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Edge irregular reflexive labeling on sun graph and corona of cycle and null graph with two vertices

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Abstract

Let $G(V, E)$ be a simple and connected graph which set of vertices is V and set of edges is E. Irregular reflexive k-labeling f on $G(V, E)$ is assignment that carries the numbers of integer to elements of graph, such that the positive integer $\{1, 2, 3, ..., k_e\}$ assignment to edges of graph and the even positive integer $\{0, 2, 4, ..., 2k_v\}$ assignment to vertices of graph. Then, we called as edge irregular reflexive k-labelling if every edges has different weight with $k = max\{k_e, 2k_v\}$. Besides that, there is definition of reflexive edge strength of $G(V, E)$ denoted as $res(G)$, that is a minimum k that using for labeling f on $G(V, E)$. This paper will discuss about edge irregular reflexive k-labeling for sun graph and corona of cycle and null graph, denoted by $C_n \odot N_2$ and make sure about their reflexive edge strengths.

Keywords: Edge irregular reflexive labeling, reflexive edge strength, sun graph, corona of cycle and null graph
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1. Introduction

All graph discuss on this paper are simple, connected, and undirected. First, let simplified $G(V,E)$ become G, vertex of graph denoted as v, and edge of graph denoted as e for easily on investigation. By definition from Wallis and Marr [10], graph labeling is a map that carries graph elements to numbers, usually to positive integer. The kinds of graph labeling divided as vertex-labelings, edge-labelings, and total labeling (vertex and edge-labelings). Based on survey that launch by Galian [5], there are many kinds of graph labelling. One of them is irregular total k-labeling.

According to Bača *et al.*[8], irregular total k-labeling divided be two kinds, that is edge irregular total k-labeling and vertex irregular total k-labeling. In 2017, Ryan *et al.* [6] has introduced new concept about irregular total k-labeling, that is vertex irregular reflexive total k-labeling and edge irregular reflexive total k-labeling. We call edge irregular reflexive total k-labeling if labeling f on G caries the positive integers 1 until k_e to edges of graph and caries the even positive integers 2 until $2k_v$ to vertices of graph with $k = max\{k_e, 2k_v\}$. The other spesifications edge irregular reflexive total k-labeling is all of edges have different weight. The mapping positive integers to edge xy of graph denoted as $f(xy)$ and the mapping even positive integers to vertex x of graph denoted as $f(x)$. Moreover, the weight edge xy of graph denoted as $wt(xy)$, where $wt(xy) = f(x) + f(xy) + f(y)$.

This paper also make investigate about reflexive edge strength of G, denoted as $res(G)$. To determine a lower bound of $res(G)$ Ryan [6] gave a $Lemma$ 1.1 for all graph G,

Lemma 1.1.

$$res(G) \geq \begin{cases} \lceil \frac{|E(G)|}{3} \rceil, & \text{if } |E(G)| \not\equiv 2,3 \pmod 6, \\ \lceil \frac{|E(G)|}{3} \rceil + 1, & \text{if } |E(G)| \equiv 2,3 \pmod 6. \end{cases}$$

Some of $res(G)$ has been determined, such as prisms graph D_n [2], cycle C_n [7], wheels graph W_n [2], corona of path and other graph, $P_n \odot K_1$ and $P_n \odot P_2$ [1], and other. This paper will investigate about sun graph and corona of cycle and null graph N_2.

2. The sun graph

Based on definition of sun graph by Wallis and Marr [10], an n-sun is a cycle C_n with an edge terminating in a vertex of degree 1 attached to each vertex and by Boulet [9] denoted as Sun_n.Then we will denote vertices in the cycle as x_i and pendant vertices denoted as y_i. So set of vertices $V(sun_n) = \{x_i, y_i : 1 \leq i \leq n\}$, consequently set of edges $E(sun_n) = \{x_i x_{i+1}, x_i y_i : 1 \leq i \leq n\}$. As a result sun_n has $2n$ edges and vertices. $Res(G)$ of sun_n can be obtained by $Theorem$ 2.1.

Theorem 2.1. *For sun_n with $n \geq 3$,*

$$res(sun_n) \doteq \begin{cases} 3, & \text{if } n \doteq 3, \\ 2\lceil \frac{n}{3} \rceil, & \text{if } n > 3. \end{cases}$$

Proof. First, we prove the lower bound of $res(sun_n)$. Since $|E|$ of sun_n is $2n$, then by $Lemma$ 1.1 we get

$$res(sun_n) \geq \begin{cases} \lceil \frac{2n}{3} \rceil, & \text{if } 2n \not\equiv 2,3 \pmod 6, \\ \lceil \frac{2n}{3} \rceil + 1, & \text{if } 2n \equiv 2,3 \pmod 6. \end{cases} \tag{1}$$

Let us prove the condition if $n \doteq 3$,
Graph sun_3 has 6 vertices and 6 edges. By (1) we get the lower bound of $res(sun_3) = 2$. Let us assume $res(sun_3) = 2$, we get maximum label of vertex and label of edges is 2, consequently the possibility of vertices and edges label for 6 edges $x_iy_i(1 \leq i \leq 6)$ with the edge weight from 1 until 6 are,

$$\begin{aligned} wt_f(x_1y_1) = \quad & f(x_1) + f(x_1y_1) + f(y_1) = & 0+1+0 = \quad & 1 \\ wt_f(x_2y_2) = \quad & f(x_2) + f(x_2y_2) + f(y_2) = & 0+2+0 = \quad & 2 \\ wt_f(x_3y_3) = \quad & f(x_3) + f(x_3y_3) + f(y_3) = & 0+1+2 = \quad & 3 \\ wt_f(x_4y_4) = \quad & f(x_4) + f(x_4y_4) + f(y_4) = & 0+2+2 = \quad & 4 \\ wt_f(x_5y_5) = \quad & f(x_5) + f(x_5y_5) + f(y_5) = & 2+1+2 = \quad & 5 \\ wt_f(x_6y_6) = \quad & f(x_6) + f(x_6y_6) + f(y_6) = & 2+2+2 = \quad & 6 \end{aligned}$$

But, its form cannot be applied. Then, let us make assume that $res(sun_n) = 3$, we get max $f(x_i) = 2$ and max $f(x_iy_i) = 3$ for $1 \leq i \leq 6$, consequently the possibility label of vertices and edges as follows,

$$\begin{aligned} wt_f(x_1y_1) = \quad & f(x_1) + f(x_1y_1) + f(y_1) = & 0+1+0 = \quad & 1 \\ wt_f(x_2y_2) = \quad & f(x_2) + f(x_2y_2) + f(y_2) = & 0+2+0 = \quad & 2 \\ wt_f(x_3y_3) = \quad & f(x_3) + f(x_3y_3) + f(y_3) = & 0+1+2 = \quad & 3 \\ wt_f(x_4y_4) = \quad & f(x_4) + f(x_4y_4) + f(y_4) = & 0+2+2 = \quad & 4 \\ wt_f(x_5y_5) = \quad & f(x_5) + f(x_5y_5) + f(y_5) = & 0+3+2 = \quad & 5 \\ wt_f(x_6y_6) = \quad & f(x_6) + f(x_6y_6) + f(y_6) = & 2+2+2 = \quad & 6 \end{aligned}$$

Its form can be applied, so 3 is sufficient to become the lower bound of $res(sun_3)$. Then, we will prove for condition if $n > 3$. There are three cases for this condition.
Firstly if $n \equiv 0 \pmod 3$. For this case we get,

$$\lceil \frac{n}{3} \rceil = \frac{n}{3}. \tag{2}$$

By (1), if $n \equiv 0 \pmod 3$ we get $res(sun_n) \geq \lceil \frac{2n}{3} \rceil$. Consequently by (2),

$$\begin{aligned} \lceil \frac{2n}{3} \rceil \quad &= \quad \frac{2n}{3} \\ &= \quad 2\lceil \frac{n}{3} \rceil. \end{aligned}$$

Secondly if $n \equiv 1 \pmod 3$. For this case we get,

$$\lceil \frac{n}{3} \rceil = \frac{n-1}{3} + 1. \tag{3}$$

By (1), if $n \equiv 1 \pmod 3$ we get $res(sun_n) \geq \lceil \frac{2n}{3} \rceil + 1$. Consequently by (3),

$$\begin{aligned}\lceil\frac{2n}{3}\rceil+1 &= \frac{2(n-1)}{3}+1+1\\ &= \frac{2(n-1)}{3}+2\\ &= 2(\frac{n-1}{3}+1)\\ &= 2\lceil\frac{n}{3}\rceil.\end{aligned}$$

Thirdly if $n \equiv 2 \pmod 3$,
By (1), if $n \equiv 2 \pmod 3$ we get $res(sun_n) \geq \lceil\frac{2n}{3}\rceil$. For $n \equiv 2 \pmod 3$, $\lceil\frac{2n}{3}\rceil$ has the same value as $2\lceil\frac{n}{3}\rceil$.

Next, we will prove the upper bound of $res(sun_n)$. To prove this part we construct k-labeling f with $k = 3$ if $n = 3$ and $k = 2\lceil\frac{n}{3}\rceil$ if $n > 3$ on sun_n as follows,
For $n \geq 3$,

$$\begin{aligned}
f(x_i) &= \begin{cases} 0, & \text{if } i = 1 \text{ and } 2,\\ \frac{4i-2}{3}, & \text{if } i = 3,4,...,\lceil\frac{n}{2}\rceil \text{ and } i \equiv 2 \pmod 3,\\ 4\lfloor\frac{i}{3}\rfloor, & \text{if } i = 3,4,...,\lceil\frac{n}{2}\rceil \text{ and } i \not\equiv 2 \pmod 3.\end{cases}\\
f(y_i) &= \begin{cases} \frac{4i-4}{3}, & \text{if } i = 1,2,...,\lceil\frac{n}{2}\rceil \text{ and } i \equiv 1 \pmod 3,\\ 2+4\lfloor\frac{i-2}{3}\rfloor, & \text{if } i = 1,2,...,\lceil\frac{n}{2}\rceil \text{ and } i \not\equiv 1 \pmod 3.\end{cases}\\
f(y_{n-(i-1)}) &= \begin{cases} \frac{4i}{3}, & \text{if } i = 1,2,...,\lfloor\frac{n}{2}\rfloor \text{ and } i \equiv 0 \pmod 3,\\ 2+4\lfloor\frac{i}{3}\rfloor & \text{if } i = 1,2,...,\lfloor\frac{n}{2}\rfloor \text{ and } i \not\equiv 0 \pmod 3.\end{cases}\\
f(x_iy_i) &= \begin{cases} 2, & \text{if } i = 1 \text{ and } 2.\\ \frac{4i-6}{3} & \text{if } i = 3,4,...,\lceil\frac{n}{2}\rceil \text{ and } i \equiv 0 \pmod 3,\\ 4\lfloor\frac{i}{3}\rfloor, & \text{if } i = 3,4,...,\lceil\frac{n}{2}\rceil \text{ and } i \not\equiv 0 \pmod 3.\end{cases}\\
f(x_1x_n) &= 1.
\end{aligned}$$

For $n \geq 3$ and $n \equiv 0 \pmod 6$,

$$f(x_{n-(i-1)}) = \begin{cases} \frac{4i}{3}, & \text{if } i = \frac{n}{2},\\ \frac{4i+4}{3}, & \text{if } i = 1,2,...,(\frac{n}{2}-1) \text{ and } i \equiv 2 \pmod 3,\\ 2+4\lfloor\frac{i}{3}\rfloor, & \text{if } i = 1,2,...,(\frac{n}{2}-1) \text{ and } i \not\equiv 2 \pmod 3.\end{cases}$$

For $n \geq 3$ and $n \not\equiv 0 \pmod 6$,

$$f(x_{n-(i-1)}) = \begin{cases} \frac{4i+4}{3}, & \text{if } i = 1,2,...,\lfloor\frac{n}{2}\rfloor \text{ and } i \equiv 2 \pmod 3,\\ 2+4\lfloor\frac{i}{3}\rfloor, & \text{if } i = 1,2,...,\lfloor\frac{n}{2}\rfloor \text{ and } i \not\equiv 2 \pmod 3.\end{cases}$$

For $n \geq 3$ and $n \equiv 2 \pmod 6$,

$$f(x_{n-(i-1)}y_{n-(i-1)}) = \begin{cases} \frac{4i-4}{3}, & \text{if } i = \frac{n}{2}, \\ \frac{4i+2}{3}, & \text{if } i = 1,2,...,(\frac{n}{2}-1) \text{ and } i \equiv 1 \pmod 3, \\ 4\lceil \frac{i}{3} \rceil, & \text{if } i = 1,2,...,(\frac{n}{2}-1) \text{ and } i \not\equiv 1 \pmod 3. \end{cases}$$

For $n \geq 3$ and $n \equiv 4 \pmod 6$,

$$f(x_{n-(i-1)}y_{n-(i-1)}) = \begin{cases} \frac{4i-2}{3}, & \text{if } i = \frac{n}{2}, \\ \frac{4i+2}{3}, & \text{if } i = 1,2,...,(\frac{n}{2}-1) \text{ and } i \equiv 1 \pmod 3, \\ 4\lceil \frac{i}{3} \rceil, & \text{if } i = 1,2,...,(\frac{n}{2}-1) \text{ and } i \not\equiv 1 \pmod 3. \end{cases}$$

For $n \geq 3$ and $n \not\equiv 2,4 \pmod 6$,

$$f(x_{n-(i-1)}y_{n-(i-1)}) = \begin{cases} \frac{4i+2}{3}, & \text{if } i = 1,2,...,(\lfloor \frac{n}{2})\rfloor \text{ and } i \equiv 1 \pmod 3, \\ 4\lceil \frac{i}{3} \rceil, & \text{if } i = 1,2,...,(\lfloor \frac{n}{2})\rfloor \text{ and } i \not\equiv 1 \pmod 3. \end{cases}$$

For $n = 3$,

$$f(x_i x_{i+1}) \;=\; 2i-1, \text{if } i = 1,2.$$

For $n > 3$ and $n \equiv 3 \pmod 6$,

$$f(x_i x_{i+1}) = \begin{cases} 1, & \text{if } i = 1 \text{ and } 2, \\ \frac{4i-5}{3}, & \text{if } i = \lceil \frac{n}{2} \rceil, \\ \frac{4i-9}{3}, & \text{if } i = 3,4,...,(\lceil \frac{n}{2} \rceil - 1) \text{ and } i \equiv 0 \pmod 3, \\ 3+4\lfloor \frac{i-4}{3} \rfloor, & \text{if } i = 3,4,...,(\lceil \frac{n}{2} \rceil - 1) \text{ and } i \not\equiv 0 \pmod 3. \end{cases}$$

For $n > 3$ and $n \not\equiv 3 \pmod 6$,

$$f(x_i x_{i+1}) = \begin{cases} 1, & \text{if } i = 1 \text{ and } 2, \\ \frac{4i-9}{3}, & \text{if } i = 3,4,...,(\lceil \frac{n}{2} \rceil) \text{ and } i \equiv 0 \pmod 3, \\ 3+4\lfloor \frac{i-4}{3} \rfloor, & \text{if } i = 3,4,...,\lceil \frac{n}{2} \rceil) \text{ and } i \not\equiv 0 \pmod 3. \end{cases}$$

For $n \geq 3$ and $n \equiv 0 \pmod 6$

$$f(x_{n-(i-1)}x_{n-i}) = \begin{cases} \frac{4i+1}{3}, & \text{if } i = (\frac{n}{2}-1), \\ \frac{4i-3}{3}, & \text{if } i = 1,2,...,(\frac{n}{2}-2) \text{ and } i \equiv 0 \pmod 3, \\ 1+4\lfloor \frac{i}{3} \rfloor, & \text{if } i = 1,2,...,(\frac{n}{2}-2) \text{ and } i \not\equiv 0 \pmod 3. \end{cases}$$

For $n \geq 3$ and $n \not\equiv 0 \pmod 6$

$$f(x_{n-(i-1)}x_{n-i}) = \begin{cases} \frac{4i-3}{3}, & \text{if } i = 1,2,...,(\lfloor \frac{n}{2} \rfloor - 1) \text{ and } i \equiv 0 \pmod 3, \\ 1+4\lfloor \frac{i}{3} \rfloor, & \text{if } i = 1,2,...,(\lfloor \frac{n}{2} \rfloor - 1) \text{ and } i \not\equiv 0 \pmod 3. \end{cases}$$

Then, by f-labeling we know that label of vertices is even positive integer. So the upper bound of $res(sun_n)$ has same value as the lower bound of $res(sun_n)$. The weight of edges are,
For even n,

$$\begin{aligned}
wt(x_iy_i) &= \begin{cases} 2, & \text{if } i = 1, \\ 4(i-1), & \text{if } i = 2, 3, ..., (\frac{n}{2}+1). \end{cases} \\
wt(x_{n-(i-1)}y_{n-(i-1)}) &= 4i+2, \text{if } i = 1, 2, ..., (\frac{n}{2}-1). \\
wt(x_ix_{i+1}) &= 4i-3, \text{if } i = 1, 2, ..., \lceil\frac{n}{2}\rceil. \\
wt(x_{n-(i-1)}x_{n-i}) &= 4i+3, \text{if } i = 1, 2, ..., \lfloor\frac{n}{2}\rfloor. \\
wt(x_1x_n) &= 3, \text{if } n \geq 3.
\end{aligned}$$

For odd n,

$$\begin{aligned}
wt(x_iy_i) &= \begin{cases} 2, & \text{if } i = 1, \\ 4(i-1), & \text{if } i = 2, 3, ..., \lceil\frac{n}{2}\rceil. \end{cases} \\
wt(x_{n-(i-1)}y_{n-(i-1)}) &= 4i+2, \text{if } i = 1, 2, ..., \lfloor\frac{n}{2}\rfloor. \\
wt(x_ix_{i+1}) &= 4i-3, \text{if } i = 1, 2, ..., \lceil\frac{n}{2}\rceil. \\
wt(x_{n-(i-1)}x_{n-i}) &= 4i+3, \text{if } i = 1, 2, ..., \lfloor\frac{n}{2}\rfloor. \\
wt(x_1x_n) &= 3, \text{if } n \geq 3.
\end{aligned}$$

By this investigate, we can conclude that every edges has different weight, consequently f is edge irregular reflexive k-labeling. So, the proof of $res(sun_n)$ is completed. $\square$

An illustration of edge irregular reflexive k-labeling on sun_n for even n can be seen on Figure 1. The black color is label, red color is weight and blue color is name of vertices.

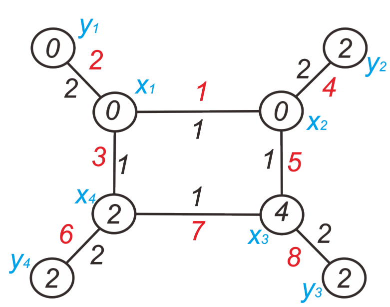

Figure 1. Edge irregular reflexive 4-labeling of sun_4.

3. The corona of cycle and null graph with two vertices

According Dwivedi [4], a graph $G(V, E)$ is called null graph which does not have any edges, in other words every vertex are isolated, denoted as N_m with m the number of vertices. Then, based on definition corona of two graph by Harray and Frunct [3], corona of cycle and null graph denoted as $C_n \odot N_m$ is a graph that formed by one copy of graph C_n and n-copy graph of N_m with $i-th$ vertex from C_n is connected to all of vertices from $i-th$ copy of graph N_m. It is consequently that $C_n \odot N_m$ has $|E| = n(m+1)$. First, we know that $C_n \odot N_1$ same as sun graph. Therefore, in this paper we continue to discuss of $C_n \odot N_2$ for C_n with $n \geq 3$. We will denote vertices in the cycle of $C_n \odot N_2$ as x_i and pendant vertices denoted as $y_{i,j}$. So set of vertices $V(C_n \odot N_2) = \{x_i, y_{i,j} : 1 \leq i \leq n$ and $j = 1,2\}$, consequently set of edges $E(C_n \odot N_2) = \{x_i x_{i+1}, x_i y_{i,j} : 1 \leq i \leq n$ and $j = 1,2\}$. Then reflexive edge strength of $C_n \odot N_2$ can be found on $Theorem$ 3.1.

Theorem 3.1. *For $C_n \odot N_2$ with $n \geq 3$,*

$$res(C_n \odot N_2) = 2\lfloor \frac{n+1}{2} \rfloor, \textit{if } n \geq 3.$$

Proof. First, we prove the lower bound of $res(C_n \odot N_2)$. By section (3) we get $|E|$ of $C_n \odot N_2$ is $3n$, then by $Lemma$ 1.1 we get

$$res(C_n \odot N_2) \geq \begin{cases} \lceil \frac{3n}{3} \rceil, & \text{if } 3n \not\equiv 2,3 \pmod 6, \\ \lceil \frac{3n}{3} \rceil + 1, & \text{if } 3n \equiv 2,3 \pmod 6. \end{cases} \tag{4}$$

It is equivalent with,

$$res(C_n \odot N_2) \geq \begin{cases} \text{n}, & \text{if } 3n \not\equiv 2,3 \pmod 6, \\ \text{n +1}, & \text{if } 3n \equiv 2,3 \pmod 6. \end{cases} \tag{5}$$

For $n \equiv 0 \pmod 2$ we get,

$$\lfloor \frac{n+1}{2} \rfloor = \frac{n}{2}. \tag{6}$$

By (5), if $n \equiv 0 \pmod 2$ we get $res(C_n \odot N_2) \geq n$. Consequently by (6) we get,

$$\begin{aligned} n &= \frac{2n}{2} \\ &= 2\lfloor \frac{n+1}{2} \rfloor. \end{aligned}$$

For $n \equiv 1 \pmod 2$ we get,

$$\lfloor \frac{n+1}{2} \rfloor = \frac{n+1}{2}. \tag{7}$$

By (5), if $n \equiv 1 \pmod 2$ we get $res(C_n \odot N_2) \geq n+1$. Consequently by (7) we get,

$$\begin{aligned} n+1 &= 2(\frac{n+1}{2}) \\ &= 2\lfloor \frac{n+1}{2} \rfloor. \end{aligned}$$

Next, we will prove the upper bound of $res(C_n \odot N_2)$ for $n \geq 3$. To prove this part we construct k-labeling f with $k = 2\lfloor \frac{n+1}{2} \rfloor$ if $n \geq 3$ on $C_n \odot N_2$ as follows,
For $n \geq 3$,

$$\begin{aligned}
f(x_i) &= \begin{cases} 2(i-1), & \text{if } i = 1 \text{ and } 2, \\ 2i, & \text{if } i = 3, 4, ..., \lceil \frac{n}{2} \rceil. \end{cases} \\
f(x_{n-(i-1)}) &= 2i, \text{if } i = 1, 2, ..., \lfloor \frac{n}{2} \rfloor. \\
f(y_{i,j}) &= \begin{cases} 0, & \text{if } i = 1 \text{ and } j = 1, 2, \\ 2i, & \text{if } i = 2, 3, ..., \lceil \frac{n}{2} \rceil \text{ and } j = 1, 2. \end{cases} \\
f(y_{(i-1),j}) &= 2i, \text{if } i = 1, 2, ..., \lfloor \frac{n}{2} \rfloor \text{ and } j = 1, 2. \\
f(x_{n-(i-1)}x_{n-i}) &= 2i + 1, \text{if } i = 1, 2, ..., (\lfloor \frac{n}{2} \rfloor - 1). \\
f(x_{n-(i-1)}y_{n-(i-1)}) &= 2i + j - 2, \text{if } i = 1, 2, ..., \lfloor \frac{n}{2} \rfloor \text{ and } j = 1, 2. \\
f(x_1 x_n) &= 1.
\end{aligned}$$

For $n = 3$,

$$f(x_i y_{i,j}) = \begin{cases} j, & \text{if } i = 1, \\ j+1, & \text{if } i = 2. \end{cases}$$

For $n > 3$ and $n \equiv 0 \pmod 2$,

$$f(x_i y_{i,j}) = \begin{cases} j, & \text{if } i = 1, 2 \text{ and } j = 1, 2, \\ 2i + j - 6, & \text{if } i = 3, 4, ..., \lceil \frac{n}{2} \rceil \text{ and } j = 1, 2. \end{cases}$$

For $n > 3$ and $n \equiv 1 \pmod 2$,

$$f(x_i y_{i,j}) = \begin{cases} j, & \text{if } i = 1, 2 \text{ and } j = 1, 2, \\ 2i + j - 6, & \text{if } i = 3, 4, ..., \lfloor \frac{n}{2} \rfloor \text{ and } j = 1, 2, \\ 2i + j - 5, & \text{if } i = \lceil \frac{n}{2} \rceil \text{ and } j = 1, 2. \end{cases}$$

For $n \geq 3$ and $n \equiv 0 \pmod 2$,

$$f(x_i x_{i+1}) = \begin{cases} 2, & \text{if } i = 1, \\ 4, & \text{if } i = 2 \text{ and } n = 4, \\ 2, & \text{if } i = 2 \text{ and } n \neq 4, \\ 2i - 4, & \text{if } i = 3, 4, ..., (\frac{n}{2} - 1) \text{ and } n > 4, \\ 2i - 2, & \text{if } i = \frac{n}{2} \text{ and } n > 4. \end{cases}$$

For $n \geq 3$ and $n \equiv 1 \pmod 2$,

$$f(x_i x_{i+1}) = \begin{cases} 2, & \text{if } i = 1, \\ 3, & \text{if } i = 2 \text{ and } n = 3, \\ 2, & \text{if } i = 2 \text{ and } n \neq 3, \\ 2i - 4, & \text{if } i = 3, 4, ..., (\lceil \frac{n}{2} \rceil - 1) \text{ and } n > 3, \\ 2i - 3, & \text{if } i = \lceil \frac{n}{2} \rceil \text{ and } n > 3. \end{cases}$$

Then, by f-labeling we know that label of vertices is even positive integer. From the above formula, we get that the upper bound of $res(C_n \odot N_2)$ same as the lower bound and the weight of edges we get,
For even n,

$$\begin{aligned} wt(x_i y_{i,j}) &= 6i + j - 6, \text{if } i = 1, 2, ..., \frac{n}{2} \text{ and } j = 1, 2. \\ wt(x_{n-(i-1)} y_{n-(i-1),j}) &= 6i + j - 2, \text{if } i = 1, 2, ..., \frac{n}{2} \text{ and } j = 1, 2. \\ wt(x_i x_{i+1}) &= 6i - 2, \text{if } i = 1, 2, ..., \frac{n}{2}. \\ wt(x_{n-(i-1)} x_{n-i}) &= 6i + 3, \text{if } i = 1, 2, ..., (\frac{n}{2} - 1). \\ wt(x_1 x_n) &= 3, \text{if } n \geq 3. \end{aligned}$$

For odd n,

$$\begin{aligned} wt(x_i y_{i,j}) &= \begin{cases} 6i + j - 6, & \text{if } i = 1, 2, ..., (\lceil \frac{n}{2} \rceil - 1) \text{ and } j = 1, 2, \\ 6i + j - 5, & \text{if } i = \lceil \frac{n}{2} \rceil \text{ and } j = 1, 2. \end{cases} \\ wt(x_{n-(i-1)} y_{n-(i-1),j}) &= 6i + j - 2, \text{if } i = 1, 2, ..., \lfloor \frac{n}{2} \rfloor \text{ and } j = 1, 2. \\ wt(x_i x_{i+1}) &= \begin{cases} 6i - 2, & \text{if } i = 1, 2, ..., (\lceil \frac{n}{2} \rceil - 1), \\ 6i - 5, & \text{if } i = \lceil \frac{n}{2} \rceil. \end{cases} \\ wt(x_{n-(i-1)} x_{n-i}) &= 6i + 3, \text{if } i = 1, 2, ..., (\lfloor \frac{n}{2} \rfloor - 1)\ . \\ wt(x_1 x_n) &= 3, \text{if } n \geq 3. \end{aligned}$$

By this investigate, we can conclude that every edges has different weight, consequently f is edge irregular reflexive k-labeling. So, the proof of $res(C_n \odot N_2)$ is completed. □

An illustration of edge irregular reflexive k-labeling on $C_n \odot N_2$ can be seen on Figure 2. The black color is label, red color is weight and blue color is name of vertices.

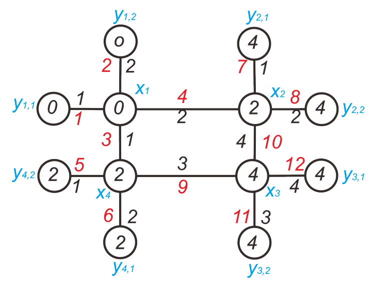

Figure 2. Edge irregular reflexive 4-labeling of $C_4 \odot N_2$.

4. Concluding remark

As result the discussion we get conclude $res(sun_n)$ or $res(C_n \odot N_1)$ are 3 for $n = 3$ and $2\lceil \frac{n}{2} \rceil$ for $n > 3$, while $res(C_n \odot N_2)$ is $2\lfloor \frac{n+1}{2} \rfloor$ for $n \geq 3$. Moreover, there is open problem for next research about this graph, which still on progress to investigate.

Open problem: What is reflexive edge strength of $C_n \odot N_m$ for $n \geq 3$ and $m \geq 3$.

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