Indonesian Journal of Combinatorics **5** (2) (2021), 94–101

INDONESIAN JOURNAL OF COMBINATORICS

On odd harmonious labeling of $P_n \trianglerighteq C_4$ and $P_n \trianglerighteq D_2(C_4)$

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Abstract

A graph G with q edges is said to be odd harmonious if there exists an injection $\tau : V(G) \to \mathbb{Z}_{2q}$ so that the induced function $\tau^* : E(G) \to \{1, 3, \ldots, 2q-1\}$ defined by $\tau^*(xy) = \tau(x) + \tau(y)$ is a bijection. Here we show that graphs constructed by edge comb product of path P_n and cycle on four vertices C_4 or shadow of a cycle of order four $D_2(C_4)$ are odd harmonious.

Keywords: Odd harmonious labeling, edge comb product, path, cycle, shadow graph.
Mathematics Subject Classification: 05C78
DOI: 10.19184/ijc.2021.5.2.5

1. Introduction

Throughout this paper we consider simple, finite, connected and undirected graph. A harmonious labeling was first introduced in 1980 by Graham and Sloane [4]. A harmonious labeling on a graph G with q edges is a one-to-one function $\tau : V(G) \to \mathbb{Z}_q$, such that the induced function $\tau^* : E(G) \to \mathbb{Z}_q$, defined by $\tau^*(e) = \tau^*(xy) = \tau(x) + \tau(y)$ for each edge $e = xy \in E(G)$ is a bijective function. One of various of harmonious labeling is an odd harmonious labeling. In 2019, Liang and Bai [12] was introduced an odd harmonious labeling. They defined that a graph G with q edges is said to be odd harmonious if there exists a one-to-one function $\tau : V(G) \to \{0, 1, \ldots, 2q-1\}$ so that the induced function $\tau^* : E(G) \to \{1, 3, \ldots, 2q-1\}$

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Received: 26 February 2021, Revised: 29 October 2021, Accepted: 31 October 2021.

defined by $\tau^*(xy) = \tau(x) + \tau(y)$ for each $uv \in E(G)$ is a bijection. Liang and Bai [12] proved that if G is an odd harmonious graph, then G is bipartite. They gave a relation between order and size of a harmonious graph, namely if G is an odd harmonious graph with p vertices and q edges, then p is on a closed interval $[2\sqrt{q}, 2q-1]$. In the same paper, they also proved that a cycle C_n is an odd harmonious graph if and only if $n \equiv 0(\text{mod}4)$.

There are many papers deal with odd harmonious labeling. In 2011, Vaidya and Shah [17] proved that the shadow graph of path P_n and star graph $K_{1,n}$ are odd harmonious graphs. Furthermore Vaidya and Shah [18] investigate odd harmonious labeling of the shadow graph and the splitting graph of bistar $B_{n,n}$, the arbitrary supersubdivision of path P_n, the joint sum of two copies of cycle C_n for $n \equiv 0(\bmod 4)$ and the graph $H_{n,n}$. Let G be a connected graph. The *shadow graph* $D_2(G)$ is constructed by taking two copies of G say G' and G'', and join each vertex $u' \in V(G')$ to the neighbours of the corresponding vertex u' in $V(G'')$.

Abdel-Aal [2] studied odd harmonious labelings of cyclic snakes. Alyani *et al.* [3] gave an odd harmonious labeling of kC_4-snake and kC_8-snake graphs. Abdel-Aal and Seoud [1] proved that m-shadow path is odd harmonious. Sugeng *et al.* [16] discussed about odd harmonious labeling of m-shadow of cycle, gear with pendant and shuriken graphs.

In their some papers, Jeyanthi and Philo studied odd harmonious labeling of some graphs, namely plus graphs [8], some cycle related graphs [9], the shadow and splitting of graph $K_{2,n}, C_n$ for $n \equiv 0(\text{mod}4)$ [10] and gird graph [6], super subdivision graphs [5], and some certain graphs [7]. Next, Jeyanthi *et al.* [11] proved that banana tree and the path union of cycles C_n for $n = 0(\text{mod}4)$ are odd harmonious.

Pujiwati *et al.* [13] gave an odd harmonious labeling of the double stars $S_{m,n}$. They also investigated whether the graphs obtained by an identification operation of a cycle and star, are odd harmonious or not. Srividya and Govindarajan [15] discussesd about an odd harmonious labelling of even cycles with parallel chords and dragons with parallel chords. Saputri *et al.* [14] proved that the dumbbell $D_{n,k,2}$ for $n \equiv k \equiv 0(\text{mod } 4)$ and the generalized prims graphs are odd harmonious.

Here we discuss an odd harmonious labeling of graphs formed by edge comb product of path P_n and the cycle C_4 or the shadow of a cycle on four vertices $D_2(C_4)$, namely $P_n \trianglerighteq C_4$ and $P_n \trianglerighteq D_2(C_4)$ for each $n \geq 2$. Let G and H be graphs. An *edge comb product* of two graphs G and H, denoted by $G \trianglerighteq H$, is a graph formed by taking one copy of G and $|E(G)|$ copies of H, then attaching the i-th copy of H at the edge e to the i-th edge of G.

2. Main Results

In this section, we prove that $P_n \trianglerighteq C_4$ and $P_4 \trianglerighteq D_2(C_4)$ are odd harmonious graphs. First, we consider a graph $P_n \trianglerighteq C_4$. A graph $P_n \trianglerighteq C_4$ has $3n-2$ vertices and $4(n-1)$ edges. Let

$$V(P_n \trianglerighteq C_4) = \{u_i | 1 \leq i \leq n\} \cup \{v_{i1}, v_{i2} | 1 \leq i \leq n-1\}$$

and

$$E(P_n \trianglerighteq C_4) = \{u_i v_{i1}, v_{i1} v_{i2}, u_i u_{i+1}, u_{i+1} v_{i2} | 1 \leq i \leq n-1\}$$

be the set of vertices and edges of $P_n \trianglerighteq C_4$, respectively. As an illustration, in Figure 1, we can see that the notation of vertices and edges of $P_5 \trianglerighteq C_4$.

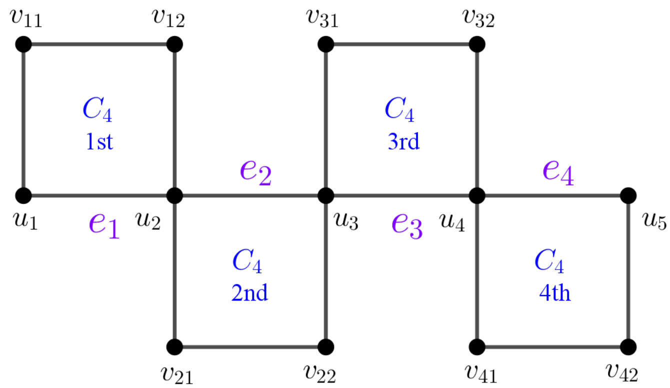

Figure 1. The notation of vertices and edges of $P_5 \trianglerighteq C_4$.

Theorem 2.1. *$P_n \trianglerighteq C_4$ is an odd harmonious graph for all $n \geq 2$.*

Proof. We define a vertex labeling $\tau : V(P_n \trianglerighteq C_4) \to \{0, 1, \ldots, 8n-9\}$ by

$$\tau(u_i) = \begin{cases} 4i-4, & \text{for odd } i, \\ 4i-3, & \text{for even } i, \end{cases}$$

for each $i = 1, 2, \ldots, n$, and

$$\tau(v_{ij}) = \begin{cases} 4i-3, & \text{for odd } i \text{ and } j = 1, \\ 4i-4, & \text{for even } i \text{ and } j = 1, \\ 4i-2, & \text{for odd } i \text{ and } j = 2, \\ 4i-1, & \text{for even } i \text{ and } j = 2, \end{cases}$$

for each $i = 1, 2, \ldots, n-1$. It is easily seen that each vertex of $V(P_n \trianglerighteq C_4)$ get distinct label. So, the vertex labeling $\tau : V(P_n \trianglerighteq C_4) \to \{0, 1, \ldots, 8n-9\}$ is an injective function. Next, by the vertex label, we obtain the edge labeling $\tau^* : E(P_n \trianglerighteq C_4) \longrightarrow \{1, 3, \ldots, 8n-9\}$ as follows.
For $i = 1, 2, \ldots, n-1$,

$$\begin{aligned} \tau^*(u_i u_{i+1}) &= 2(4i-2)+1, \\ \tau^*(v_{i1} v_{i2}) &= 2(4i-3)+1, \\ \tau^*(u_i v_{i1}) &= 2(4i-4)+1, \\ \tau^*(u_{i+1} v_{i2}) &= 2(4i-1)+1. \end{aligned}$$

We can see that all edges get odd distinct labels from $1, 3, \ldots, 8n-9$. Since the cardinality of the set $\{1, 3, \ldots, 8n-9\}$ is the same as the number of edges $E(P_n \trianglerighteq C_4)$, namely $4n-4$ and each edge obtain distinct labels, then $\tau^* : E(P_n \trianglerighteq C_4) \longrightarrow \{1, 3, \ldots, 8n-9\}$ is a bijection. Hence, $P_n \trianglerighteq C_4$ is an odd harmonious graph for all $n \geq 2$. □

An odd harmonious labeling of $P_7 \trianglerighteq C_4$ is depicted in Figure 2.

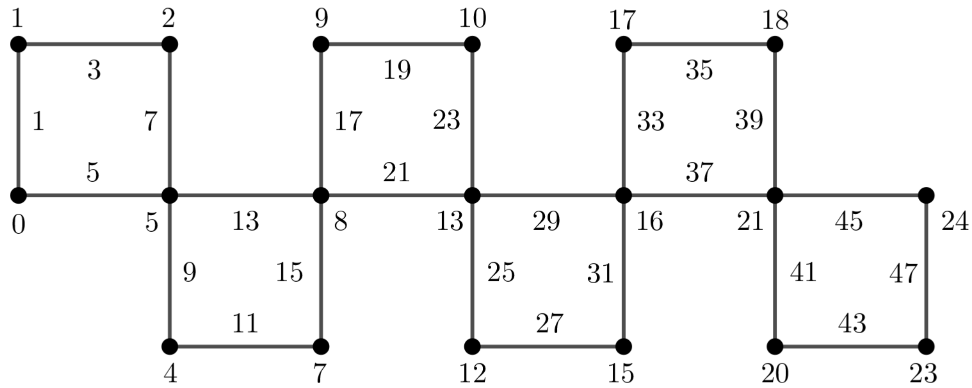

Figure 2. An odd harmonious labeling of $P_7 \trianglerighteq C_4$.

Furthermore, we consider a graph $P_n \trianglerighteq D_2(C_4)$. A graph $P_n \trianglerighteq D_2(C_4)$ has $7n-6$ vertices and $16(n-1)$ edges. We denote the vertex-set and edge-set of a graph $P_n \trianglerighteq D_2(C_4)$ as follows.

For $i = 1, 2, \ldots, n-1$,

$$V(P_n \trianglerighteq D_2(C_4)) = \{u_1, u_2, \ldots, u_n\} \cup \{v_{ij}, x_{ij}, y_{ij} |\ j = 1, 2\}$$

and

$$\begin{aligned} E(P_n \trianglerighteq D_2(C_4)) &= \{u_iu_{i+1}, v_{i1}v_{i2}, x_{i1}x_{i2}, y_{i1}y_{i2}\} \cup \{u_iv_{i1}, x_{i1}y_{i1}, x_{i2}y_{i2}, u_{i+1}v_{i2}\} \cup \\ &\quad \{u_ix_{i1}, u_iy_{i2}, v_{i1}x_{i2}, v_{i1}y_{i1}\} \cup \{u_{i+1}x_{i2}, u_{i+1}y_{i1}, v_{i2}x_{i1}, v_{i2}y_{i2}\}. \end{aligned}$$

Figure 3 shows the vertices and edges notation of the $P_5 \trianglerighteq D_2(C_4)$.

Theorem 2.2. *$P_n \trianglerighteq D_2(C_4)$ is an odd harmonious graph for all $n \geq 2$.*

Proof. We define the vertex labeling of $V(P_n \trianglerighteq D_2(C_4))$, $\tau : V(P_n \trianglerighteq D_2(C_4)) \to \{0, 1, \ldots, 32n-33\}$ as follows. For $i = 1, 2, \ldots, n$,

$$\tau(u_i) = \begin{cases} 16i - 16, & \text{for odd } i, \\ 16i - 25, & \text{for even } i, \end{cases}$$

and for $i = 1, 2, \ldots, n-1$,

$$\tau(v_{ij}) = \begin{cases} 16i - 15, & \text{for odd } i \text{ and } j = 1, \\ 16i - 6, & \text{for even } i \text{ and } j = 1, \\ 16i + 8, & \text{for odd } i \text{ and } j = 2, \\ 16i - 1, & \text{for even } i \text{ and } j = 2, \end{cases}$$

$$\tau(x_{ij}) = \begin{cases} 16i - 13, & \text{for odd } i \text{ and } j = 1, \\ 16i - 4, & \text{for even } i \text{ and } j = 1, \\ 16i, & \text{for odd } i \text{ and } j = 2, \\ 16i - 9, & \text{for even } i \text{ and } j = 2, \end{cases}$$

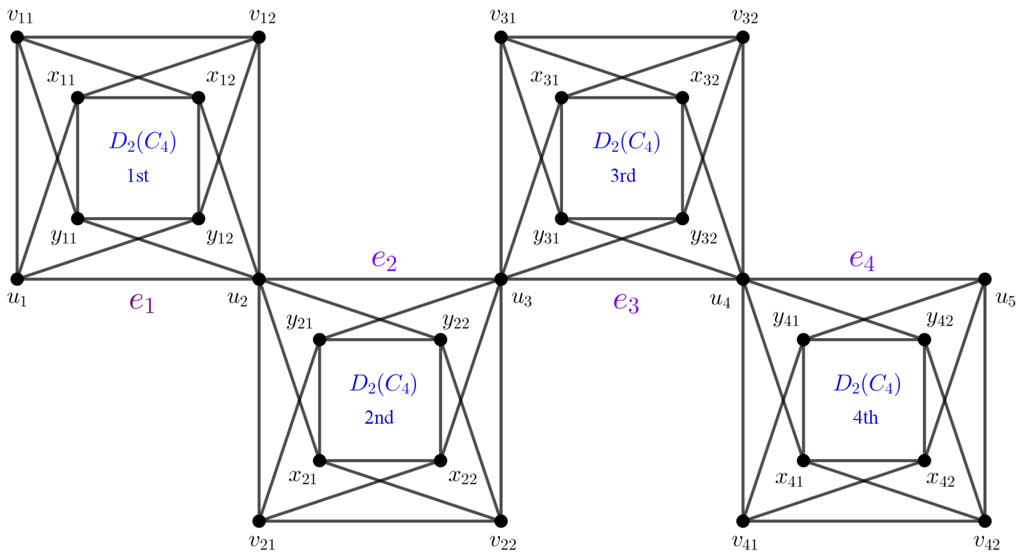

Figure 3. The notation of vertices dan edges of $P_5 \trianglerighteq D_2(C_4)$.

$$\tau(y_{ij}) = \begin{cases} 16i-8, & \text{for odd } i \text{ and } j=1, \\ 16i-17, & \text{for even } i \text{ and } j=1, \\ 16i-11, & \text{for odd } i \text{ and } j=2, \\ 16i-2, & \text{for even } i \text{ and } j=2. \end{cases}$$

We see that each vertex of $V(P_n \trianglerighteq D_2(C_4))$ has distinct label. So, the vertex labeling τ is injective. By the vertex labeling τ, we obtain the edge label by the formula $\tau^*(xy) = \tau(x) + \tau(y)$ for each $xy \in E(P_n \trianglerighteq D_2(C_4))$ and prove that every edge gets the distinct odd label.
For $i = 1, 2, \ldots, n-1$,

$$\begin{array}{lcllcl} \tau^*(u_iu_{i+1}) & = & 32i-25, & \tau^*(v_{i1}v_{i2}) & = & 32i-7, \\ \tau^*(x_{i1}x_{i2}) & = & 32i-13, & \tau^*(y_{i1}y_{i2}) & = & 32i-19, \\ \tau^*(u_iv_{i1}) & = & 32i-31, & \tau^*(x_{i1}y_{i1}) & = & 32i-21, \\ \tau^*(x_{i2}y_{i2}) & = & 32i-11, & \tau^*(u_{i+1}v_{i2}) & = & 32i-1 \\ \tau^*(u_ix_{i1}) & = & 32i-29, & \tau^*(u_iy_{i2}) & = & 32i-27, \\ \tau^*(v_{i1}x_{i2}) & = & 32i-15, & \tau^*(v_{i1}y_{i1}) & = & 32i-23, \\ \tau^*(u_{i+1}x_{i2}) & = & 32i-9, & \tau^*(u_{i+1}y_{i1}) & = & 32i-17, \\ \tau^*(v_{i2}x_{i1}) & = & 32i-5, & \tau^*(v_{i2}y_{i2}) & = & 32i-3. \end{array}$$

It is easily seen that each edge obtains the distinct odd label. Thus, τ is an odd harmonious labeling. Therefore $P_n \trianglerighteq D_2(C_4)$ is odd harmonious for all $n \geq 2$. □

For an illustration, an odd harmonious labeling of $P_5 \trianglerighteq D_2(C_4)$ as depicted in Figure 4.

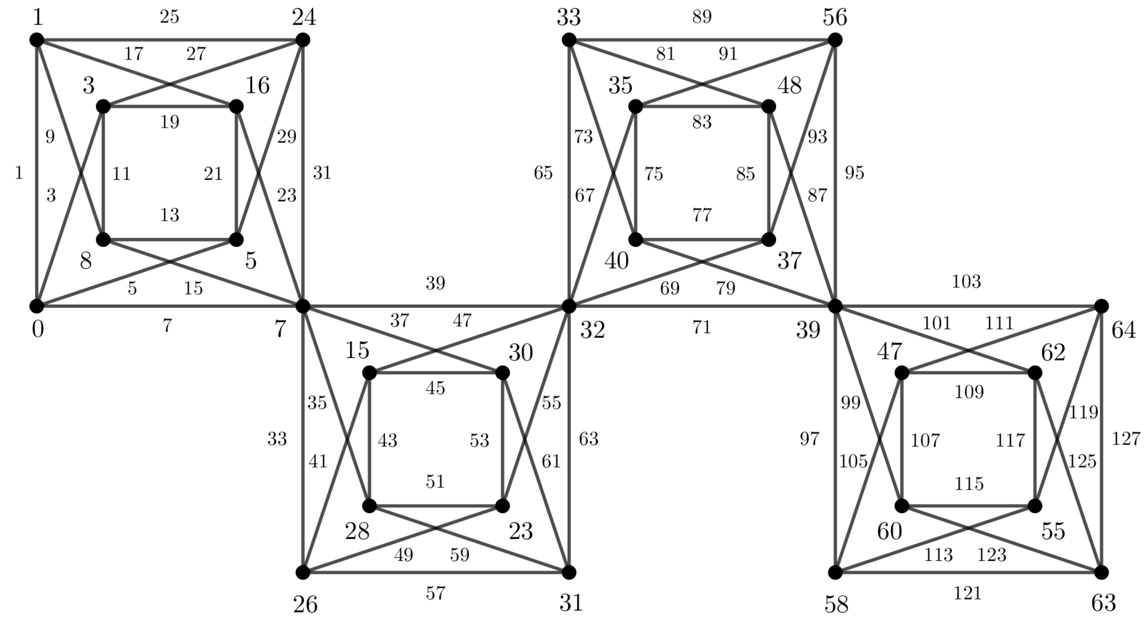

Figure 4. An odd harmonious labeling of $P_5 \trianglerighteq D_2(C_4)$.

3. Concluding Remarks

We conclude this paper by giving some open problems.

1. Whether edge comb product of path P_n and a cycle C_m is an odd harmonious graph or not, for each $n \geq 2$, $m \geq 5$.
2. Investigate the odd harmonious labeling of edge comb product of path P_n and shadow of a cycle $D_2(C_m)$, namely $P_n \trianglerighteq D_2(C_m)$ for all $n \geq 2$, $m \geq 5$.

Acknowledgments

This research has been supported by ”Stimulus Penelitian, Universitas Jember, Tahun Anggaran 2021”

References

[1] M.E. Abdel-Aal and M.A. Seoud, Futher results on odd harmonious graphs, *International Journal on Applications of Graph Theory in Wireless Ad hoc Networks and Sensor Networks (GRAPH-HOC*, **8(3-4)**, (2016), 1–14, https://doi.org/10.5121/jgraphoc.2016.8401

[2] M. E. Abdel-Aal, Odd harmonious labelings of cyclic snakes, *International Journal on Applications of Graph Theory in Wireless Ad hoc Networks and Sensor Networks (GRAPH-HOC)*, **5(3)**, (2013), 1–11, https://doi.org/10.5121/jgraphoc.2013.5301

[3] F. Alyani, F. Firmansah, W. Giyarti, and K.A. Sugeng, The odd harmonious labeling of kC_n-snake graphs for spesific values of n, that is, for $n = 4$ and $n = 8$, *Proceeding of IICMA 2013*, (2013), 225–230.

[4] R.L. Graham and N.J.A. Sloane, On additive bases and harmonious graphs, *SIAM J. Alg. Disc. Meth.*, **1(4)**, (1980), 382–404, https://doi.org/10.1137/0601045

[5] P. Jeyanthi, S. Philo, and M.K. Siddiqui, Odd harmonious labeling of super subdivision graphs, *Proyecciones J. Math.* **38(1)**, (2019), 1–11, https://doi.org/10.4067/S0716-09172019000100001

[6] P. Jeyanthi, S. Philo, and M. Youssef, Odd harmonious labeling of grid graph, *Proyecciones J. Math.* **38**, (2019), 411–428, https://doi.org/10.22199/issn.0717-6279-2019-03-0027

[7] P. Jeyanthi and S. Philo, Odd harmonious labeling of certain graphs, *Journal of Applied Science and Computations*, **6(4)**, (2019), 1224–1232.

[8] P. Jeyanthi and S. Philo, Odd harmonious labeling of plus graphs, *Bull. Int. Math. Virtual Inst.*, **7**, (2017), 515–526, DOI : 10.7251/BIMVI1703515J

[9] P. Jeyanthi and S. Philo, Odd harmonious labeling of some cycle related graphs, *Proyecciones J. Math.* **35(1)**, (2016), 85–98, https://doi.org/10.4067/S0716-09172016000100006

[10] P. Jeyanthi and S. Philo, Odd harmonious labeling of some new families of graphs, *Electron. Notes Discrete Math.* **48**, (2015), 165 –168, https://doi.org/10.1016/j.endm.2015.05.024

[11] P. Jeyanthi, S. Philo, and K.A. Sugeng, Odd harmonious labeling of some new families of graphs, *SUT J. Math.* **51(2)**, (2015), 181–193.

[12] Z. Liang and Z. Bai, On the odd harmonious graphs with applications, *J. Appl. Math. Comput.*, **29**, (2009), 105–116, https://doi.org/10.1007/s12190-008-0101-0

[13] D.A. Pujiwati, I. Halikin, and K. Wijaya, Odd harmonious labeling of two graphs containing star, *AIP Conference Proceedings* **2326**, 020019 (2021), https://doi.org/10.1063/5.0039644

[14] G.A. Saputri, K.A. Sugeng, and D. Froncek, The odd harmonious labeling of dumbbell and generalized prims graphs, *AKCE Int, J. Graphs Comb.*, **10(2)**, (2013), 221–228, https://doi.org/10.1080/09728600.2013.12088738

[15] V. Srividya and R. Govindarajan, On odd harmonious labelling of even cycles with parallel chords and dragons with parallel chords, *International Journal of Computer Aided Engineering and Technology* **13(4)**, (2020), https://doi.org/10.1504/IJCAET.2020.110475

[16] K.A. Sugeng, S. Surip, and R. Rismayati, On odd harmonious labeling of m-shadow of cycle, gear with pendant and shuriken graphs, *AIP Conference Proceedings* **2192**, 040015 (2019), https://doi.org/10.1063/1.5139141

[17] S.K. Vaidya and N.H. Shah, Some new odd harmonious graphs, *International Journal of Mathematics Soft Computing*, **1(1)**, (2011), 9–16.

[18] S.K. Vaidya and N.H. Shah, Odd harmonious labeling of some graphs, *International J. Math. Combin.* **3**, (2012), 105–112, https://doi.org/10.5281/ZENODO.9410