Indonesian Journal of Combinatorics **7** (2) (2023), 88–93

INDONESIAN JOURNAL OF COMBINATORICS

On inclusive distance vertex irregularity strength of book graphs

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Abstract

The concept of distance vertex irregular labeling of graphs was introduced by Slamin in 2017. The *distance vertex irregular labeling* on a graph G with v vertices is defined as an assignment $f : V \to \{1, 2, \cdots, k\}$ so that the weights calculated at vertices are distinct. The weight of a vertex x in G is defined as the sum of the labels of all the vertices adjacent to x (distance 1 from x). The distance vertex irregularity strength of graph G, denoted by $dis(G)$, is defined as the minimum value of the largest label k over all such irregular assignments. Bong, Lin and Slamin generalized the concept to inclusive and non-inclusive distance vertex irregular labeling. The difference between them depends on the way to calculate the weight on the vertex whether the vertex label we calculate its weight is included or not. The *inclusive distance vertex irregularity strength* of G, denoted by $\widehat{\mathrm{dis}}(G)$ is defined as the minimum of the largest label k over all such inclusive irregular assignments. In this paper, we determine the inclusive distance vertex irregularity strength of the book graph.

Keywords: inclusive distance vertex irregular labeling, inclusive distance vertex irregularity strength, book graph
Mathematics Subject Classification: 05C78

1. Introduction

Let $G(V, E)$ be a simple, finite and undirected graph with V vertices and E edges. Irregular labeling is a given label or mapping set of number (usually positive integers), not necessarily distinct, such that the weight of each vertex graph G is different. In 2014, Slamin introduced the

Received: 19 July 2020, Revised: 28 June 2022, Accepted: 18 July 2022.

concept of the distance irregular labeling of a graph. The distance from a vertex u to a vertex v in the graph G is defined to be the length of the shortest path between vertices u and v. If vertices u and v are adjacent, then the distance between u and v is 1. The distance irregular labeling of a graph G is a mapping $f : V(G) \to \{1, 2, 3, \cdots, k\}$ such that the weights calculated at vertices are distinct. The weight of the vertex x, denoted by $w(x)$, is the sum of the labels of all vertices adjacent to vertex x, that is,

$$w(x) = \sum_{y \in N(x)} f(y)$$

where $N(x)$ is the open neighborhood of x, i.e., the set of vertices adjacent to x in G. The minimum k for which the graph G has the distance irregular labeling is called the distance vertex irregularity strength of G and denoted by $dis(G)$.

A labeling f of a graph G is called an inclusive irregular distance labeling, if there are no two vertices having the same weight, where the weight of a vertex x is now calculated by summing all the vertices in the close neighborhood of x, that is,

$$w(x) = f(x) + \sum_{y \in N(x)} f(y)$$

where $N[x]$ is the close neighborhood of x, i.e., the set of vertex x and all its adjacent vertices. The minimum k for which the graph G has the inclusive distance vertex irregular labeling is called the inclusive distance vertex irregularity strength of G and denoted by $\widehat{\text{dis}}(G)$. This variation of the distance irregular labeling was introduced by Bača *et al.* [2].

The lower bound on the inclusive distance vertex irregularity strength given by Bong *et al.* [3] as follows.

Lemma 1.1. *Let G be a graph with minimum degree δ and maximum degree Δ. Then*

$$\widehat{\text{dis}}(G) \geq \lceil \frac{\delta + |V(G)|}{\Delta + 1} \rceil$$

Susanto *et al.* [4] proposed the lower bound on the inclusive distance vertex irregularity strength of a graph G by considering the degrees of vertices in G in the following theorem.

Theorem 1.1. *Let G be a graph with minimum degree δ and maximum degree Δ. let n_i be the number of vertices of degree i in G for every $\delta \leq i \leq \Delta$*

$$\widehat{dis}(G) \geq \max_{\delta \leq i \leq \Delta} \{ \lceil \frac{\delta + \sum_{j=\delta}^{i} n_j}{i + 1} \rceil \}$$

In this paper, we determine the inclusive distance vertex irregularity strength of a book graph. The book graph, denoted by B_m, is defined as the graph obtained by Cartesian product of a star S_m for $m \geq 3$ and a path on two nodes P_2. The vertex set of the book graph is

$$V(B_m) = \{x_1, x_2\} \cup \{y_i, z_i : 1 \leq i \leq m\}$$

and the edge set of book graph is

$$E(B_m) = \{x_1 x_2\} \cup \{x_1 y_i, x_2 z_i, y_i z_i : 1 \leq i \leq m\}.$$

2. Result

We start this section with the lower bound on the inclusive distance vertex irregularity strength of the book graph B_m for $m \geq 3$ as presented in the following corollary.

Corollary 2.1. *If B_m, for $m \geq 3$, is the book graph, then*

$$\widehat{dis}(B_m) \geq \lceil \frac{2m+2}{3} \rceil$$

Proof. Let B_m be a book graph for $m \geq 3$. Then the minimum degree of B_m is 2 and the maximum degree of B_m is m. Furthermore, the number of vertices of degree 2 in B_m is $2m$, while the number of vertices of degree m in B_m is 2. By Theorem 1.1, we have

$$\widehat{dis}(G) \geq \max\{\lceil \frac{2m+2}{3} \rceil, \lceil \frac{2m+2}{4} \rceil, \ldots, \lceil \frac{2m+2}{m} \rceil, \lceil \frac{4}{m+1} \rceil\}$$

Thus, we conclude that $\widehat{dis}(G) \geq \lceil \frac{2m+2}{3} \rceil$. □

The following two lemmas present the properties of the inclusive distance vertex irregular labeling for vertices in the book graph B_m for $m \geq 3$.

Lemma 2.1. *Let B_m be a book graph for $m \geq 3$. If f is an inclusive distance vertex irregular labeling of B_m, then $f(x_1) \neq f(x_2)$, for $x_1, x_2 \in V(B_m)$.*

Proof. Let B_m be a book graph for $m \geq 3$. Then the vertex set of B_m is $V(B_m) = \{x_1, x_2\} \cup \{y_i, z_i : 1 \leq i \leq m\}$. Let f be an inclusive distance vertex irregular labeling of B_m. Then the weight of vertices y_i and z_i, for $1 \leq i \leq m$, are $w(y_i) = f(y_i) + f(z_i) + f(x_1)$ and $w(z_i) = f(z_i) + f(y_i) + f(x_2)$, respectively. Since the weight of every vertex in B_m must be distinct, then $w(y_i) \neq w(z_i)$. Consequently $f(x_1) \neq f(x_2)$. □

Lemma 2.2. *Let B_m be a book graph for $m \geq 3$. If f is an inclusive distance vertex irregular labeling of B_m, then $f(y_i) + f(z_i) \neq f(y_j) + f(z_j)$, for $y_i, z_i, y_j, z_j \in V(B_m)$ and $1 \leq i \neq j \leq m$.*

Proof. Let B_m be a book graph for $m \geq 3$. Then the vertex set of B_m is $V(B_m) = \{x_1, x_2\} \cup \{y_i, z_i : 1 \leq i \leq m\}$. Let f be an inclusive distance vertex irregular labeling of B_m. Then the weight of vertices y_i and z_j, for $1 \leq i \neq \leq m$, are $w_f(y_i) = f(y_i) + f(z_i) + f(x_1)$ and $w_f(y_j) = f(y_j) + f(z_j) + f(x_1)$, respectively. Since the weight of every vertex in B_m must be distinct, then $w(y_i) \neq w(y_j)$. Consequently $f(y_i) + f(z_i) \neq f(y_j) + f(z_j)$. □

By the corollary and two lemmas above, we now present the main result of the paper in the following theorem.

Theorem 2.1. *If B_m, for $m \geq 3$, is the book graph, then the inclusive distance vertex irregularity strength of B_m is*

$$\lceil \frac{2m+2}{3} \rceil \leq \widehat{dis}(B_m) \leq \begin{cases} m+1, & \text{if } m = 3 \\ m, & \text{if } m > 3 \end{cases}$$

Proof. Let B_m be a book graph for $m \geq 3$. They Corollary 2.1, we have the lower bound of the inclusive distance vertex irregularity strength of B_m,

$$\widehat{dis}(B_m) \geq \lceil \frac{2m+2}{3} \rceil$$

We now prove the upper bound of the inclusive distance vertex irregularity strength of B_m for $m \geq 3$. Let $V(B_m) = \{x_1, x_2\} \cup \{y_i, z_i : 1 \leq i \leq m\}$ be the vertex set of B_m. There are two cases to be considered, namely, for $m = 3$ and $m > 3$.

Case 1, for $m = 3$. Let f be a labeling of vertices in B_m as defined in the following formula.

$$\begin{aligned} f(x_1) &= 2, \\ f(x_2) &= 3, \\ f(y_i) &= i, && \text{for } 1 \leq i \leq m, \\ f(z_i) &= i+1, && \text{for } 1 \leq i \leq m. \end{aligned}$$

The corresponding weights are:

$$\begin{aligned} w(x_1) &= f(x_1) + f(x_2) + \sum_{i=1}^{m} f(y_i) = \frac{m(m+1)}{2} + 5, \\ w(x_2) &= f(x_1) + f(x_2) + \sum_{i=1}^{m} f(z_i) = \frac{m(m+3)}{2} + 5, \\ w(y_i) &= f(y_i) + f(z_i) + f(x_1) = 2i+3, \quad \text{for } 1 \leq i \leq m, \\ w(z_i) &= f(y_i) + f(z_i) + f(x_2) = 2i+4, \quad \text{for } 1 \leq i \leq m \end{aligned}$$

Thus, for $m = 3$, the labeling f gives the weight of vertices in B_m are $\{5, 6, 7, 8, 9, 10, 11, 14\}$. This results in that $\widehat{dis}(B_m) \leq m + 1$.

Case 2, for $m > 3$. Let f be a labeling of vertices in B_m as defined in the following formula.

$$\begin{aligned} f(x_1) &= 1, \\ f(x_2) &= m-1, \\ f(y_i) &= \begin{cases} \lceil \frac{i}{2} \rceil, & \text{for } 1 \leq i \leq m-2 \\ m-1, & \text{for } m-1 \leq i \leq m \end{cases} \\ f(z_i) &= \begin{cases} \lceil \frac{i+1}{2} \rceil, & \text{for } 1 \leq i \leq m-2 \\ i, & \text{for } m-1 \leq i \leq m \end{cases} \end{aligned}$$

The corresponding weights are:

$$w(x_1) = \begin{cases} f(x_1) + f(x_2) + \sum_{i=1}^{m-2} f(y_i) + \sum_{i=m-1}^{m} f(y_i) = \frac{m(m-2)}{4} + 3m - 2, & \text{for even } m \\ f(x_1) + f(x_2) + \sum_{i=1}^{m-2} f(y_i) + \sum_{i=m-1}^{m} f(y_i) = \frac{(m-3)(m-1)+2(m-1)}{4} + 3m - 2, & \text{for odd } m \end{cases}$$

$$w(x_2) = \begin{cases} f(x_1) + f(x_2) + \sum_{i=1}^{(m-2)} f(z_i) + \sum_{i=m-1}^{m} f(z_i) = \frac{(m-4)(m+2)+2m}{4} + 3m, & \text{for even } m \\ f(x_1) + f(x_2) + \sum_{i=1}^{(m-2)} f(z_i) + \sum_{i=m-1}^{m} f(z_i) = \frac{(m-3)(m+3)}{4} + 3m, & \text{for odd } m \end{cases}$$

$$w(y_i) = \begin{cases} f(y_i) + f(z_i) + f(x_1) = i + 2, & \text{for } 1 \le i \le m-2 \\ f(y_i) + f(z_i) + f(x_1) = m + i, & \text{for } m-1 \le i \le m \end{cases}$$

$$w(z_i) = \begin{cases} f(y_i) + f(z_i) + f(x_2) = m + i, & \text{for } 1 \le i \le m-2 \\ f(y_i) + f(z_i) + f(x_2) = 2m - 2 + i, & \text{for } m-1 \le i \le m. \end{cases}$$

Thus, for $m > 3$, the labeling f gives the weight of vertices in B_m are distinct. This results in that $\widehat{dis}(B_m) \le m$. We conclude that

$$\lceil \frac{2m+2}{3} \rceil \le \widehat{dis}(B_m) \le \begin{cases} m+1, & \text{if } m = 3 \\ m, & \text{if } m > 3 \end{cases}$$

□

3. Conclusion

We have determined the lower and the upper bounds of the book graph B_m, for $m \ge 3$. We conclude this paper with the following open problem.

Problem 1. *Determine the exact value of the inclusive distance vertex irregularity strength of the book graph B_m, for $m \ge 3$.*

Acknowledgement

This research is supported by DRPM Ditjen Penguatan Risbang Kemenristekdikti under World Class Research 2019 research grant.

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