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A strict upper bound for size multipartite Ramsey numbers of paths versus stars

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Abstract

Let P_n represent the path of size n. Let $K_{1,m-1}$ represent a star of size m and be denoted by S_m. Given a two coloring of the edges of a complete graph $K_{j\times s}$ we say that $K_{j\times s} \rightarrow (P_n, S_{m+1})$ if there is a copy of P_n in the first color or a copy of S_{m+1} in the second color. The size Ramsey multipartite number $m_j(P_n, S_{m+1})$ is the smallest natural number s such that $K_{j\times s} \rightarrow (P_n, S_{m+1})$. Given j, n, m if $s = \left\lceil \frac{n+m-1-k}{j-1} \right\rceil$, in this paper, we show that the size Ramsey numbers $m_j(P_n, S_{m+1})$ is bounded above by s for $k = \left\lceil \frac{n-1}{j} \right\rceil$. Given $j \geq 3$ and s, we will obtain an infinite class (n, m) that achieves this upper bound s. In the later part of the paper, will also investigate necessary and sufficient conditions needed for the upper bound to hold.

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Introduction

All graphs considered in this paper finite graphs without loops and multiple edges. Let $K_{n,m}$ represent the complete bipartite graph of nm vertices, partitioned in to two sets of size n and m. Let $K_{j\times s}$ represent the complete balance multipartie graph having j uniform multipartite sets of size s. If for every two coloring (red and blue) of the edges of a complete graph K_n, there exists a copy of G in the first color (red) or a copy of H in the second color (blue), we say $K_n \rightarrow (G, H)$. The

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Ramsey number $r(G, H)$ is defined as the smallest positive integer n such that $K_n \to (G, H)$. The classical Ramsey number $r(n, m)$ is defined as $r(K_n, K_m)$. However, even in the case of diagonal classical Ramsey numbers corresponding to $r(n, n)$ the exact determination (see Radziszowski, 2004 for a survey) of the has abruptly halted at $r(5, 5)$ (at present known to be between 43 and 49). One of the first variations of the classical Ramsey numbers namely size Ramsey numbers was introduced by Erdös, Faudree, Rousseau and Shelph (i.e., Erdös et al., 1978; Rousseau et al. 1978). In the last decade, using this idea of the original classical Ramsey numbers and of the size Ramsey numbers, the notion of size multipartite Ramsey numbers were introduced by Burger et al. and Syafrizal et al. (i.e., Burger et al., 2004; Syfrizal et al., 2005) by exploring the two colorings of multipartite graphs $K_{j\times s}$ instead of the complete graph. More formally, they defined *size Ramsey multipartite number* $m_j(G, H)$ to be the smallest natural number s such that $K_{j\times s} \to (G, H)$. A few classes of such size Ramsey multipartite number have been studied by Syafrizal Sy, et al. (i.e., Syfrizal et al., 2005; Syfrizal et al., 2007; Syfrizal et al., 2009; Syfrizal et al., 2012; Syfrizal, 2010 and Syfrizal, 2011). However, it is unfortunate that unlike in the cases of Ramsey numbers or size bipartite Ramsey numbers, the search has been restricted to a few Ramsey multipartite numbers between classes of graphs. In most cases, even in the case when such Ramsey numbers are found it has been limited to restricted classes of graphs. Motivated by this fact, in this paper we try to extend the list of size Ramsey multipartite numbers for pairs of classes of graphs, by finding the exact values of "Size multipartite Ramsey numbers for paths versus stars".

1. Notation

The *order* of the graph $G = (V, E)$ is denoted by $|V(G)|$ and the number of edges in the graph is denoted by $|E(G)|$. The *neighborhood* of a vertex $v \in V$ is defined as the set of vertices adjacent to v and is denoted by $N(v)$. The *degree* of v, $d(v)$, is defined as the cardinality of $N(v)$. Also, $\delta(G)$ is defined as the minimum degree of graph G. A *k regular graph* on n vertices is a graph G on n vertices which satisfies $d(v) = k$ for all $v \in V(G)$. Given a red/blue coloring of $K_{j\times s} = H_R \oplus H_B$, define the red degree and blue degree of any vertex $v \in V(K_{j\times s})$ denoted by $d_R(v)$ and $d_B(v)$ as the degree of vertex v in H_R and H_B respectively. Define δ_R and δ_B as the minimum degree of H_R and H_B respectively. If C is a set of vertices of $K_{j\times s}$ and $x \in C$ then the set of vertices of C belonging to the partite set x belongs to is denoted by C_x. Define k as $\left\lceil \frac{n-1}{j} \right\rceil$ and k_{n-2} as $\left\lceil \frac{n-2}{j} \right\rceil$. Unless stated otherwise let $n \geq 3$ and $m \geq 2$.

2. Some Lemmas

Lemma 2.1. *If any red/blue coloring of $K_{j\times s}$ given by $K_{j\times s} = H_R \oplus H_B$ has a red P_l then there exists a partite set that contains at least $\left\lceil \frac{l}{j} \right\rceil$ points of P_l in it.*

Proof. The proof follows from the pigeon-hole principle and is left to the reader. □

Lemma 2.2. *Let $j \geq 3$. If $a_n = n - \left\lceil \frac{n}{j} \right\rceil$ then $(a_n)_{n \in N}$ is monotonically increasing sequence and $a_{n-1} - a_{n-3} \geq 1$ for all $n \geq 3$. Also,*

$$a_{n-1} - a_{n-2} = \begin{cases} 1 & \text{if } (n-2) \neq 0 \bmod(j) \\ \\ 0 & \text{otherwise} \end{cases}$$

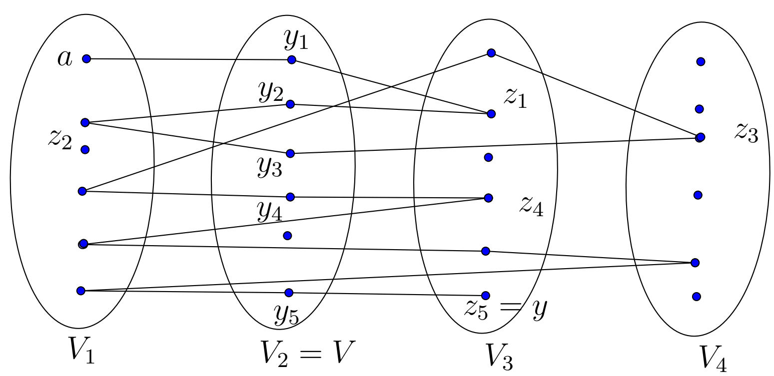
Proof. The proof is left to the reader. □

Lemma 2.3. *Let $k_l = \left\lceil \frac{l}{j} \right\rceil$. Consider any red/blue, coloring of $K_{j \times s}$ containing the red path of size l. If $\delta_R \geq l - k_l + 1$, then there exists a red path of size $l+1$.*

Proof. Assume that such a red path of size $l+1$ does not exist.

Claim: There exists a red P_l with say x as its end point such that C_x contains at least k_l number of vertices of P_l.

Proof of Claim: Let a, y be the end vertices of the red path containing l points. By the previous lemma, let V be a partite set containing at least k_l vertices of this P_l. Let $C = V(P_l)$. If $a \in V$ or $y \in V$ the claim follows. Suppose that $a \notin V$ and $y \notin V$ as shown in the following diagram. Let P_l be given by $a, ..., y_1, z_1, ..., y_2, z_2, ..., y_{k_l}, z_{k_l}, ..., y$, where $\{y_1, y_2, y_3, ..., y_{k_l}\} \subseteq V$. Note that in general z_{k_l} may coincide with y. As P_l is a path of length l, a cannot be adjacent in red to any vertices outside $V(P_l)$. Then there are two possibilities.



Example of a situation where $l = 17, j = 4$, and $k_l = 5$

Case 1: $(C_a)^c \cap \{z_1, z_2, z_3, ..., z_{k_l - 1}\} \neq \phi$.
Then $|C_a \cup \{z_1, z_2, z_3, ..., z_{k_l-1}\}| \geq k_l$. That is, the vertex a is adjacent in red to some vertex z_i for some $i \in \{1, 2, ..., k_l - 1\}$.
Then the path $y_i..., z_{i-1}, y_{i-1}, ..., z_{i-2}, y_{i-2}, ...z_1, y_1, ..., a, z_i, ..., y_{i+1}, z_{i+1}, ...y_{k_l}, z_{k_l}, ..., y$ is a red P_l with y_i as its end vertex such that $y_i \in V$, which contains at least k_l number of vertices of this P_l.

Case 2: $\{z_1, z_2, z_3, ..., z_{k_l-1}\} \subseteq C_a$
Then it follows that P_l is a longest red path with a as its end point such that C_a contains at least k_l number of vertices of P_l.

Hence we get claim from the two cases.

But then as $\delta_R \geq l - k_l + 1$, this path obtained from the claim can be extended to a path of size $l + 1$, a contradiction. Hence, the lemma.

□

Lemma 2.4. *$m_j(P_n, S_m) \leq m_j(P_n, S_{m+1})$ for any n, m.*

Proof. We skip this proof as its an elementary result in Ramsey Theory . □

Theorem 2.1. *$m_j(P_n, S_{m+1}) \leq \left\lceil \dfrac{n+m-1-k}{j-1} \right\rceil$ for $j \geq 3$.*

Proof. Let $j \geq 3$ and $k_l = \left\lceil \frac{l}{j} \right\rceil$. Also let $s = \left\lceil \dfrac{n+m-1-k}{j-1} \right\rceil$. Consider any red/blue, red P_n - free and blue S_{m+1} - free coloring given by $K_{j\times s} = H_R \oplus H_B$. Let the longest red path be P_l where $l \leq n-1$. Then for any vertex $x \in K_{j\times s}$,

$$d_R(x) = s(j-1) - d_B(x) \geq n + m - k - 1 - (m-1) = n - k \geq l - k_l + 1$$

Therefore, by lemma 2.3, we will obtain a red path of size $l + 1$, a contradiction.

□

Lemma 2.5. *Suppose that $j \geq 3$ and $n + 1 - k = s(j-1) - p$ where $p \in \{0, 1, ..., j-2\}$ and $s = \left\lceil \dfrac{n+1-k}{j-1} \right\rceil$. Then, $m_j(P_n, S_3) = s$ if $(s-1)j$ can be expressed as a linear combination of $n-2$ if $n-2 = 0 \ mod \ j$ and $n-1$.*

Proof. It suffices to show that $m_j(P_n, S_3) \geq s$. Let $k_1 = \left\lceil \frac{n-2}{j} \right\rceil$.
Construct a red/blue coloring of $K_{j\times(s-1)} = H_R \oplus H_B$, by partitioning $(s-1)j$ (using a lexicographical type ordering with respect to rows) in to sets satisfying either one of the following categories.
a) Consists of size $n-1$ such that any such set intersected with any partite set will have k or $k-1$ elements.
b) Consists of size $n-2$ where $n-2 = 0 \ mod \ j$ such that any such set intersected with any partite set will have k_1 elements. However, since $n-2 = 0 \ mod \ j$ we get that any such set intersected with any partite set will in fact have $k-1$ elements.

Color the edge (a, b) in red if a and b belong to the same set, else color (a, b) in blue. By construction H_R is P_n - free. Let x be any point of $K_{j\times(s-1)}$. Then by the above construction,
a) If x is contained in a set of size $n-1$ then the number of elements of that set, contained in the partite set which x belongs to will be k or $k-1$. Then,

$$d_B(x) = (s-1)(j-1) - ((n-1)-k) = n+2-k+p-j-(n-k-1) \leq 1$$

or

$$d_B(x) = (s-1)(j-1) - ((n-1)-(k-1)) = n+2-k+p-j-(n-k) \leq 1$$

b) If x is contained in a set of size $n-2$ then the number of elements of that set, contained in the partite set which x belongs to will be $k-1$. Then,

$$d_B(x) = (s-1)(j-1) - ((n-2)-(k-1)) = n+2-k+p-j-(n-k-1) \leq 1$$

Therefore, H_B will not contain a blue S_3. Hence the result.

□

Lemma 2.6. *Let $j \geq 3$ and $n+m-1-k = s(j-1)-p$ where $p = j-2$, $s = \left\lceil \frac{n+m-1-k}{j-1} \right\rceil$. Then, $m_j(P_n, S_{m+1}) = s$ if $(s-1)j$ can be expressed as a linear combination of $n-2$ if $n-2 = 0$ $mod\ j$ and $n-1$.*

Proof. It suffices to show that $m_j(P_n, S_{m+1}) \geq s$. Let $k_1 = \left\lceil \frac{n-2}{j} \right\rceil$.
Construct a red/blue coloring of $K_{j\times(s-1)} = H_R \oplus H_B$, by partitioning $(s-1)j$ (using a lexicographical type ordering with respect to rows) in to sets satisfying either one of the following categories.
a) Consists of size $n-1$ such that any such set intersected with any partite set will have k or $k-1$ elements.
b) Consists of size $n-2$ where $n-2 = 0\ mod\ j$ such that any such set intersected with any partite set will have k_1 elements. However, since $n-2 = 0\ mod\ j$ we get that any such set intersected with any partite set will in fact have $k-1$ elements.

Color the edge (a, b) in red if a and b belong to the same set, else color (a, b) in blue. By construction H_R is P_n - free. Let x be any point of $K_{j\times(s-1)}$. Then by the above construction,
a) If x is contained in a set of size $n-1$ then the number of elements of that set, contained in the partite set which x belongs to will be k or $k-1$. Then,

$$d_B(x) = (s-1)(j-1) - ((n-1)-k) = n+m-2-k-(n-k-1) = m-1$$

or

$$d_B(x) = (s-1)(j-1) - ((n-1)-(k-1)) = n+m-2-k-(n-k) = m-2$$

b) If x is contained in a set of size $n-2$ then the number of elements of that set, contained in the partite set which x belongs to will be $k-1$. Then,

$$d_B(x) = (s-1)(j-1) - ((n-2)-(k-1)) = n+m-2-k-(n-k-1) = m-1$$

Therefore, H_B will not contain a blue S_{m+1}. Hence the result.

□

Lemma 2.7. *Let $j \geq 3$, $m \geq 2$ and $n+m-1-k = s(j-1)-p$ where $p \in \{0,1,...,j-2\}$, $s = \left\lceil \frac{n+m-1-k}{j-1} \right\rceil$. Then, $m_j(P_n, S_{m+1}) = s$ if $(s-1)j$ can be expressed as a linear combination of $n-2$ if $n-2 = 0 \ mod \ j$ and $n-1$.*

Proof. Let $j \geq 3$. Suppose that $(s-1)j$ can be expressed as a linear combination of $n-2$ if $n-2 = 0 \ mod \ j$ and $n-1$.

Case 1: Given n and m we can find a $m' \geq 2$ such that

$$n+m'-1-k = s(j-1)-(j-2) \ \ \texttt{and} \ \ 0 \leq m-m' < j-1$$

where $s = \left\lceil \frac{n+m'-1-k}{j-1} \right\rceil$.
From lemma 2.6, we get that $m_j(P_n, S_{m'+1}) = s$. Applying lemma 2.4 to this result, gives us $m_j(P_n, S_{m+1}) \geq s$. Also we can observe that $s = \left\lceil \frac{n+m'-1-k}{j-1} \right\rceil = \left\lceil \frac{n+m-1-k}{j-1} \right\rceil$.
Hence by theorem 2.1, we also get $m_j(P_n, S_{m+1}) \leq s$.
Therefore, we can conclude that $m_j(P_n, S_{m+1}) = s$.

Case 2: If no such m' satisfying $m' \geq 2$ exists, then we get that $2 \leq m < j$ and

$$n+2-1-k = s(j-1)-(j-2)$$

where $s = \left\lceil \frac{n+2-1-k}{j-1} \right\rceil$.
From lemma 2.5, we get that $m_j(P_n, S_3) = s$. Applying lemma 2.4 to this result, gives us $m_j(P_n, S_{m+1}) \geq s$. Also we can observe that $s = \left\lceil \frac{n+m-1-k}{j-1} \right\rceil = \left\lceil \frac{n+2-1-k}{j-1} \right\rceil$. Hence by theorem 2.1, we also get $m_j(P_n, S_{m+1}) \leq s$.
Therefore, we can conclude that $m_j(P_n, S_{m+1}) = s$.

□

3. Some related results

In this section we first try to find a necessary and sufficient condition needed for $m_j(P_n, S_{m+1}) \leq s$ corresponding to $p = j-2$. Given any positive integer s and $j \geq 3$, in the later part of the theorem we will obtain an infinite class of pairs (n, m) achieving the upper bound s for $m_j(P_n, S_{m+1})$.

Lemma 3.1. *Let $n+m-1-k = s(j-1)-p$ where $p = j-2$, $s = \left\lceil \frac{n+m-1-k}{j-1} \right\rceil$ and $n \geq 3$. Then $m_j(P_n, S_{m+1}) \leq s-1$ if for any red/blue coloring of $K_{j\times s-1} = H_R \oplus H_B$, H_R cannot be partitioned in to a combination of connected components of size $(n-2)$ satisfying the additional condition $n-2 = 0 \ mod\ j$ or else of size $(n-1)$.*

Proof. Suppose that $m_j(P_n, S_{m+1}) \leq s-1$ is false. That is $m_j(P_n, S_{m+1}) \geq s$. Hence, by theorem 2.1, $m_j(P_n, S_{m+1}) = s$. Therefore, there exists a blue S_{m+1} - free and a red P_n - free, red/blue, coloring of $K_{j\times s-1} = H_R \oplus H_B$, containing a red path of size l where $l \leq n-1$. If the longest path of the graph is of size less than or equal to $n-3$, then as $(s-1)(j-1)-(m-1) = n-1-k \geq (n-3)-k_{n-3}+1$ by lemma 2.2 and lemma 2.3 we can obtain a path of size $l+1$, a contradiction. Similarly if the longest path is of size $n-2$ with $n-2$ satisfying the additional condition that $n-2 \neq 0 \ mod\ j$ again by lemma 2.2 and lemma 2.3, we can obtain a path of size $n-1$, as

$$(s-1)(j-1)-(m-1) = n-1-k = (n-2)-k_{n-2}+1$$

a contradiction. Hence the longest red path is of size either equal to $n-1$ or else equal to $n-2$ with the additional condition that $n-2 = 0 \ mod\ j$. Let C be the set of vertices of the longest red path.

If there exists a longest red path of size $n-1$, let x, y be its end vertices. By lemma 2.1, let V be the partite set containing at least k vertices (namely $y_1, y_2, ..., y_k$) of this P_{n-1}. Then as there is no blue $K_{1,m}$,

$$d_R(x) = (s-1)(j-1) - d_B(x) \geq n-k-1$$

Case 1) If $x \in V$ and $y \in V$. In the scenario that the path is contained in two partite sets we will get that in fact these two partite sets will contain $|C_x|$ and $|C_x|-1$ elements of the P_{n-1} vertices in them, this gives us $n \geq 2k$. But then as x is not adjacent to any vertices by a red edge outside of P_{n-1}, $n-k-1 \leq |C_x|-1$. That is $n \leq 2k$. Hence $n = 2k$. However,

$$\frac{n}{2} = \left\lceil \frac{n-1}{j} \right\rceil < \frac{n-1}{j} + 1 \leq \frac{n-1}{3} + 1$$

Using n is even, we get $n < 3$. A contradiction. Hence, this scenario is not possible.

Therefore, there is an edge (a, b) of $P_{n-1} = x, ..., a, b, ..., y$ where $a, b \notin V$. But then as $d_R(x) \geq n-k-1$, (x, b) and (y, a) will be red edges. This will result in a cycle containing all the vertices of P_{n-1}. Thus all the vertices of this longest path cannot be adjacent to any vertices outside the path in red.

Case 2) If $x \in V$ and $y \notin V$ then let P_{n-1} be $x, ..., y_1, z_1, ..., y_2, z_2, ..., y_3, z_3, ..., y_k, z_k, ..., y$ where z_k may coincide with y and $\{y_1, y_2, z_3, ..., y_k\} \subseteq V$. But since $d_R(x) \geq n-k-1$, (x, y) is red. This will give a red cycle of size $n-1$. Therefore, will get that the vertices of this longest path cannot be adjacent to any vertices outside the path in red.

Case 3) If $x \notin V$ and $y \notin V$ let P_{n-1} be $x, ..., y_1, z_1, ..., y_2, z_2, ..., y_3, z_3, ..., y_k, z_k, ..., y$, where z_k may coincide with y and $\{y_1, y_2, z_3, ..., y_k\} \subseteq V$. If x is not connected to any z_i in red then x can be adjacent in red to at most $n-k-2$ vertices of $V(P_{n-1})$. Also x cannot be adjacent in red to any vertices outside of $V(P_{n-1})$. This leads to a contradiction as $d_R(x) \geq n-k-1$. Therefore, x is adjacent in red to one of the vertices of $\{z_1, z_2, z_3, ..., z_k\}$. But this will give us a red path of size $n-1$ satisfying the conditions of case 2.

Next remove this $n-1$ size red component of H_R and consider the remaining vertices in H_R, if this too has a longest path of size $n-1$ we will obtain another red $n-1$ size component. Removing this component and repeating this process we will come to a stage where there are no red $n-1$ size paths left in the remaining vertices.
Thus in the remaining red graph there will be a longest path of size $n-2$ with the additional condition that $n-2 = 0 \ mod \ j$. Let x, y be the end vertices of the longest red path containing $n-2$ points. By lemma 2.3, let V be the partite set containing at least $k_{n-2} = k-1$(namely $y_1, y_2, ...y_{k-1}$) vertices of this P_{n-2}. Then as there is no blue S_{m+1},

$$d_R(x) = (s-1)(j-1) - d_B(x) \geq n-k-1$$

Case 1) If $x \in V$ and $y \in V$. In the scenario that the path is contained in two partite sets we will get that in fact these two partite sets will contain $|C_x|$ and $|C_x|-1$ where $C = V(P_{n-2})$. That is $n \geq 2k+1$. But then as x is not adjacent by a red edge to any vertices outside of P_{n-2}, $n-k-1 \leq |C_x|-1$. That is $n \leq 2k-1$. A contradiction.
Therefore, there is an edge (a, b) of $P_{n-2} = x, ..., a, b, ..., y$ where $a, b \notin V$. But then as $d_R(x) \geq n-k-1 = (n-2)-k_{n-2}$, (x, b) and (y, a) will be red edges. This will result in a cycle containing all the vertices of P_{n-2}. Thus all the vertices of this longest path cannot be adjacent to any vertices outside the path by a red edge.

Case 2) If without loss of generarity, say $x \in V$ and $y \notin V$, then
P_{n-2} can be represented as $x = y_1, z_1, ..., y_2, z_2, ..., y_3, z_3, ..., y_{k-1}, z_{k-1}, ..., y$ where z_{k-1} may coincide with y and $\{y_1, y_2, z_3, ..., y_{k-1}\} \subseteq V$. But then x must be adjacent to a point outside $V(P_{n-2})$ or adjacent to y. The second possibility will result in a red cycle of of size $n-2$. The first possibility leads to a contradiction and from the second possibility we get that all the vertices of this longest path cannot be adjacent to any vertices outside the path.

Case 3) If $x \notin V$ and $y \notin V$ let P_{n-2} be $x, ..., y_1, z_1, ..., y_2, z_2, ..., y_3, z_3, ..., y_{k-1}, z_{k-1}, ..., y$, where z_k may coincide with y and $\{y_1, y_2, z_3, ..., y_{k-1}\} \subseteq V$. If x is not connected to any z_i in red then x can be adjacent in red to at most $n-k-2$ vertices of $V(P_{n-2})$. Also x cannot be adjacent in red to any vertices outside of $V(P_{n-2})$. This leads to a contradiction as $d_R(x) \geq n-k-1$. Therefore, x is adjacent in red to one of the vertices of $\{z_1, z_2, z_3, ..., z_{k-1}\}$. But this will give us a red path of size $n-2$ satisfying the conditions of case 2.

Therefore removing this component and repeating this process we will come to a stage where there are no such red $n-2$ size paths remaining. Therefore, we will get that H_R can be partitioned in to connected components of size either $(n-1)$ or else of size $(n-2)$ sets satisfying the additional condition that $n-2 = 0 \ mod \ j$.
Hence the result. □

From lemma 2.6 and lemma 3.1 we obtain the following theorem giving the necessary and sufficient condition for $m_j(P_n, S_{m+1}) \leq s$ corresponding to $p = j - 2$.

Theorem 3.1. *Let* $j \geq 3$ *and* $n + m - 1 - k = s(j - 1) - p$ *where* $p = j - 2$ *and* $s = \left\lceil \frac{n+m-1-k}{j-1} \right\rceil$. *Then,* $m_j(P_n, S_{m+1}) = s$ *if* $(s-1)j$ *can be expressed as a linear combination of* $n - 2$ *if* $n - 2 = 0 \ mod \ j$ *and* $n - 1$.

Theorem 3.2. *Given any positive integer* s *and* $j \geq 3$, *there exists a pair* (n, m) *such that* $m_j(P_n, S_{m+1}) = s$.

Proof. Let $n = j + 1$ and $m = (j - 1)(s - 1) - 1$, then theorem 2.1 and lemma 2.7 gives $m_j(P_{j+1}, S_{(j-1)(s-1)}) = s$. Hence the theorem. □

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