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# On the locating chromatic number of barbell shadow path graphs

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**Abstract**

The locating chromatic number was introduced by Chartrand in 2002. The locating chromatic number of a graph is a combined concept between the coloring and partition dimension of a graph. The locating chromatic number of a graph is defined as the cardinality of the minimum color classes of the graph. In this paper, we discuss about the locating chromatic number of shadow path graphs and barbell graph containing shadow graph.

*Keywords:* the locating-chromatic number, shadow path graph, barbell graph
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## 1. Introduction

The locating chromatic number of a graph was introduced by Chatrand et al.[6] by combining two concepts in graph theory, which are vertex coloring and partition dimension of a graph. Let $G = (V, E)$ be a connected graph. A $k$-coloring of $G$ is a function $c : V(G) \longrightarrow \{1, 2, \cdots, k\}$, where $c(u) \neq c(v)$ for any two adjacent vertices $u$ and $v$ in $G$. Thus, the coloring $c$ induces a partition $\Pi$ of $V(G)$ into $k$ color classes (independent sets) $C_1, C_2, \cdots, C_k$, where $C_i$ is the set of all vertices colored by the color $i$ for $1 \leq i \leq k$. The color code $c_\Pi(v)$ of a vertex $v$ in $G$ is defined

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as the $k-$ordinate $(d(v, C_1), d(v, C_2), \cdots, d(v, C_k))$, where $d(v, C_i) = \min\{d(v, x); x \in C_i\}$ for $1 \leq i \leq k$. The $k-$coloring $c$ of $G$ such that all vertices have different color codes is called a *locating coloring* of $G$. The *locating chromatic number* of $G$, denoted by $\chi_L(G)$, is the minimum $k$ such that $G$ has a locating coloring.

The following two theorems are useful to determine the lower bound of the locating chromatic of a graph. The set of neighbors of a vertex $q$ in $G$, denoted by $N(q)$.

**Theorem 1.1.** *(see [6]). Let $c$ be a locating coloring in a connected graph $G$. If $x$ and $y$ are distinct vertices of $G$ such that $d(p, w) = d(q, w)$ for all $w \in V(G) - \{p, q\}$, then $c(p) \neq c(q)$. In particular, if $p$ and $q$ are non-adjacent vertices such that $N(p) \neq N(q)$, then $c(p) \neq c(q)$.*

**Theorem 1.2.** *(see [6]). The locating chromatic number of a cycle graph $C_n(n \geq 3)$ is 3 for odd n and 4 for even n.*

The locating chromatic number of a graph is an interesting topic to study because there is no general algorithm for determining the locating chromatic number of any graphs and there are only a few results related to determining of the locating chromatic number of some graphs. Chartrand *et al.* [6] determined all graphs of order $n$ with locating number $n$, namely a complete multipartite graph of $n$ vertices. Moreover, Chartrand *et al.*[7] provided a tree construction of $n$ vertices, $n \geq 5$, with locating chromatic number varying from 3 to $n$, except for $(n-1)$. Next, Asmiati *et al.* [1] obtained the locating chromatic number of amalgamation of stars and non-homogeneous caterpillars and firecracker graphs [2]. In [5] Welyyanti *et al.* determined the locating chromatic number of complete $n$-ary trees. Next, Sofyan *et al.* [4] determined the locating chromatic number of homogeneous lobster. Recently, Ghanem *et al.* [8] found the locating chromatic number of powers of the path and cycles.

Let $P_n$ be a path with $V(P_n) = \{x_i \mid 1 \leq i \leq n\}$ and $E(P_n) = \{x_i x_{i+1} \mid 1 \leq i \leq n-1\}$. The shadow path graph $D_2(P_n)$ is a graph with the vertex set $\{u_i, v_i \mid 1 \leq i \leq n\}$ where $u_i u_j \in E(D_2(P_n))$ if and only if $x_i x_j \in E(P_n)$ and $u_i v_j \in E(D_2(P_n))$ if and only if $x_i x_j \in E(P_n)$. *A barbell graph containing shadow path graph*, denoted by $B_{D_2(P_n)}$ is obtained by copying a shadow path graph (namely, $D_2^{'}(P_n)$) and connecting the two graphs with a bridge. We assume that $\{u_i', v_i' \mid 1 \leq i \leq n\}$ is a vertex set of $D_2^{'}(P_n)$ and a bridge in $B_{D_2(P_n)}$ connecting $\{u'_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\}$ for odd $n$ and $\{u'_{\frac{n}{2}} v_{\frac{n}{2}}\}$ for even $n$.

Motivated by the result of Asmiati *et al.* [3] about the determination of the locating chromatic number of certain barbell graphs, in this paper we determine the locating chromatic number of shadow path graphs and barbell graph containing shadow path for $n \geq 3$.

## 2. Main Results

The following theorem gives the locating chromatic number for shadow path graph $D_2(P_n)$ for $n \geq 3$.

**Lemma 2.1.** *Let $c$ be a locating-chromatic number for shadow path graph $D_2(P_n)$, with $u_i \in P_i^1$ and $v_i \in P_i^2$. Then $c(u_i) \neq c(v_i)$.*

*Proof.* On the shadow path graph $D_2(P_n)$, we can see that $d(u_i, x) = d(v_i, x)$, $i \in [1, n-1]$ for every $x \in ((D_2(P_n)) \setminus \{u_i, v_i\})$. By Theorem 1.1, we have $c(u_i) \neq c(v_i)$. □

**Theorem 2.1.** *The locating chromatic number of a shadow path graph for* $n \geq 3, D_2(P_n)$ *is* $6$.

*Proof.* First, we determine the lower bound for the locating-chromatic number of shadow path graph $D_2(P_n)$ for $n \geq 3$. The Shadow path graph $D_2(P_n)$ for $n \geq 3$ consists of minimal two cycles $C_4$. Pick the first cycle $C_4$, then by Theorem 1.2, we could assign $4$ colors, $\{1, 2, 3, 4\}$ to the first cycle's vertices. Next, in the second $C_4$, we have two vertices, which intersect with two vertices in the first $C_4$. By Lemma 2.1, we must assign two different colors to the remaining vertices of the second $C_4$. Therefore, we have $\chi_L(G) \geq 6$.

Next, we determine the upper bound of the locating chromatic number of the shadow path graph for $n \geq 3$. Let $c$ be a coloring using 6 colors as follow:

$$c(u_i) = \begin{cases} 1, & \text{for } i = 1, \\ 2, & \text{for } i = 2n\ , n \geq 1, \\ 3, & \text{for } i = 2n+1\ , n \geq 1, \end{cases}$$

$$c(v_i) = \begin{cases} 4, & \text{for } i = 1, \\ 5, & \text{for } i = 2n\ , n \geq 1, \\ 6, & \text{for } i = 2n+1\ , n \geq 1. \end{cases}$$

The color codes of $D_2(P_n)$ are :

$$c_\pi(u_i) = \begin{cases} i-1, & \text{for } 1^{st} \text{ ordinate}, i \geq 1, \\ & \text{for } 4^{th}\text{ordinate}, i \geq 2, \\ 0, & \text{for } 2^{nd} \text{ ordinate, even i}\ , 2 \leq i \leq n, \\ & \text{for } 3^{rd} \text{ ordinate, odd i}\ , 3 \leq i \leq n, \\ 2, & \text{for } 5^{th} \text{ ordinate, even i}\ , 2 \leq i \leq n, \\ & \text{for } 6^{th} \text{ ordinate, odd i}\ , 1 \leq i \leq n, \\ & \text{for } 3^{rd} \text{ ordinate}, i = 1, \\ & \text{for } 4^{th} \text{ ordinate}, i = 1, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(v_i) = \begin{cases} i-1, & \text{for } 1^{st} \text{ ordinate}, i \geq 2, \\ & \text{for } 4^{th}\text{ordinate}, i \geq 1, \\ 0, & \text{for } 5^{nd} \text{ ordinate, even i}\ , 2 \leq i \leq n, \\ & \text{for } 6^{th} \text{ ordinate, odd i}\ , 3 \leq i \leq n; \\ 2, & \text{for } 2^{nd} \text{ ordinate, even i}\ , 2 \leq i \leq n, \\ & \text{for } 3^{rd} \text{ ordinate, odd i}\ , 3 \leq i \leq n, \\ & \text{for } 1^{st} \text{ ordinate}, i = 1, \\ & \text{for } 6^{th} \text{ ordinate}, i = 1, \\ 1, & \text{otherwise}. \end{cases}$$

 

Since all vertices in $D_2(P_n)$ for $n \geq 3$ have distinct color codes, then $c$ is a locating coloring using 6 colors. As a result $\chi_L D_2(P_n) \leq 6$. Thus $\chi_L D_2(P_n) = 6$. □

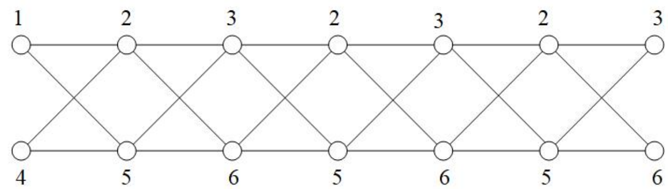


Figure 1. A minimum locating coloring of $D_2(P_7)$.

**Theorem 2.2.** *The locating chromatic number of a barbell graph containing shadow path for* $n \geq 3$ *is* 6.

*Proof.* First, we determine the lower bound of $\chi_L B_{D_2(P_n)}$ for $n \geq 3$. Since the barbell graph $B_{D_2(P_n)}$ containing $D_2(P_n)$, then by Theorem 2.3 we have $\chi_L(B_{D_2(P_n)}) \geq 6$. To prove the upper bound, consider the following three cases.

CASE 1 ($n = 3$). Let $c$ be a locating coloring using 6 colors as follows :

$$c(u_i) = \begin{cases} 1, & \text{for } i = 2, \\ 2, & \text{for } i = 1, \\ 3, & \text{for } i = 3, \end{cases}$$

$$c(u'_i) = \begin{cases} 4, & \text{for } i = 2, \\ 5, & \text{for } i = 1, \\ 6, & \text{for } i = 3, \end{cases}$$

$$c(v_i) = \begin{cases} 1, & \text{for } i = 1, \\ 5, & \text{for } i = 3, \\ 6, & \text{for } i = 2, \end{cases}$$

$$c(v'_i) = \begin{cases} 2, & \text{for } i = 1, \\ 3, & \text{for } i = 3, \\ 4, & \text{for } i = 2. \end{cases}$$

The color codes of $B_{D_2(P_3)}$ are

$$c_\pi(u_i) = \begin{cases} 0, & \text{for } 1^{st} \text{ ordinate}, \ i = 2, \\ & \text{for } 2^{nd} \text{ ordinate}, \ i = 1, \\ & \text{for } 3^{rd} \text{ ordinate}, \ i = 3, \\ 2, & \text{for } 3^{th} \text{ ordinate}, \ i = 1, \\ & \text{for } 4^{th} \text{ ordinate}, \ i = 2, \\ & \text{for } 2^{nd} \text{ ordinate}, \ i = 3, \\ & \text{for } 5^{th} \ and \ 6^{th} \text{ ordinate}, \ i = 1 \text{ and } 3, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(u'_i) = \begin{cases} 0, & \text{for } 5^{th} \text{ ordinate}, \; i = 1, \\ & \text{for } 4^{th} \text{ ordinate}, \; i = 2, \\ & \text{for } 6^{th} \text{ ordinate}, \; i = 3, \\ 2, & \text{for } 6^{th} \text{ ordinate}, \; i = 1, \\ & \text{for } 1^{nd} \text{ ordinate}, \; i = 2, \\ & \text{for } 5^{th} \text{ ordinate}, \; i = 3, \\ & \text{for } 2^{nd} \; and \; 3^{rd} \text{ ordinate}, \; i = 1 \text{ and } 3, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(v_i) = \begin{cases} 0, & \text{for } 1^{st} \text{ ordinate}, \; i = 1, \\ & \text{for } 6^{th} \text{ ordinate}, \; i = 2, \\ & \text{for } 5^{th} \text{ ordinate}, \; i = 3, \\ 2, & \text{for } 5^{th} \text{ ordinate}, \; i = 1, \\ & \text{for } 1^{st} \text{ ordinate}, \; i = 3, \\ & \text{for } 2^{nd} \; and \; 3^{rd} \text{ ordinate}, \; i = 1 \text{ and } 3, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(v'_i) = \begin{cases} 0, & \text{for } 2^{nd} \text{ ordinate}, \; i = 1, \\ & \text{for } 4^{th} \text{ ordinate}, \; i = 2, \\ & \text{for } 3^{rd} \text{ ordinate}, \; i = 3, \\ 2, & \text{for } 1^{st} \text{ ordinate}, \; i = 1 \text{ and } 3, \\ & \text{for } 6^{th} \text{ ordinate}, \; i = 2, \\ & \text{for } 2^{nd} \text{ ordinate}, \; i = 3, \\ & \text{for } 3^{rd} \; and \; 5^{th} \text{ ordinate}, \; i = 3, \\ 1, & \text{otherwise}. \end{cases}$$

Since all vertices in $B_{D_2(P_3)}$ have distinct color codes, then $c$ is a locating coloring using 6 colors. As a result $\chi_L B_{D_2(P_3)} \leq 6$. Thus $\chi_L B_{D_2(P_3)} = 6$.

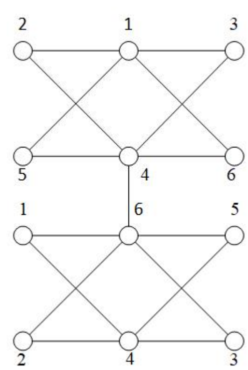


Figure 2. A minimum locating coloring of $B_{D_2(P_3)}$.

CASE 2 ($n$ odd). Let $c$ be a locating coloring using 6 colors as follows :

$$c(u_i)=\begin{cases}1, & \text{for } i=\frac{n+1}{2},\\ 2, & \text{for odd i}; i<\frac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+1, j\geq 1,\\ & \text{for even 1}; i<\frac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+3, j\geq 1,\\ 3, & \text{for odd i}; i<\frac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+3, j\geq 1,\\ & \text{for even i}; i<\frac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+1, j\geq 1,\end{cases}$$

$$c(u_i')=\begin{cases}4, & \text{for } i=\frac{n+1}{2},\\ 5, & \text{for odd i}; i<\frac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+1, j\geq 1,\\ & \text{for even 1}; i<\frac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+3, j\geq 1,\\ 6, & \text{for odd i}; i<\frac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+3, j\geq 1,\\ & \text{for even i}; i<\frac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+1, j\geq 1,\end{cases}$$

$$c(v_i)=\begin{cases}1, & \text{for odd i}; i<\frac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+1, j\geq 1,\\ & \text{for even i}; i<\frac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+3, j\geq 1,\\ 5, & \text{for odd i}; i<\frac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+3, j\geq 1,\\ & \text{for even i}; i<\frac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+1, j\geq 1,\\ 6, & \text{for } i=\frac{n+1}{2},\end{cases}$$

$$c(v_i')=\begin{cases}2, & \text{for odd i}; i<\frac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+1, j\geq 1,\\ & \text{for even i}; i<\frac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+3, j\geq 1,\\ 4, & \text{for } i=\frac{n+1}{2},\\ 3, & \text{for odd i}; i<\frac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+3, j\geq 1,\\ & \text{for even i}; i<\frac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+1, j\geq 1.\end{cases}$$

The color codes of $B_{D_2(P_n)}$ are

$$c_\pi(u_i) = \begin{cases} (\frac{n+1}{2}) - i, & \text{for } 1^{st} \text{ ordinate}, i \leq \frac{n+1}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i < \frac{n+1}{2}, \\ i - (\frac{n+1}{2}), & \text{for } 1^{st} \text{ ordinate}, i > \frac{n+1}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i > \frac{n+1}{2}, \\ 0, & \text{for } 2^{nd} \text{ ordinate, odd i }, i < \frac{n+1}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, even i }, i < \frac{n+1}{2}, \\ & \text{for } 2^{th} \text{ ordinate, even i }, i > \frac{n+1}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, odd i }, i > \frac{n+1}{2}, \\ 2, & \text{for } 5^{th} \text{ ordinate, odd i }, i < \frac{n+1}{2}, \\ & \text{for } 5^{th} \text{ ordinate, even i }, i > \frac{n+1}{2}, \\ & \text{for } 6^{th} \text{ ordinate, odd i }, i < \frac{n+1}{2}, \\ & \text{for } 6^{th} \text{ ordinate, even i }, i > \frac{n+1}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i = \frac{n+1}{2}, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(u'_i) = \begin{cases} (\frac{n+1}{2}) - i, & \text{for } 1^{st} \text{ ordinate}, i < \frac{n+1}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i \leq \frac{n+1}{2}, \\ i - (\frac{n+1}{2}), & \text{for } 1^{st} \text{ ordinate}, i > \frac{n+1}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i > \frac{n+1}{2}, \\ 0, & \text{for } 5^{th} \text{ ordinate, odd i }, i < \frac{n+1}{2}, \\ & \text{for } 5^{th} \text{ ordinate, even i }, i > \frac{n+1}{2}, \\ & \text{for } 6^{th} \text{ ordinate, odd i }, i > \frac{n+1}{2}, \\ & \text{for } 6^{th} \text{ ordinate, even i }, i < \frac{n+1}{2}, \\ 2, & \text{for } 2^{nd} \text{ ordinate, odd i }, i < \frac{n+1}{2}, \\ & \text{for } 2^{nd} \text{ ordinate, even i }, i > \frac{n+1}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, odd i }, i > \frac{n+1}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, even i }, i < \frac{n+1}{2}, \\ & \text{for } 1^{st} \text{ ordinate}, i = \frac{n+1}{2}, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(v_i) = \begin{cases} (\frac{n+1}{2}) - i, & \text{for } 4^{th} \text{ ordinate}, i < \frac{n+1}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i \leq \frac{n+1}{2}, \\ i - (\frac{n+1}{2}), & \text{for } 4^{th} \text{ ordinate}, i > \frac{n+1}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i > \frac{n+1}{2}, \\ 0, & \text{for } 1^{st} \text{ ordinate, odd i }, i < \frac{n+1}{2}, \\ & \text{for } 1^{st} \text{ ordinate, even i }, i > \frac{n+1}{2}, \\ & \text{for } 5^{th} \text{ ordinate, odd i }, i > \frac{n+1}{2}, \\ & \text{for } 5^{th} \text{ ordinate, even i }, i < \frac{n+1}{2}, \\ 2, & \text{for } 2^{nd} \text{ ordinate, odd i }, i < \frac{n+1}{2}, \\ & \text{for } 2^{nd} \text{ ordinate, even i }, i > \frac{n+1}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, odd i }, i > \frac{n+1}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, even i }, i < \frac{n+1}{2}, \\ & \text{for } 1^{st} \text{ ordinate}, i = \frac{n+1}{2}, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(v'_i) = \begin{cases} (\frac{n+1}{2}) - i, & \text{for } 4^{th} \text{ ordinate}, i \leq \frac{n+1}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i > \frac{n+1}{2}, \\ i - (\frac{n+1}{2}), & \text{for } 4^{th} \text{ ordinate}, i > \frac{n+1}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i > \frac{n+1}{2}, \\ 0, & \text{for } 2^{nd} \text{ ordinate, odd i}, i < \frac{n+1}{2}, \\ & \text{for } 2^{nd} \text{ ordinate, even i}, i > \frac{n+1}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i > \frac{n+1}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, even i}, i < \frac{n+1}{2}, \\ 2, & \text{for } 1^{st} \text{ ordinate, odd i}, i < \frac{n+1}{2}, \\ & \text{for } 1^{st} \text{ ordinate, even i}, i > \frac{n+1}{2}, \\ & \text{for } 5^{th} \text{ ordinate, odd i}, i > \frac{n+1}{2}, \\ & \text{for } 5^{th} \text{ ordinate, even i}, i < \frac{n+1}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i = \frac{n+1}{2}, \\ 1, & \text{otherwise}. \end{cases}$$

Since all vertices in $B_{D_2(P_n)}, n > 3$ for odd $n$ have distinct color codes, then $c$ is a locating coloring using 6 colors. As a result $\chi_L B_{D_2(P_n)} \leq 6$. Thus $\chi_L B_{D_2(P_n)} = 6$.

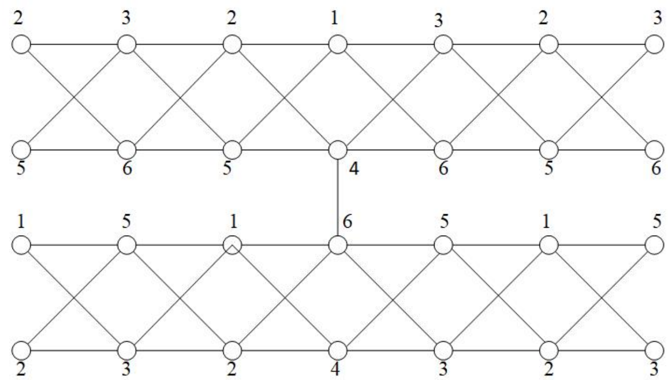


Figure 3. A minimum locating coloring of $B_{D_2(P_7)}$

CASE 3 ($n$ even). Let $c$ be a locating coloring using 6 colors as follows:

$$c(u_i) = \begin{cases} 1, & \text{for } i = \frac{n}{2}, \\ 2, & \text{for odd i; } i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \\ & \text{for even i; } i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ 3, & \text{for odd i; } i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ & \text{for even i; } i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \end{cases}$$

$$c(u_i') = \begin{cases} 4, & \text{for } i = \frac{n}{2}, \\ 5, & \text{for odd i; } i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \\ & \text{for even i; } i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ 6, & \text{for odd i; } i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ & \text{for even i; } i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \end{cases}$$

$$c(v_i) = \begin{cases} 1, & \text{for odd i; } i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \\ & \text{for even i; } i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ 5, & \text{for odd i; } i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ & \text{for even i; } i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \\ 6, & \text{for } i = \frac{n}{2}, \end{cases}$$

$$c(v_i') = \begin{cases} 2, & \text{for odd i; } i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \\ & \text{for even i; } i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ 3, & \text{for odd i; } i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ & \text{for even i; } i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \\ 4, & \text{for } i = \frac{n}{2}. \end{cases}$$

The color codes of $B_{D_2(P_n)}$ are

$$c_\pi(u_i) = \begin{cases} (\frac{n}{2}) - i, & \text{for } 1^{st} \text{ ordinate}, i \le \frac{n}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i < \dfrac{n}{2}; \\ i - (\frac{n}{2}), & \text{for } 1^{st} \text{ ordinate}, i > \frac{n}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i > \dfrac{n}{2}; \\ 0, & \text{for } 2^{nd} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, even i}, i < \frac{n}{2}, \\ & \text{for } 2^{th} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ 2, & \text{for } 5^{th} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 5^{th} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i = \frac{n}{2}, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(u'_i) = \begin{cases} (\frac{n}{2}) - i, & \text{for } 1^{st} \text{ ordinate}, i < \frac{n}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i \le \frac{n}{2}, \\ i - (\frac{n}{2}), & \text{for } 1^{st} \text{ ordinate}, i > \frac{n}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i > \frac{n}{2}, \\ 0, & \text{for } 5^{th} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 5^{th} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate, even i}, i < \frac{n}{2}, \\ 2, & \text{for } 2^{nd} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 2^{nd} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, even i}, i < \frac{n}{2}, \\ & \text{for } 1^{st} \text{ ordinate}, i = \frac{n}{2}, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(v_i) = \begin{cases} (\dfrac{n}{2}) - i, & \text{for } 4^{th} \text{ ordinate}, i < \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i \leq \frac{n}{2}, \\ i - (\dfrac{n}{2}), & \text{for } 4^{th} \text{ ordinate}, i > \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i > \frac{n}{2}, \\ 0, & \text{for } 1^{st} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 1^{st} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 5^{th} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ & \text{for } 5^{th} \text{ ordinate, even i}, i < \frac{n}{2}, \\ 2, & \text{for } 2^{nd} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 2^{nd} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, even i}, i < \frac{n}{2}, \\ & \text{for } 1^{st} \text{ ordinate}, i = \frac{n}{2}, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(v'_i) = \begin{cases} (\dfrac{n}{2}) - i, & \text{for } 4^{th} \text{ ordinate}, i \leq \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i > \frac{n}{2}, \\ i - (\dfrac{n}{2}), & \text{for } 4^{th} \text{ ordinate}, i > \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i > \frac{n}{2}, \\ 0, & \text{for } 2^{nd} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 2^{nd} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, even i}, i < \frac{n}{2}, \\ 2, & \text{for } 1^{st} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 1^{st} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 5^{th} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ & \text{for } 5^{th} \text{ ordinate, even i}, i < \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i = \frac{n}{2}, \\ 1, & \text{otherwise}. \end{cases}$$

Since all vertices in $B_{D_2(P_n)}$, $n \geq 3$ for even $n$ have distinct color codes, then $c$ is a locating coloring using 6 colors. As a result $\chi_L B_{D_2(P_n)} \leq 6$. Thus $\chi_L B_{D_2(P_n)} = 6$. □

## 3. Concluding Remarks

The locating chromatic number of a shadow path graphs and the barbell graph containing a shadow path graph is similar, which is 6.

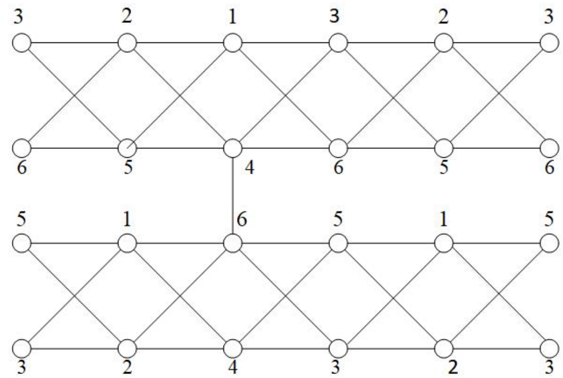


Figure 4. A minimum locating coloring of $B_{D_2(P_6)}$

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