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# The degree sequences of a graph with restrictions

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**Abstract**

Two finite sequences $s_1$ and $s_2$ of nonnegative integers are called bigraphical if there exists a bipartite graph $G$ with partite sets $V_1$ and $V_2$ such that $s_1$ and $s_2$ are the degrees in $G$ of the vertices in $V_1$ and $V_2$, respectively. In this paper, we introduce the concept of 1-graphical sequences and present a necessary and sufficient condition for a sequence to be 1-graphical in terms of bigraphical sequences.

*Keywords:* degree, degree sequence, graphical sequence, bigraphical sequence, 1-graphical sequence
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## 1. Introduction

We generally follow the notation and terminology pertaining to graphs of [1]. If $F$ is a nonempty subset of the edge set $E(G)$ of a graph $G$, then the subgraph $\langle F \rangle$ *induced by* $F$ is

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the graph whose vertex set consists of those vertices of $G$ incident with at least one edge of $F$ and whose edge set is $F$.

The *degree of* a *vertex* $v$ in a graph $G$ is the number of edges of $G$ incident with $v$, which is denoted by $\deg v$. A vertex is called *even* or *odd* according to whether its degree is even or odd.

A sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers is called a *degree sequence* of a graph $G$ if the vertices of $G$ can be labeled $v_1, v_2, \ldots, v_n$ so that $\deg v_i = d_i$ for all $i$. We adopt the convention that the vertices have been labeled so that $d_1 \geq d_2 \geq \cdots \geq d_n$. We call a sequence of nonnegative integers *graphical* if it is the degree sequence of some graph. A necessary and sufficient condition for a sequence to be graphical was found by Havel [4] and later rediscovered by Hakimi [3].

**Theorem 1.1.** *A sequence $s : d_1, d_2, \ldots, d_n$ of nonnegative integers with $d_1 \geq d_2 \geq \cdots \geq d_n, n \geq 2, d_1 \geq 1$ is graphical if and only if the sequence $s' : d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1$, $d_{d_1+2}, \ldots, d_n$ is graphical.*

According to the definition of a simple graph, two distinct vertices are joined by at most one edge. If we allow more than one edge (but a finite number) to join pairs of vertices, the resulting structure is called a *multigraph*. If two or more edges join the same two vertices in a multigraph, then these edges are referred to as *multiple edges*. Hakimi [3] extended the preceding result to multigraphs.

**Theorem 1.2.** *Let $d_1, d_2, \ldots, d_n$ be nonnegative integers with $d_1 \geq d_2 \geq \cdots \geq d_n$ and $n \geq 2$. Then there exists a multigraph with degree sequence $s : d_1, d_2, \ldots, d_n$ if and only if $\sum_{i=1}^{n} d_i$ is even and $d_1 \leq d_2 + d_3 + \cdots + d_n$.*

Two finite sequences $s_1$ and $s_2$ of nonnegative integers are called *bigraphical* if there exists a bipartite graph $G$ with partite sets $V_1$ and $V_2$ such that $s_1$ and $s_2$ are the degrees in $G$ of the vertices in $V_1$ and $V_2$, respectively. The following result is an analog of Theorem 1.1 for graphs (see [1, p. 16]).

**Theorem 1.3.** *The sequences $s_1 : a_1, a_2, \ldots, a_r$ and $s_2 : b_1, b_2, \ldots, b_t$ of nonnegative integers with $r \geq 2, a_1 \geq a_2 \geq \cdots \geq a_r$, $b_1 \geq b_2 \geq \cdots \geq b_t$, $0 < a_1 \leq t, 0 < b_1 \leq r$ are bigraphical if and only if $s'_1 : a_2, a_3, \ldots, a_r$ and $s'_2 : b_1 - 1, b_2 - 1, \ldots, b_{a_1} - 1$, $b_{a_1+1} - 1, \ldots, b_t$ are bigraphical.*

The *outdegree* $\text{od}\, v$ of a vertex $v$ of a digraph $D$ is the number of vertices of $D$ that are adjacent from $v$, while the *indegree* $\text{id}\, v$ of $v$ is the number of vertices of $D$ adjacent to $v$. The *degree* $\deg v$ of a vertex $v$ of $D$ is defined by

$$\deg v = \text{od}\, v + \text{id}\, v.$$

A *loop* is an edge that joins a vertex to itself and contributes to the degree of a vertex twice. A graph $G$ is called a $1$-*graph* if it has at most one loop attached at each vertex and at most two multiple edges joining each pair of vertices. A sequence $s$ is called $1$-*graphical* if there exists a 1-graph that realizes $s$.

For the sake of notational convenience, we will denote the interval of integers $x$ such that $a \leq x \leq b$ by simply writing $[a, b]$.

In this paper, we present a necessary and sufficient condition for a sequence to be 1-graphical in terms of bigraphical sequences. To this end, we use the following theorem, due to Veblen [7], which characterizes eulerian graphs in terms of their cycle structures.

**Theorem 1.4.** *A nontrivial connected graph $G$ is eulerian if and only if $E(G)$ can be partitioned into subsets $E_i$, $i \in [1,k]$, where each subgraph $\langle E_i \rangle$ is a cycle.*

To conclude this introduction, it is worth to mention that López and Muntaner-Batle [5] completely characterized the degree sequences of graphs with at most one loop attached at each vertex and no multiple edges. Hence, the work conducted in this paper would be a natural continuation of their work.

## 2. Characterization of 1-graphical sequences

We are now ready to state and prove the following theorem.

**Theorem 2.1.** *A sequence $s : d_1, d_2, \ldots, d_n$ of nonnegative integers with $d_1 \geq d_2 \geq \cdots \geq d_n$ and $n \geq 2$ is 1-graphical if and only if there exist bigraphical sequences $s_1 : a_1, a_2, \ldots, a_n$ and $s_2 : b_1, b_2, \ldots, b_n$ such that $a_i = b_i = d_i/2$ for even $d_i$ and $a_i + b_i = d_i$ for odd $d_i$, where $i \in [1,n]$.*

*Proof.* By assumption, there exists a 1-graph that realizes $s$. Assume that the vertices of $G$ are labeled $v_1, v_2, \ldots, v_n$ so that $\deg v_i = d_i$ for all $i$, and construct a new graph $H$ with vertex set

$$V(H) = V(G) \cup \{u\} \quad \text{and edge set} \quad E(H) = E(G) \cup \{uv_i \,|d_i \text{ is odd}\}.$$

Since in any graph, there is an even number of odd vertices, it follows that all vertices in $H$ are even vertices. Therefore, since every component of $H$ is eulerian, it follows from Theorem 1.4 that $E(H)$ can be partitioned into subsets $E_i$, $i \in [1,k]$, where each subgraph $\langle E_i \rangle$ is a cycle. If we orient each cycle in $\langle E_i \rangle$ cyclically, then we obtain a digraph $D$ with the property that $\mathrm{od}\ v = \mathrm{id}\ v$ for every $v \in V(D)$. Now, let $D'$ be the digraph obtained by deleting the vertex $u$ from $D$. Certainly, $d_i = \mathrm{od}\ v + \mathrm{id}\ v$ for each $v \in V(D')$. Furthermore, the vertices of $D'$ have the properties that $\mathrm{od}\ v = \mathrm{id}\ v = d_i/2$ for even $d_i$ and $|\mathrm{od}\ v - \mathrm{id}\ v| = 1$ for odd $d_i$, where $i \in [1,n]$. Let $A(D') = [\alpha_{ij}]$ be the adjacency matrix of $D'$, and construct the bipartite digraph $D^*$ with partite sets

$$X = \{x_i \,|1 \leq i \leq n\} \text{ and } Y = \{y_i \,|1 \leq i \leq n\},$$

and with the arcs in such a way that $(x_i, y_j) \in E(D^*)$ if and only if $\alpha_{ij} = 1$. It remains to observe that the sequences of outdegrees and of indegrees satisfy the required properties.

Let $G$ be the bipartite graph with partite sets

$$X = \{x_i \,|i \in [1,n]\} \text{ and } Y = \{y_i \,|i \in [1,n]\}$$

such that $s_1$ and $s_2$ are the degrees in $G$ of the vertices in $X$ and $Y$, respectively. Further, consider the digraph $D$ obtained from $G$, and let $[\beta_{ij}]$ be the $n \times n$ matrix with $\beta_{ij} = 1$ if and only if $(x_i, y_j)$ is an arc of $D$ and $\beta_{ij} = 0$ otherwise. Let $D'$ be the digraph with the vertex set $V(D') = \{w_i \,|1 \leq i \leq n\}$ and adjacency matrix $[\beta_{ij}]$ so that $\mathrm{od}\ w_i + \mathrm{id}\ w_i = a_i + b_i = d_i$. Then the graph obtained by replacing each arc $(u,v)$ of $D'$ by the edge of $uv$ is a 1-graph that realizes a sequence $s$. □

This result has the following consequences.

**Corollary 2.1.** *Let $s : d_1, d_2, \ldots, d_n$ be a sequence of nonnegative even integers with $d_1 \geq d_2 \geq \cdots \geq d_n$ and $n \geq 2$. Then $s$ is 1-graphical if and only if the sequences*

$$s_1 = s_2 : \frac{d_1}{2}, \frac{d_2}{2}, \ldots, \frac{d_n}{2}$$

*are bigraphical.*

**Corollary 2.2.** *Let $s : d_1, d_2, \ldots, d_n$ be a sequence of nonnegative integers with $d_1 \geq d_2 \geq \cdots \geq d_n$, $n \geq 2$ and the properties that $d_i = d_{i+1}$ for $k \leq i \leq k+l-1$, $d_i$ is even for all $i \in [1, k-1] \cup [k+l+1, n]$ and $d_i$ is odd for all $i \in [1, k+l]$. Then $s$ is 1-graphical if and only if the sequences*

$$\begin{aligned} s_1 \;=\; s_2 &: \frac{d_1}{2}, \frac{d_2}{2}, \ldots, \frac{d_{k-1}}{2}, \left\lceil \frac{d_k}{2} \right\rceil, \left\lceil \frac{d_{k+1}}{2} \right\rceil, \ldots, \left\lceil \frac{d_{k+(l-1)/2}}{2} \right\rceil, \\ &\left\lfloor \frac{d_{k+(l+1)/2}}{2} \right\rfloor, \ldots, \left\lfloor \frac{d_{k+l}}{2} \right\rfloor, \ldots, \frac{d_{k+l+1}}{2}, \frac{d_{k+l+2}}{2}, \ldots, \frac{d_n}{2} \end{aligned}$$

*are bigraphical.*

**Corollary 2.3.** *Let $s : d_1, d_2, \ldots, d_n$ be a sequence of nonnegative integers with $d_1 \geq d_2 \geq \cdots \geq d_n$, $n \geq 2$ and the properties that there exist some integers $k$ and $l$, $1 \leq k < l \leq n$, so that $d_k$ and $d_l$ are odd, and $d_i$ is even for all $i \in [1, n] \setminus \{k, l\}$. Then $s$ is 1-graphical if and only if the sequences*

$$s_1 : \frac{d_1}{2}, \frac{d_2}{2}, \ldots, \frac{d_{k-1}}{2}, \left\lceil \frac{d_k}{2} \right\rceil, \frac{d_{k+1}}{2}, \ldots, \frac{d_{l-1}}{2}, \left\lfloor \frac{d_l}{2} \right\rfloor, \frac{d_{l+1}}{2}, \ldots, \frac{d_n}{2}$$

*and*

$$s_2 : \frac{d_1}{2}, \frac{d_2}{2}, \ldots, \frac{d_{k-1}}{2}, \left\lfloor \frac{d_k}{2} \right\rfloor, \frac{d_{k+1}}{2}, \ldots, \frac{d_{l-1}}{2}, \left\lceil \frac{d_l}{2} \right\rceil, \frac{d_{l+1}}{2}, \ldots, \frac{d_n}{2}$$

*are bigraphical.*

## 3. Conclusions

The bipartite realization problem formulates as follows: Given two finite sequences $s_1$ and $s_2$ of nonnegative integers, is there a labeled bipartite graph such that the pair of $s_1$ and $s_2$ is the degree sequence of some bipartite graph? This classical decision problem belongs to the complexity class of P. This can be proven by two known approaches established in 1957 by Gale [2] and also by Ryser [6]. In this paper, we have extended Theorem 1.2 by presenting necessary and sufficient conditions for a sequence of nonnegative integers to be 1-graphical in terms of bigraphical sequences. These together with the bipartite realization problem imply that the decision problem associated with determining whether a given sequence of nonnegative integers is 1-graphical remains to be the complexity class of P.

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