{L}_U^{-3} - 54\mathcal{L}_U^{-3}) - \frac{1003112i}{1331} + \phi(U \setminus Y.4)' \quad (3.52)$$

where

$$\phi(U \setminus Y.4)' = -8n(11i\mathcal{L}_U + 7 - 50\mathcal{L}_U^{-1} + 23i\mathcal{L}_U^{-3} - 54\mathcal{L}_U^{-3}) \quad (3.53)$$

(use (3.52) and observe [by elementary computation and by using (3.52)] that

$$((2(n-1)+1)^2 - 1 - \mathcal{L}_U^2 + 7\mathcal{L}_U) = ((2n+1)^2 - 1 - \mathcal{L}_U^2 + 7\mathcal{L}_U) - 8n).$$

So

$$\phi(U \setminus Y.4) = \phi(U.4) - 8n(11i\mathcal{L}_U + 7 - 50\mathcal{L}_U^{-1} + 23i\mathcal{L}_U^{-3} - 54\mathcal{L}_U^{-3}) \quad (3.54)$$

(use (3.52) and (3.53) and (3.8) of Definitions 3.3) and clearly

$$\phi(U \setminus Y.4) - \phi(U.4) = -8n(11i\mathcal{L}_U + 7 - 50\mathcal{L}_U^{-1} + 23i\mathcal{L}_U^{-3} - 54\mathcal{L}_U^{-3})$$

(use (3.54)). Property (3.8.4) follows and Remark 3.8 immediately follows.□

Remark 3.9. *Let n be an integer ≥ 2 and let U be uniform such that $\omega(U) = n$; consider $\mathcal{L}_U$ (use Recalls and Definitions 3.2). Now let $\Xi(U)$ be the canonical coloration of U (use Definition 2.3), and let $Y \in \Xi(U)$; consider $(U \setminus Y, \mathcal{L}_{U\setminus Y})$ [where $U \setminus Y$ is the induced subgraph of U by $V(U) \setminus Y$ and $\mathcal{L}_{U\setminus Y} = 2b(U \setminus Y) + 2$]. Look at $(\nu(U), \epsilon(U))$ introduced in Definitions 3.3, and via $(\nu(U), \epsilon(U))$, consider $(\nu(U \setminus Y), \epsilon(U \setminus Y))$ (this consideration gets sense, since U is uniform such that $\omega(U) = n$ with $n \geq 2$ and therefore $U \setminus Y$ is uniform such that $\omega(U \setminus Y) = n - 1$ [use property (2.11.4) of Assertion 2.11], with $n - 1 \geq 1$). If $\mathcal{L}_U \leq 2n$, then we have the following two properties.*
(3.9.1.) $\nu(U \setminus Y) - \nu(U) = 32n - 8\mathcal{L}_U + 8i + 16$.
(3.9.2.) $\epsilon(U \setminus Y) - \epsilon(U) = -32n - 8\mathcal{L}_U + 8i + 16$.

Proof. Indeed let $(\nu(U \setminus Y), \epsilon(U \setminus Y))$; clearly

$$\nu(U \setminus Y) = (i\mathcal{L}_{U\setminus Y} - 4i(n-1) - 4i + 1)^2 - 1 + \mathcal{L}_{U\setminus Y}^2 \quad (3.55)$$

(use (3.9) of Definitions 3.3 and observe that $\omega(U \setminus Y) = \omega(U) - 1 = n - 1$) and

$$\epsilon(U \setminus Y) = 4(\,(2(n-1)+1)^2 - 1 - \mathcal{L}_{U\setminus Y}^2 + \mathcal{L}_{U\setminus Y}\,) + (\,i\mathcal{L}_{U\setminus Y} - 2i(n-1) + 1\,)(\,4i\mathcal{L}_{U\setminus Y} + 4 - 8i\,) \quad (3.56)$$

(use (3.10) of Definitions 3.3 and observe that $\omega(U \setminus Y) = \omega(U) - 1 = n - 1$). So

$$\nu(U \setminus Y) = (i\mathcal{L}_U - 4i(n-1) - 4i + 1)^2 - 1 + \mathcal{L}_U^2 \quad (3.57)$$

(use (3.55) and notice [by observing that $\mathcal{L}_U \leq 2n$ and by using Remark 3.7] that $\mathcal{L}_U = \mathcal{L}_{U \setminus Y}$) and

$$\epsilon(U \setminus Y) = 4(\ (2(n-1)+1)^2 - 1 - \mathcal{L}_U^2 + \mathcal{L}_U\ ) + (\ i\mathcal{L}_U - 2i(n-1) + 1\ )(\ 4i\mathcal{L}_U + 4 - 8i\ ) \quad (3.58)$$

(use (3.56) and notice [by observing that $\mathcal{L}_U \leq 2n$ and by using Remark 3.7] that $\mathcal{L}_U = \mathcal{L}_{U \setminus Y}$).
That being said, we now prove easily property (3.9.1) and property (3.9.2).
(3.9.1.) Indeed observing (by elementary computation and the fact that $i^2 = -1$) that

$$(\ i\mathcal{L}_U - 4i(n-1) - 4i + 1\ )^2 - 1 + \mathcal{L}_U^2 = (\ i\mathcal{L}_U - 4in - 4i + 1\ )^2 - 1 + \mathcal{L}_U^2 + 32n - 8\mathcal{L}_U + 8i + 16 \quad (3.59)$$

then clearly

$$\nu(U \setminus Y) = (\ i\mathcal{L}_U - 4in - 4i + 1\ )^2 - 1 + \mathcal{L}_U^2 + 32n - 8\mathcal{L}_U + 8i + 16 \quad (3.60)$$

(use (3.57) and (3.59)). So

$$\nu(U \setminus Y) = \nu(U) + 32n - 8\mathcal{L}_U + 8i + 16 \quad (3.61)$$

(use (3.60) and (3.9) of Definitions 3.3) and clearly

$$\nu(U \setminus Y) - \nu(U) = 32n - 8\mathcal{L}_U + 8i + 16$$

(use (3.61)). Property (3.9.1.) follows.
(3.9.2.) Indeed observing (by elementary computation and the fact that $i^2 = -1$) that

$$4(\ (2(n-1)+1)^2 - 1 - \mathcal{L}_U^2 + \mathcal{L}_U\ ) = 4(\ (2n+1)^2 - 1 - \mathcal{L}_U^2 + \mathcal{L}_U\ ) - 32n \quad (3.62)$$

and

$$(\ i\mathcal{L}_U - 2i(n-1) + 1\ )(\ 4i\mathcal{L}_U + 4 - 8i\ ) = (i\mathcal{L}_U - 2in + 1)(4i\mathcal{L}_U + 4 - 8i) + 2i(4i\mathcal{L}_U + 4 - 8i) \quad (3.63),$$

then clearly

$$\epsilon(U \setminus Y) = 4(\ (2n+1)^2 - 1 - \mathcal{L}_U^2 + \mathcal{L}_U\ ) + (\ i\mathcal{L}_U - 2in + 1\ )(\ 4i\mathcal{L}_U + 4 - 8i\ ) - 32n + 2i(\ 4i\mathcal{L}_U + 4 - 8i\ ) \quad (3.64)$$

(use (3.58) and (3.62) and (3.63)). So

$$\epsilon(U \setminus Y) = \epsilon(U) - 32n + 2i(\ 4i\mathcal{L}_U + 4 - 8i\ ) \quad (3.65)$$

(use (3.64) and (3.10) of Definitions 3.3). Clearly

$$\epsilon(U \setminus Y) - \epsilon(U) = -32n - 8\mathcal{L}_U + 8i + 16$$

(use (3.65) and observe [by elementary computation and the fact that $i^2 = -1$] that

$$-32n + 2i(\ 4i\mathcal{L}_U + 4 - 8i\ ) = -32n - 8\mathcal{L}_U + 8i + 16).$$

Property $(3.9.2)$ follows and Remark 3.9 immediately follows. □

Remark 3.10. *Let n be an integer ≥ 2 and let U be uniform such that $\omega(U) = n$. Now let $\Xi(U)$ be the canonical coloration of U (use Definition 2.3), and let $Y \in \Xi(U)$; consider $U \setminus Y$ [where $U \setminus Y$ is the induced subgraph of U by $V(U) \setminus Y$]. Look at $\phi(U)$ introduced in Definitions 3.3, and via $\phi(U)$, consider $\phi(U \setminus Y)$ (this consideration gets sense, since U is uniform such that $\omega(U) = n$ with $n \geq 2$ and therefore $U \setminus Y$ is uniform such that $\omega(U \setminus Y) = n - 1$ [use property $(2.11.4)$ of Assertion 2.11], with $n - 1 \geq 1$). Then*

$$\phi(U \setminus Y) - \phi(U) = \sum_{j=1}^{4}(\phi(U \setminus Y.j) - \phi(U.j)).$$

Proof. Indeed let $\phi(U)$; clearly

$$\phi(U \setminus Y) = \sum_{j=1}^{4} \phi(U \setminus Y.j) \tag{3.66}$$

(use (3.4) of Definitions 3.3). So

$$\phi(U \setminus Y) - \phi(U) = \sum_{j=1}^{4} \phi(U \setminus Y.j) - (\sum_{j=1}^{4} \phi(U.j)) \tag{3.67}$$

(use (3.66) and (3.4) of Definitions 3.3) and clearly

$$\phi(U \setminus Y) - \phi(U) = \sum_{j=1}^{4}(\phi(U \setminus Y.j) - \phi(U.j))$$

(use (3.67)). Remark 3.10 follows.□

Now using the previous four Remarks, then Proposition 3.6 becomes elementary to prove.

Proof of Proposition 3.6. Indeed look $(\, \phi(U \setminus Y) - \phi(U), \nu(U \setminus Y) - \nu(U), \epsilon(U \setminus Y) - \epsilon(U)\,)$. Clearly

$$\phi(U \setminus Y) - \phi(U) = \sum_{j=1}^{4}(\phi(U \setminus Y.j) - \phi(U.j)) \tag{3.68}$$

(use Remark 3.10), where

$$\phi(U \setminus Y.1) - \phi(U.1) = 0 \tag{3.69}$$

(use property $(3.8.1)$ of Remark 3.8), and

$$\phi(U \setminus Y.2) - \phi(U.2) = -8n(16\mathcal{L}_U^{-3} + 50\mathcal{L}_U^{-2} + 11i\mathcal{L}_U^{-3} - 13i + 5i\mathcal{L}_U) + 8\mathcal{L}_U - 8i - 16 \tag{3.70}$$

(use property (3.8.2) of Remark 3.8), and

$$\phi(U \setminus Y.3) - \phi(U.3) = -8n(34i\mathcal{L}_U^{-2} - 70i\mathcal{L}_U^{-3} - 11i + 5 + 6i\mathcal{L}_U) \quad (3.71)$$

(use property (3.8.3) of Remark 3.8), and

$$\phi(U \setminus Y.4) - \phi(U.4) = -8n(11i\mathcal{L}_U + 7 - 50\mathcal{L}_U^{-1} + 23i\mathcal{L}_U^{-3} - 54\mathcal{L}_U^{-3}) \quad (3.72)$$

(use property (3.8.4) of Remark 3.8), and

$$\nu(U \setminus Y) - \nu(U) = 32n - 8\mathcal{L}_U + 8i + 16 \quad (3.73)$$

(use property (3.9.1) of Remark 3.9), and

$$\epsilon(U \setminus Y) - \epsilon(U) = -32n - 8\mathcal{L}_U + 8i + 16 \quad (3.74)$$

(use property (3.9.2) of Remark 3.9). That being so, let (x, y, k) such that

$$x = -8\mathcal{L}_U + 8i + 16;\ y = 32n;\ and\, k = 16n\mathcal{L}_U^{-3} - 16n\mathcal{L}_U^{-2} + 16n\mathcal{L}_U^{-1} \quad (3.75).$$

Now let $(\ \phi(U \setminus Y) - \phi(U), \nu(U \setminus Y) - \nu(U), \epsilon(U \setminus Y) - \epsilon(U), x, y, k\ )$ where (x, y, k) is explicted in (3.75) and where

$$(\ \phi(U \setminus Y) - \phi(U), \nu(U \setminus Y) - \nu(U), \epsilon(U \setminus Y) - \epsilon(U))$$

is explicited above (use (3.68) for $\phi(U \setminus Y) - \phi(U)$; and (3.73) for $\nu(U \setminus Y) - \nu(U)$; and (3.74) for $\epsilon(U \setminus Y) - \epsilon(U)$). Now consider $(\nu(U \setminus Y) - \nu(U), \epsilon(U \setminus Y) - \epsilon(U), x, y\ )$; then using (3.73) and (3.74) and the first two equalities of (3.75), we easily check (by elementary computation and the fact that $i^2 = -1$) that

$$x + y - (\nu(U \setminus Y) - \nu(U)) = 0\, and\, x - y - (\epsilon(U \setminus Y) - \epsilon(U)) = 0 \quad (3.76).$$

That being so, look again at $(\ \phi(U \setminus Y) - \phi(U), \nu(U \setminus Y) - \nu(U), \epsilon(U \setminus Y) - \epsilon(U), x, y, k\ )$ and consider $(\ \phi(U \setminus Y) - \phi(U), x, y, k\ )$; using the three equalities of (3.75), then it becomes very easy to check (by elementary computation and the fact that $i^2 = -1$) that

$$x + 3iy\mathcal{L}_U^{-1} + k(6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i) + \phi(U \setminus Y) - \phi(U) = 0;\ k \in \mathcal{R}$$

and

$$\phi(U \setminus Y) - \phi(U) = \sum_{j=1}^{4}(\phi(U \setminus Y.j) - \phi(U.j)) \quad (3.77)$$

(use (3.75) for (x, y, k); (3.69) for $\phi(U \setminus Y.1) - \phi(U.1)$; (3.70) for $\phi(U \setminus Y.2) - \phi(U.2)$; (3.71) for $\phi(U \setminus Y.3) - \phi(U.3)$; (3.72) for $\phi(U \setminus Y.4) - \phi(U.4)$; and (3.68) for $\phi(U \setminus Y) - \phi(U)$). Clearly $(\ \phi(U \setminus Y) - \phi(U), \nu(U \setminus Y) - \nu(U), \epsilon(U \setminus Y) - \epsilon(U)\ )$ tackles $(1, 3i\mathcal{L}_U^{-1})$ around $6\mathcal{L}_U - 19 - i\mathcal{L}_U + 11i\mathcal{L}_U^2 - 18i$ (use (3.76) and (3.77) and the notion of tackle introduced in Definitions and Recalls 3.2 [observe that $(x, y) \in \mathcal{C}^2$ and $k \in \mathcal{R}$; so $(x, y, k) \in \mathcal{C}^2 \times \mathcal{R}$]). Proposition 3.6 immediately follows. □

References

[1] I. Annouk, Around the Hadwiger conjecture and the Berge problem, *International Journal of Mathematical Combinatorics* **3** (2012), 72–82.

[2] C. Berge, *Graphs (Chap.* 16*), Third revised edition*, North Holland Mathematical Library, (1991).

[3] M. Chudnovsky, N. Roberston, P. Seymour and R. Thomas, The strong perfect graph theorem, *Annals of Mathematics* **164(1)** (2006), 51–229.