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On the number of caterpillars

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Abstract

A caterpillar is a tree obtained from a path by attaching pendent vertices. The number of caterpillars of size n is a well-known result. In this work we extend this result exploring the number of caterpillars of size n together with the cardinality of the stable sets as well as the diameter. Three closed formulas are presented, giving the number of caterpillars of size n with: (i) smaller stable set of cardinality k, (ii) diameter d, and (iii) diameter d and smaller stable set of cardinality k.

Keywords: caterpillar, tree, enumeration
Mathematics Subject Classification: 05C30, 05C05

1. Introduction

We use here standard notation and terminology, as in [3]. All graphs considered in this work are finite and simple, that is, with no loops or multiple edges. A graph G of order m and size n is a graph where its vertex set V has cardinality m and its edge set E has cardinality n. A vertex v of G is said to be a *leaf* when its degree is one, otherwise v is called an *internal vertex*. A graph G is *bipartite* if V is the union of two *independent sets* or *stable sets* of G. If G is a connected graph and $u, v \in V$, the *distance* between u and v, denoted by $d(u, v)$, is the length (or size) of the shortest path from u to v; the *diameter* of G is $\max\{d(u, v) : u, v \in V\}$. For each $v \in V$, $\deg(v)$ denotes the *degree* of v, i.e., the number of edges of G incident to v.

For every $m \geq 1$, P_m denotes the *path* of order m, where $P_1 \cong K_1$ and for all $m \geq 2$, P_m is the tree with exactly two leaves, equivalently, the tree of order m with $m - 2$ interior vertices. Paths can be seen as members of a larger family of trees, the family of caterpillars. A *caterpillar*

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is a tree of order $m \geq 3$ with the property that the deletion of all its leaves results in a path; the paths P_1 and P_2 are considered caterpillars in this work.

Harary and Schewnk [2] applied Pólya's enumeration theorem to determine the number of non-isomorphic caterpillars of order m, finding the generating function for the number of caterpillars. A closed formula for the number of caterpillar of order m is $2^{m-4} + 2^{\lfloor \frac{m-4}{2} \rfloor}$. The present work includes three enumerations of caterpillars, where the size of a caterpillar is combined with two other parameters: the diameter and the cardinality of the smaller stable set. In Section 2 we introduce the basic results needed to perform the calculations in the upcoming sections. In Section 3 we determine the number $a(n,k)$ of caterpillars of size n with stable sets of cardinality k and $n+1-k$; in Section 4 we study the number $b(n,d)$ of caterpillars of size n and diameter d; we conclude this work in Section 5, where we give the number $c(n,d,k)$ of caterpillars of size n, diameter d, and stable sets of cardinalities k and $n+1-k$.

In this work we use the symbol $C(s,r)$ to represent the binomial coefficients, that is,

$$C(s,r) = \frac{s!}{r!(s-r)!}.$$

2. The Essential Results

Suppose that $\mathscr{Z}$ is a finite set of integers and $\mathscr{S}$ is the set of all sequences of length r formed with elements of $\mathscr{Z}$. Let $S_1 = \{a_i\}_i^r$ and $S_2 = \{b_i\}_i^r$ be elements of $\mathscr{S}$, we say that $S_1 \approx S_2$ if for every $i \in \{1, 2, \ldots, r\}$ at least one of the following conditions hold:

(a) $a_i = b_i$,

(b) $a_i = b_{r+1-i}$.

It is not difficult to see that $\approx$ is an equivalence relation on the set $\mathscr{S}$. We are interested in finding the number of equivalence classes induced by $\approx$. Let $S \in \mathscr{S}$, the equivalence class of S is denoted by $[S]$. If $S = \{a_i\}_i^r$, its reverse sequence is defined as $S^{-1} = \{a_{r+1-i}\}_i^r$. Thus, both S and S^{-1} are in the same equivalence class. The sequence S is said to be ***reversible*** if $S = S^{-1}$. Therefore $[S] = \{S\}$ if and only if S is reversible, otherwise $[S] = \{S, S^{-1}\}$. Thus, using Pólya's enumeration theorem, we know that the number of equivalence classes induced by $\approx$ on $\mathscr{S}$, is half of the addition of the total number of sequences and the total number of reversible sequences. This is the technique used in this work to find the closed formulas for $a(n,k)$, $b(n,d)$, and $c(n,d,k)$.

Consider the following equation, where each x_i is an integer. In the following sections we use the number of solutions to this equation to ascertain the specific amount of caterpillars considered there.

$$x_1 + x_2 + \cdots + x_r = s. \tag{1}$$

The next two theorems can be found in Chapter 3 of the combinatorics book by Allenby and Slomson [1]. We present them omitting their proofs.

Theorem 2.1. *For each positive integer r and each nonnegative integer s, the number of nonnegative integer solutions, $(x_1, x_2, \dots, x_r)$, of equation (1) is $\sigma_0(s,r) = C(s+r-1, r-1)$.*

Theorem 2.2. *Let r, s be positive integers with $s \geq r$. Then the number of positive integer solutions, $(x_1, x_2, \dots, x_r)$, of equation (1) is $\sigma(s,r) = C(s-1, r-1)$.*

If $(x_1, x_2, \dots, x_r)$ is a solution of equation (1), where each $x_i > 0$, then $(x_1, x_2, \dots, x_r)$ is a *partition* of s; moreover, it is an ordered partition in the sense that any permutation of its entries provides a solution of (1). In this work the word partition is used to indicate an ordered partition.

In the next theorems we use the following well-known property of the addition of binomial coefficients.

Theorem 2.3. *For all positive integers, j, t with $j \leq t$,*

$$C(t+1, j+1) = C(t, j) + C(t-1, j) + \cdots + C(j, j).$$

If $(x_1, x_2, \dots, x_r)$ is a solution of equation (1), then it is an element of $\mathscr{S}$, therefore it is reversible when $x_i = x_{r+1-i}$ for each $i \in \{1, 2, \dots, r\}$. We are interested in the number of reversible solutions of equation (1). As we saw in the first two theorems, the number of solutions of (1) depends on the nature of the x_i; consequently, we need to consider two instances. Let $\rho_0(s,r)$ denote the number of reversible nonnegative integer solutions, while $\rho(s,r)$ designates the number of reversible positive integer solutions. In order to determine $\rho_0(s,r)$ and $\rho(s,r)$ we analyze two cases based on the parity of r.

Suppose first that r is even. Since $(x_1, x_2, \dots, x_r)$ is reversible, we conclude that equation (1) can be reduced to

$$x_1 + x_2 + \cdots + x_{\frac{r}{2}} = \frac{s}{2}. \tag{2}$$

But equation (2) has integer solutions if and only if s is even because each x_i is an integer. Therefore, $\rho_0(s,r) = \rho(s,r) = 0$ when s is odd and r is even. Theorems 2.1 and 2.2 can be used to determine $\rho_0(s,r)$ and $\rho(s,r)$ when s is even. Thus, we know that $\rho_0(s,r) = C\left(\frac{s+r-2}{2}, \frac{r-2}{2}\right)$ and $\rho(s,r) = C\left(\frac{s-2}{2}, \frac{r-2}{2}\right)$.

Suppose now that r is odd. Since $(x_1, x_2, \dots, x_r)$ is reversible, equation (1) can be written as

$$2(x_1 + x_2 + \cdots + x_{\frac{r-1}{2}}) + x_{\frac{r+1}{2}} = s, \tag{3}$$

which is equivalent to

$$x_1 + x_2 + \cdots + x_{\frac{r-1}{2}} = \frac{1}{2}\left(s - x_{\frac{r+1}{2}}\right). \tag{4}$$

Note that the right side of equation (4) is an integer if and only if s and $x_{\frac{r+1}{2}}$ have the same parity. This implies that in addition to the nature of each x_i we also need to consider the parity of s to calculate the value of $\rho_0(s,r)$ and $\rho(s,r)$. We analyze now the four different scenarios.

(a) *When s is even and each x_i is a nonnegative integer*. Since each x_i is a nonnegative integer, we have that $x_{\frac{r+1}{2}} \in \{0, 2, \ldots, s\} = \{2j : 0 \leq j \leq \frac{s}{2}\}$. Let $x_{\frac{r+1}{2}} = 2j$ for some $j \in \{0, 1, \ldots, \frac{s}{2}\}$, then equation (4) can be written as

$$x_1 + x_2 + \cdots + x_{\frac{r-1}{2}} = \frac{s}{2} - j. \tag{5}$$

The number of nonnegative integer solutions of this last equation can be calculated, using Theorem 2.1, to be equal to $C\left(\frac{s}{2} - j + \frac{r-3}{2}, \frac{r-3}{2}\right)$. Seeing that $j \in \{0, 1, \ldots, \frac{s}{2}\}$, Theorem 2.3 tells us that

$$\rho_0(s, r) = \sum_{j=0}^{\frac{s}{2}} C\left(\frac{s}{2} - j + \frac{r-3}{2}, \frac{r-3}{2}\right) = C\left(\frac{s}{2} + \frac{r-1}{2}, \frac{r-1}{2}\right).$$

(b) *When s is even and each x_i is a positive integer*. Considering that each x_i is a positive integer, we have that $x_{\frac{r+1}{2}} \in \{2, 4, \ldots, s\}$, but equation (4) has positive integer solutions if and only if $\frac{1}{2}(s - x_{\frac{r+1}{2}}) \geq \frac{r-1}{2}$, which is equivalent to say that $x_{\frac{r+1}{2}} \leq s - r + 1$, in other terms, $x_{\frac{r+1}{2}} \in \{2, 4, \ldots, s - r + 1\} = \{2j : 1 \leq j \leq \frac{r-1}{2}\}$. If $x_{\frac{r+1}{2}} = 2j$ for some $j \in \{1, 2, \ldots, \frac{s}{2} - \frac{r-1}{2}\}$, then the number of positive integer solutions of equation (5) is, determined using Theorem 2.2, to be $C\left(\frac{s}{2} - j - 1, \frac{r-1}{2} - 1\right)$. Using Theorem 2.3 we get

$$\rho(s, r) = \sum_{j=1}^{\frac{s-r+1}{2}} C\left(\frac{s}{2} - j - 1, \frac{r-3}{2}\right) = C\left(\frac{s-2}{2}, \frac{r-1}{2}\right).$$

(c) *When s is odd and each x_i is a nonnegative integer*. Since each x_i is a nonnegative integer and both $x_{\frac{r+1}{2}}$ and s have the same parity, we get that $x_{\frac{r+1}{2}} \in \{1, 2, \ldots, s\} = \{2j + 1 : 0 \leq j \leq \frac{s-1}{2}\}$. Let $x_{\frac{r+1}{2}} = 2j - 1$ for some $j \in \{0, 1, \ldots, \frac{s-1}{2}\}$, then equation (4) can be written as

$$x_1 + x_2 + \cdots + x_{\frac{r-1}{2}} = \frac{s-1}{2} - j. \tag{6}$$

The number of nonnegative integer solutions of equation (6) can be calculated, using Theorem 2.1, to be equal to $C\left(\frac{s-1}{2} - j + \frac{r-3}{2}, \frac{r-3}{2}\right)$. Based on the fact that $j \in \{0, 1, \ldots, \frac{s-1}{2}\}$, Theorem 2.3 tells us that

$$\rho_0(s, r) = \sum_{j=0}^{\frac{s-1}{2}} C\left(\frac{s-1}{2} - j + \frac{r-3}{2}, \frac{r-3}{2}\right) = C\left(\frac{s-1}{2} + \frac{r-1}{2}, \frac{r-1}{2}\right).$$

(d) *When s is odd and each x_i is a positive integer*. As in the previous cases, $x_{\frac{r+1}{2}}$ and s have the same parity; the fact that each x_i is a positive integer implies that the right side of equation (6) must be at least equal to the number of terms in this equation, i.e., $\frac{s-1}{2} - j \geq \frac{r-1}{2}$, or equivalently, $\frac{s-r}{2} \geq j$. Consequently, $x_{\frac{r+1}{2}} \in \{1, 3, \ldots, \frac{s-r}{2}\}$. If $x_{\frac{r+1}{2}} = 2j+1$ for some $j \in \{0, 1, \ldots, \frac{s-r}{2}\}$, then the number of positive integer solutions of equation (6) is determined by Theorem 2.2 to be $C\left(\frac{s-1}{2} - j - 1, \frac{r-1}{2} - 1\right) = C\left(\frac{s-3}{2} - j, \frac{r-3}{2}\right)$. Using Theorem 2.3 we get

$$\rho(s, r) = \sum_{j=0}^{\frac{s-r}{2}} C\left(\frac{s-3}{2} - j, \frac{r-3}{2}\right) = C\left(\frac{s-1}{2}, \frac{r-1}{2}\right).$$

Thus we have proven the next two theorems.

Theorem 2.4. *For each positive integer r and each nonnegative integer s, the number of reversible nonnegative integer solutions, $(x_1, x_2, \ldots, x_r)$, of equation (1) is:*

i. $\rho_0(s, r) = 0$ *when s is odd and r is even,*

ii. $\rho_0(s, r) = C\left(\frac{s}{2} + \frac{r-2}{2}, \frac{r-2}{2}\right)$ *when both s and r are even,*

iii. $\rho_0(s, r) = C\left(\frac{s}{2} + \frac{r-1}{2}, \frac{r-1}{2}\right)$ *when s is even and r is odd,*

iv. $\rho_0(s, r) = C\left(\frac{s-1}{2} + \frac{r-1}{2}, \frac{r-1}{2}\right)$ *when both s and r are odd.*

Theorem 2.5. *Let r, s be positive integers with $s \geq r$. Then the number of reversible positive integer solutions, $(x_1, x_2, \ldots, x_r)$, of equation (1) is:*

i. $\rho(s, r) = 0$ *when s is odd and r is even,*

ii. $\rho(s, r) = C\left(\frac{s-2}{2}, \frac{r-2}{2}\right)$ *when both s and r are even,*

iii. $\rho(s, r) = C\left(\frac{s-2}{2}, \frac{r-1}{2}\right)$ *when s is even and r is odd,*

iv. $\rho(s, r) = C\left(\frac{s-1}{2}, \frac{r-1}{2}\right)$ *when both s and r are odd.*

Suppose that s is even. If r is even, then $r \in \{2, 4, \ldots, s\}$ and the sum of all values of $\rho(s, r)$ is $\sum_{i=0}^{\frac{s-2}{2}} C\left(\frac{s-2}{2}, i\right) = 2^{\frac{s-2}{2}}$. Similarly, if r is odd, then $r \in \{1, 2, \ldots, s-1\}$ and the sum of all values of $\rho(s, r)$ is $\sum_{i=0}^{\frac{s-2}{2}} C\left(\frac{s-2}{2}, i\right) = 2^{\frac{s-2}{2}}$ too. Therefore,

$$\sum_{r=1}^{s} \rho(s, r) = 2 \sum_{i=0}^{\frac{s-2}{2}} C\left(\frac{s-2}{2}, i\right) = 2^{\frac{s-2}{2}}.$$

3. First Enumeration: Stable Sets

As we mentioned in the Introduction, we are aware of only one enumeration of caterpillars, which was performed by Harary and Schwenk [2], where they determined the number of caterpillars of order $m \geq 3$ to be $c_m = 2^{m-4} + 2^{\lfloor \frac{m-4}{2} \rfloor}$. The sequence formed by the consecutive values of c_p is A005418 in OEIS. If instead of using the order of the caterpillars we use the size, the formula given in [2] can be written for every $n \geq 3$ as:

$$c(n) = 2^{n-3} + 2^{\lfloor \frac{n-3}{2} \rfloor}.$$

In Table 1 we show the first values of $c(n)$ for $3 \leq n \leq 20$.

Table 1. Number of caterpillars of size n.

n	1	2	3	4	5	6	7	8	9	10
$c(n)$	1	1	2	3	6	10	20	36	72	136
n	11	12	13	14	15	16	17	18	19	20
$c(n)$	272	528	1056	2080	4160	8256	16512	32896	65792	131328

Let G be a caterpillar of size $n \geq 1$ where the stable sets are $A = \{u_1, u_2, \ldots, u_k\}$ and $B = \{v_1, v_2, \ldots, v_l\}$. Without loss of generality, we assume that $k \leq l$. We want to determine the number $a(n,k)$ of non-isomorphic caterpillars of size n such that the smallest stable set has cardinality k. This implies that $1 \leq k \leq \frac{n}{2}$ when n is even and $1 \leq k \leq \frac{n+1}{2}$ when n is odd. Consequently,

$$\sum_{k=1}^{\lceil \frac{n}{2} \rceil} a(n,k) = c(n).$$

Caterpillars have a characteristic that distinguish them from any other type of tree, they can be drawn in such a way that the vertices can be organized on two parallel lines (or rows), one for each stable set, and the edges connecting these vertices, represented by line segments between the two rows, never cross. We refer to this representation of a caterpillar as its *2-row representation*. In Figure 1 we show the 10 non-isomorphic caterpillars of size 6, organized in levels according to the three possible values of k. The elements of the stable set A are in black. On each caterpillar, the leftmost elements are u_1 and v_1, while the rightmost elements are u_k and v_l. The number on top of each black vertex is its degree.

If $\deg(u_i) = d_i$, then every caterpillar can be described by the sequence $d_1, d_2, \ldots, d_k$, where each $d_i \geq 1$ and $d_1 + d_2 + \cdots + d_k = n$. But the same caterpillar can also be described by the reverse sequence, that is, $d_k, d_{k-1}, \ldots, d_1$. Thus, the problem of counting non-isomorphic caterpillars of size n which smaller stable set has k elements, is equivalent to count sequences of length k which entries are positive integers where the addition of all entries equals n and the sequence and its reverse are considered the same. Note that each of these sequences is a solution of equation (1) with $r = k$ and $s = n$. Since each $d_i > 0$, the values of $\rho(n,k)$ obtained in Theorem 2.5 are used to calculate $a(n,k)$.

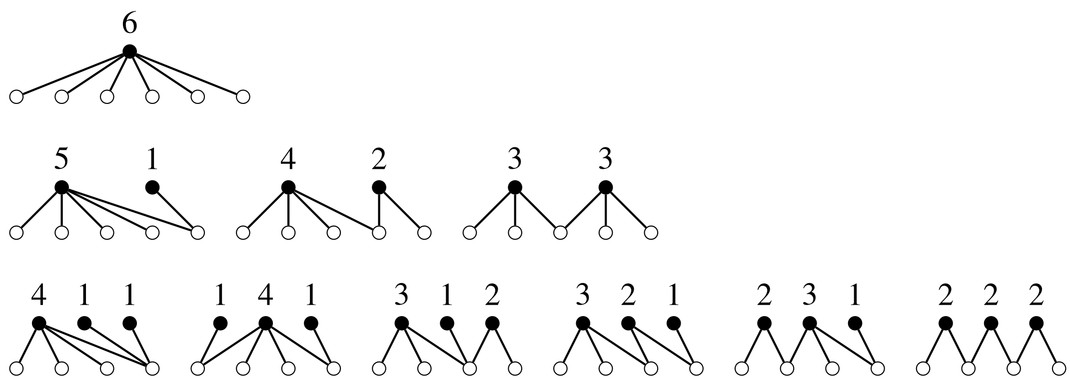

Figure 1. All non-isomorphic caterpillars of size 6

Theorem 3.1. *The number $a(n,k)$ of non-isomorphic caterpillars of size n which smaller stable sets has cardinality $k \leq \lfloor \frac{n-1}{2} \rfloor$ is $\frac{1}{2}\left(C(n-1,k-1)+\rho(n,k)\right)$.*

Proof. Let $A=\{u_1,u_2,\ldots,u_k\}$ and $B=\{v_1,v_2,\ldots,v_l\}$ be the stable sets of a caterpillar of size n, where $k \leq l$, which is assumed to be depicted using the 2-row representation. Suppose that for each $i \in \{1,2,\ldots,k\}$, $\deg(u_i)=x_i$. Since caterpillars are trees and trees are bipartite graphs, the following equation holds:

$$x_1+x_2+\cdots+x_k=n, \tag{7}$$

which has the structure of equation (1).

If $(x_1,x_2,\ldots,x_k)$ is a solution of (7), then its reverse sequence $(x_k,x_{k-1},\ldots,x_1)$ is also a solution. Thus every caterpillar is associated to two of these solutions. Since some of these solutions are reversible, the number of non-isomorphic caterpillars of size n with smaller stable set of cardinality k is given by half of the sum of the number of solutions of (7) and the number of reversible solutions.

The number of positive integer solutions of (7) is $C(n-1,k-1)$ and the number of reversible solutions is $\rho(n,k)$. Therefore, $a(n,k)=\frac{1}{2}\left(C(n-1,k-1,+\rho(n,k)\right)$. □

When n is even, the stable sets of any caterpillar of size n have different cardinality; this implies that Theorem 3.1 tells us the exact value of $a(n,k)$. The same occurs when n is odd and $k \neq \frac{n+1}{2}$. However the situation is different when n is odd and $k=\frac{n+1}{2}$. This case is different of the other cases because both stable sets have the same cardinality, this means that there are up to four solutions of equation (1) that can represent the same graph. To explain the problem, let us consider the three caterpillars of size $n=9$ shown in Figure 2.

Since both stable sets have the same cardinality, each of them can be selected to be A. This implies that each caterpillar can be associated with up to four partitions of n. For G_1 we have four partitions: $(2,1,3,2,1),(1,2,3,1,2),(1,3,1,2,2)$, and $(2,2,1,3,1)$, which are non-reversible. For G_2 we have two different reversible partitions: $(3,1,1,1,3)$ and $(1,1,5,1,1)$. For G_3 we also have two partitions but they are not reversible: $(1,1,3,1,3)$ and $(3,1,3,1,1)$.

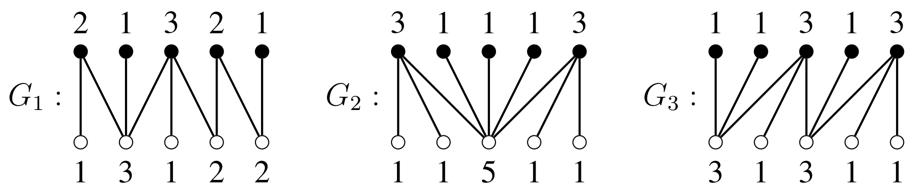

Figure 2. Two-row representation of some caterpillars of size 9

Even when graphs are not geometrical objects, consider the following rigid representations of G_1, G_2, and G_3 depicted in Figure 3.

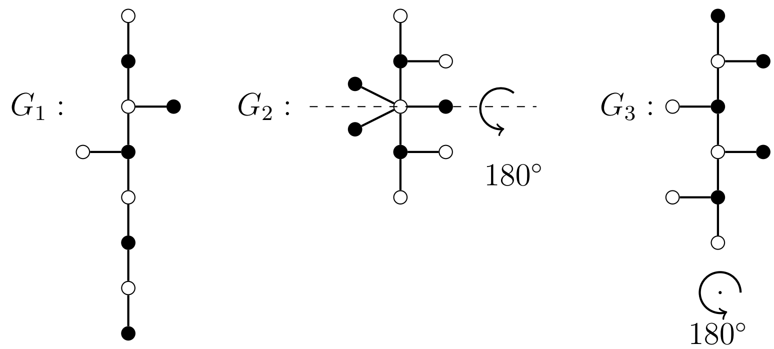

Figure 3. Rigid representations to the caterpillars G_1, G_2, and G_3

We may observe that G_1 his asymmetric, while G_2 and G_3 are symmetric; in the case of G_2 we may rotate the figure $180°$ around a horizontal axis, G_3 may be rotated $180°$ around a central point. Thus, the family of all caterpillars of odd size n with stable sets of cardinality $k = \frac{n+1}{2}$ can be partitioned into three classes according to these symmetries and the amount of partitions of n associated with them. We use the following notation: $[G_1]$ is the set that includes all the caterpillars associated with four different partitions, $[G_2]$ includes all the caterpillars associated with two different reversible partitions, and $[G_3]$ includes all the caterpillars associated with two different but not reversible partitions. Consequently, to know the value of $a\big(n, \frac{n+1}{2}\big)$ we need to determine the cardinality of these three sets. Based on the fact that the equation $d_1+d_2+\cdots+d_k = n$ has $C(n-1, k-1)$ solutions, we just need to find the cardinalities of $[G_2]$ and $[G_3]$. But the partitions associated to the elements in $[G_2]$ are reversible, hence, we just need to determine the cardinality of $[G_3]$, because the cardinality of $[G_2]$ is given by $\rho(n, k)$.

Suppose that $P = (p_1, p_2, \ldots, p_k)$ is a partition of n into k parts such that for every $1 \leq i \leq k-1$, $p_i \geq p_{i+1}$. A *Ferrer diagram* is a pictorial representation of P that uses k rows of dots, where the number of dots on the ith row is p_i. In this way, the number of dots per row does not increase as we go from any row to the one below it. Since our partitions of n not necessarily satisfy $d_i \geq d_{i+1}$, we cannot use Ferrer diagrams but we can use the same principle, that is, the ith row on the diagram contains d_i dots. Consider the partitions $(2, 1, 3, 2, 1)$ and $(1, 3, 1, 2, 2)$ of $n = 9$, because they have the same parts, they correspond to the same standard partition; in Figure 4 we show the associated Ferrer diagram together with their ordered diagram and a matrix representation of $(1, 3, 1, 2, 2)$.

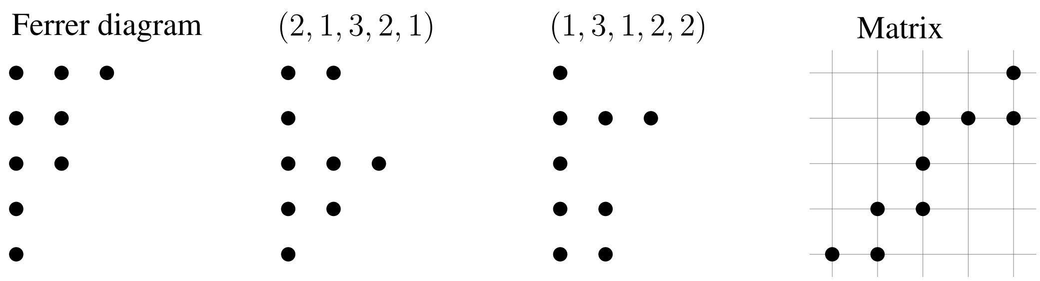

Figure 4. Different dot diagrams of the same partition of 9

Recall that when $D = (d_1, d_2, \ldots, d_k)$ is a solution of equation (1), D is an ordered partition of n. Let M be a 0-1 matrix of order $k \times k$, where $k = \frac{n+1}{2}$, such that for every $1 \leq i \leq k$, the ith row of M has exactly d_i nonzero entries that occupy consecutive cells and the nonzero entries, on each row, are organized according to:

1. the last d_1 entries on row 1 are nonzero.
2. for each $2 \leq i \leq k$, the last nonzero entry on row i occupies a cell on the same column that contains the first nonzero entry on the previous row.

As an example of this representation, the matrices associated with the caterpillars G_1, G_2, and G_3 are shown below.

$$M_1 = \begin{array}{|c|c|c|c|c|}\hline 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ \hline \end{array} \qquad M_2 = \begin{array}{|c|c|c|c|c|}\hline 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 \\ \hline \end{array} \qquad M_3 = \begin{array}{|c|c|c|c|c|}\hline 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 \\ \hline \end{array}$$

The symmetries of the rigid representations of the graph can be also seen within the corresponding matrix. Since our goal is to determine the number of caterpillars associated with exactly two non-reversible partitions, we concentrate our attention on matrices like M_3. First, observe that in the example, we have that $M_3 = M_3^t$, that is, the matrix M_3 is symmetric. In addition, M^t is the matrix associated with the same caterpillar when the stable sets are swapped and the degrees of the vertices are read from right to left. Another property that we can observe in all these matrices is that there is only one nonzero entry on each descending diagonal. Thus, the task is to find out the number of symmetric 0-1 matrices of order $k \times k$ such that the cell $m_{1,1}$ is occupied by a zero, $m_{k,1} = m_{1,k} = 1$, and on each row, the 1's are in consecutive cells.

If we outline the distribution of 1's on any of these matrices, a path is revealed, which can be understood as a symmetric path on the *x*-*y* plane from $(0, 0)$ to $(k - 1, k - 1)$, where every step of the path is a move one unit to the right or one unit up. Since the symmetry is with respect to the main diagonal of the matrix, we can calculate the number of symmetric paths from $(0, 0)$ to $(k-1, k-1)$ by adding the integers $C(k-1, i)$ over all the possible values of i, that is, the number of these paths from $(0, 0)$ to $(k-i, i)$. Since $\sum_{i=0}^{k-1} C(k-1, i) = 2^{k-1}$, we can say that the number of symmetric 0-1 matrices of order $k \times k$ satisfying the conditions is 2^{k-1}. Therefore, the number of caterpillars of size n associated with exactly two non-reversible partitions is 2^{k-1}.

Thus, when n is odd and $k = \frac{n+1}{2}$, $a(n, k) = \frac{1}{4}(C\big(n-1, k-1\big) + \rho(n, k) + 2^{k-1})$. In this way we have proven the following theorem.

Theorem 3.2. *When n is odd and $k = \frac{n+1}{2}$, the number of non-isomorphic caterpillars of size n with stable sets of cardinality k is $a(n, k) = \frac{1}{4}(C\big(n-1, k-1\big) + \rho(n, k) + 2^{k-1})$.*

In Table 2 we show the first values of $a(n, k)$, for every $1 \leq n \leq 20$ and $1 \leq k \leq 10$. Note that the entries in the last column correspond to the entries of $c(n)$ in Table 1. We conclude this section verifying this last statement.

Table 2. Number of caterpillars of size n with smaller stable set of cardinality k.

$n \backslash k$	1	2	3	4	5	6	7	8	9	10	Total
1	1										1
2	1										1
3	1	1									2
4	1	2									3
5	1	2	3								6
6	1	3	6								10
7	1	3	9	7							20
8	1	4	12	19							36
9	1	4	16	28	23						72
10	1	5	20	44	66						136
11	1	5	25	60	110	71					272
12	1	6	30	85	170	236					528
13	1	6	36	110	255	396	252				1056
14	1	7	42	146	365	651	868				2080
15	1	7	49	182	511	1001	1519	890			4160
16	1	8	56	231	693	1512	2520	3235			8256
17	1	8	64	280	924	2184	4032	5720	3299		16512
18	1	9	72	344	1204	3108	6216	9752	12190		32896
19	1	9	81	408	1548	4284	9324	15912	21942	12283	65792
20	1	10	90	489	1956	5832	13608	25236	37854	46252	131328

Suppose that n is even and $k \in \{1, 2, \ldots, \frac{n}{2}\}$. Then,

$$\begin{aligned}
\sum_{k=1}^{\frac{n}{2}} a(n,k) &= \sum_{k=1}^{\frac{n}{2}} \frac{1}{2}\left(C(n-1,k-1)+\rho(n,k)\right) \\
&= \frac{1}{2}\sum_{k=1}^{\frac{n}{2}} \frac{1}{2}C(n-1,k-1) + \frac{1}{2}\sum_{k=1}^{\frac{n}{2}} \rho(n,k) \\
&= \frac{1}{2}\cdot\frac{1}{2}\sum_{j=0}^{n-2} C(n-1,j) + \frac{1}{2}\sum_{i=0}^{\frac{n-2}{2}} C\left(\frac{n}{2},i\right) \\
&= \frac{1}{2}\cdot 2^{n-2} + \frac{1}{2}\cdot 2^{\frac{n-2}{2}} \\
&= 2^{n-4} + 2^{\frac{n-4}{2}} = c(n).
\end{aligned}$$

Since the case n odd can be proven similarly, we omit its proof.

4. Second Enumeration: Diameter

Let G be a caterpillar of size n and diameter d. If $v_0, v_1, \ldots, v_d$ are the consecutive vertices of a path of maximum length in G, then $v_1, v_2, \ldots, v_{d-1}$ are internal vertices, thus

$$\begin{aligned}
d + \sum_{i=1}^{d-1} (\deg(v_i) - 2) &= n \\
\sum_{i=1}^{d-1} (\deg(v_i) - 2) &= n - d.
\end{aligned}$$

To avoid any confusion with the terminology, we refer to v_0 and v_d as the extreme vertices and pendent vertex to any other vertex of degree 1. If for each $i \in \{1, 2, \ldots, d-1\}$, $x_i = \deg(v_i) - 2$, then x_i is the number of pendent vertices attached to v_i. Therefore,

$$x_1 + x_2 + \cdots + x_{d-1} = n - d. \tag{8}$$

In this section we want to determine the number $b(n,k)$ of non-isomorphic caterpillars of size n and diameter d. As we did in Section 3, we start counting the number of solutions of (8). Note that now each x_i is a nonnegative integer because $\deg(v_i)$ could be equal to 2, implying that $x_i = 0$. Based on the result in Theorem 2.1, we know that the number of nonnegative integer solutions of (8) is $C(n-d+d-2, d-2) = C(n-2, d-2)$. As in the previous case, if $(x_1, x_2, \ldots, x_{d-1})$ is a solution of (8) where each $x_i \geq 0$, then $(x_{d-1}, x_{d-2}, \ldots, x_1)$ is also a solution of (8). This indicates that we need to find the number of reversible solutions of this equation. But this number can be obtained using Theorem 2.4 with $s = n - d$ and $r = d - 1$. Therefore, the number $b(n,d)$, of non-isomorphic caterpillars of size n and diameter d is given by $\frac{1}{2}\left(C(n-2,d-2) + \rho_0(n-d,d-1)\right)$. We summarize this result in the following theorem.

Theorem 4.1. *The number $b(n,d)$, of non-isomorphic caterpillars of size n and diameter d is:*

i. $\frac{1}{2}C(n-2,d-2)$ *when n is even and d is odd,*

ii. $\frac{1}{2}\left(C(n-2,d-2)+C\left(\frac{n-3}{2},\frac{d-3}{2}\right)\right)$ *when both n and d are odd,*

iii. $\frac{1}{2}\left(C(n-2,d-2)+C\left(\frac{n-2}{2},\frac{d-2}{2}\right)\right)$ *when both n and d are even,*

iv. $\frac{1}{2}\left(C(n-2,d-2)+C\left(\frac{n-3}{2},\frac{d-2}{2}\right)\right)$ *when n is odd and d is even.*

As an example for this theorem, in Figure 5 we exhibit the 20 caterpillars of size 7 classified by their diameter. In addition, in Table 3 we show the first values of $b(n,d)$ for $2 \le n \le 20$ and $2 \le d \le \lceil \frac{n}{2} \rceil$. The remaining values of $b(n,d)$ can be easily obtained using the fact that $b(n,d) = b(n,n+2-d)$. We must also mention here that the sequence formed by the values of $b(n,d)$ form the rows of Losanitsch's triangle, that is, the sequence A034851 in OEIS.

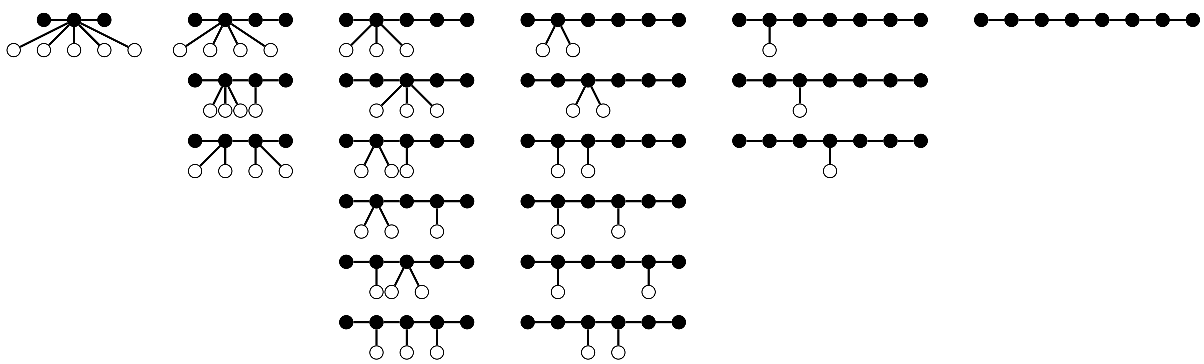

Figure 5. Caterpillars of size 7 classified by their diameter

5. Third Enumeration: Stable Sets and Diameter

In this last section we combine the ideas of the two previous sections by determining the number of caterpillars of size n, diameter d, and smaller stable set of cardinality k, this number is denoted by $c(n,d,k)$. As it is expected, this enumeration is more complicated that the previous calculations; however, the technique is essentially the same: we find first the number of ways to distribute the pendent vertices among the internal vertices, we add to this amount the number of reversible distributions, and we divide this sum by two.

Let $v_0, v_1, \dots, v_d$ be the consecutive vertices of a path P of maximum length in a caterpillar of size n, then there are $n-d$ pendent vertices that need to be attached to the interior vertices. For each $i \in \{1,2,\dots,d-1\}$, the number of pendent vertices attached to v_i is $x_{\frac{i}{2}}$ when i is even and $y_{\frac{i+1}{2}}$ when i is odd. Note that in any caterpillar of size n and diameter d, the amount of interior vertices with an even index is $\frac{d-2}{2}$ when d is even and $\frac{d-1}{2}$ when d is odd, and the amount of interior vertices with an odd index is $\frac{d}{2}$ when d is even and $\frac{d-1}{2}$ when d is odd.

Table 3. Number of caterpillars of size n and diameter d.

$n\backslash d$	2	3	4	5	6	7	8	9	10	11	Total
2	1										1
3	1										2
4	1	1									3
5	1	2									6
6	1	2	4								10
7	1	3	6								20
8	1	3	9	10							36
9	1	4	12	19							72
10	1	4	16	28	38						136
11	1	5	20	44	66						272
12	1	5	25	60	110	126					528
13	1	6	30	85	170	236					1056
14	1	6	36	110	255	395	472				2080
15	1	7	42	146	365	651	868				4160
16	1	7	49	182	511	1001	1519	1716			8256
17	1	8	56	231	693	1512	2520	3225			16512
18	1	8	64	280	924	2184	4032	5720	6470		32896
19	1	9	72	344	1204	3108	6216	9752	12190		12283
20	1	9	81	408	1548	4284	9324	15912	21942	24310	131328

Furthermore, for every $n \geq 2$, there is only one caterpillar of size n and diameter $d = 2$, this tree is the star $S_n \cong K_{1,n}$; thus, $c(n, 2, 1) = 1$. In a caterpillar of diameter $d = 3$, the $n - 3$ pendent vertices are distributed among the two interior vertices, each possible distribution corresponds to a partition of $n - 3$ into two parts. If we consider that one of these parts can be zero, then there are $\frac{n-2}{2}$ of these partitions when n is even and $\frac{n-1}{2}$ when n is odd. Therefore, when n is even, for each $k \in \{2, 3, \dots, \frac{n}{2}\}$, $c(n, 3, k) = 1$; when n is odd, for each $k \in \{2, 3, \dots, \frac{n+1}{2}\}$, $c(n, 3, k) = 1$. Consequently, starting at this point, we assume that $d \geq 4$. We analyze two major cases that depend of the parity of the parameter d. In both cases, the parameter f is the number of pendent vertices attached to the interior vertices with odd index and the parameter g is the number of pendent vertices attached to the interior vertices with even index.

Case 1: When d is odd. The following equations must be satisfied by the x_i and y_i:

$$x_1 + x_2 + \cdots + x_{\frac{d-1}{2}} = f, \tag{9}$$

$$y_1 + y_2 + \cdots + y_{\frac{d-1}{2}} = g. \tag{10}$$

For the path P that contains the vertices $v_0, v_1, \dots, v_d$, there is an automorphism ϕ that transforms v_i into v_{d-i}, that is, for each odd index i, there exists an even index j, such that $\phi(v_i) = v_j$.

This implies that all the possible distributions of the pendent vertices are included in the cases where $f \in \{0, 1, \dots, \lfloor \frac{n-d}{2} \rfloor\}$. Since the parity of $n-d$ depends on the parity of n, we consider two subcases:

Subcase 1.a: When n is even. Thus, $f \in \{0, 1, \dots, \frac{n-d-1}{2}\}$ and $g \in \{n-d, n-d-1, \dots, \frac{n-d+1}{2}\}$. Consequently, the cardinality of the smaller stable set is $k = \frac{d+1}{2} + f$, i.e., $k \in \{\frac{d+1}{2}, \frac{d+3}{2}, \dots, \frac{n}{2}\}$.

Since there is no automorphism of P that transforms a vertex with an even (or odd) index into another vertex with an even (or odd) index, it is impossible to have a reversible distribution of the pendent vertices. Thus, $c(n,d,k)$ is obtained by the product of the number of solutions of equations (9) and (10), that is, for each $k = \frac{d+1}{2} + f$, i.e., for each $k \in \{\frac{d+1}{2}, \frac{d+3}{2}, \dots, \frac{n}{2}\}$,

$$c(n,d,k) = \sigma_0\left(f, \tfrac{d-1}{2}\right)\sigma_0\left(g, \tfrac{d-1}{2}\right).$$

Since $k = \frac{d+1}{2} + f$ and $g = n-d-f$, this equation can be written in the following form

$$c(n,d,k) = \sigma_0\left(k - \tfrac{d+1}{2}, \tfrac{d-1}{2}\right)\sigma_0\left(n-k-\tfrac{d-1}{2}, \tfrac{d-1}{2}\right).$$

Using the formula for $\sigma_0(s,r)$ in Theorem 2.1, we can reduce this last expression into

$$c(n,d,k) = C\left(k-2, \tfrac{d-3}{2}\right)C\left(n-k-1, \tfrac{d-3}{2}\right).$$

Subcase 1.b: When n is odd. Now, $f \in \{0, 1, \dots, \frac{n-d}{2}\}$ and $g \in \{\frac{n-d}{2}, \frac{n-d}{2} - 1, \dots, 0\}$. Thus, $k \in \{\frac{d+1}{2}, \frac{d+3}{2}, \dots, \frac{n+1}{2}\}$. The only difference with the previous case occurs when $f = g = \frac{n-d}{2}$, i.e., when $k = \frac{n+1}{2}$. Suppose that $f = \frac{n-d}{2}$ and that $(x_1, x_2, \dots, x_{\frac{d-1}{2}})$ is a solution of equation (9), therefore $(x_{\frac{d-1}{2}}, x_{\frac{d-3}{2}}, \dots, x_1)$ is a solution of (10). In other terms,

$$x_1, x_{\frac{d-1}{2}}, x_2, x_{\frac{d-3}{2}}, \dots, x_2, x_{\frac{d-3}{2}}, x_1, x_{\frac{d-1}{2}}$$

is a reversible distribution of the pendent vertices among all the interior vertices. Thus,

$$c\left(n,d,\tfrac{n+1}{2}\right) = \tfrac{1}{2}\left(\sigma_0\left(\tfrac{n-d}{2}, \tfrac{d-1}{2}\right)\sigma_0\left(\tfrac{n-d}{2}, \tfrac{d-1}{2}\right) + \sigma_0\left(\tfrac{n-d}{2}, \tfrac{d-1}{2}\right)\right).$$

Hence, when n is odd and $k = \frac{n+1}{2}$,

$$c\left(n,d,\tfrac{n+1}{2}\right) = \tfrac{1}{2}\left[\left(C\left(\tfrac{n-3}{2}, \tfrac{d-3}{2}\right)\right)^2 + C\left(\tfrac{n-3}{2}, \tfrac{d-3}{2}\right)\right].$$

We summarize these results in the next theorem.

Theorem 5.1. *Let $d \geq 5$ be an odd integer, $k \geq \frac{d+1}{2}$, and $n \geq d$. The number of nonisomorphic caterpillars of size n, diameter d, and smaller stable set of cardinality k is:*

i. $c(n,d,k) = \frac{1}{2}\left[\left(C\left(\frac{n-3}{2}, \frac{d-3}{2}\right)\right)^2 + C\left(\frac{n-3}{2}, \frac{d-3}{2}\right)\right]$ *when n is odd and $k = \frac{n+1}{2}$.*

ii. $c(n,d,k) = C\left(k-2, \frac{d-3}{2}\right)C\left(n-k-1, \frac{d-3}{2}\right)$ *otherwise.*

Case 2: When d is even. The major difference with Case 1 is that now, in any caterpillar of even diameter, the path P contains more vertices with an even index than vertices with an odd index. Anyway, the distributions of the pendent vertices are associated to the solutions of the following equations:

$$x_1 + x_2 + \cdots + x_{\frac{d}{2}} = f, \tag{11}$$

$$y_1 + y_2 + \cdots + y_{\frac{d-2}{2}} = g. \tag{12}$$

The fact that these equations do not have the same number of terms implies that $f \in \{0, 1, \ldots, n-d\}$ and $g \in \{n-d, n-d-1, \ldots, 0\}$. We must observe that in this case, there is an automorphism of P that transform the vertex v_i into v_{d-i}; because d is even, i and $d-i$ have the same parity. In other words, a reversible solution of (11), combined with a reversible solution of (12), produces a reversible distribution of the pendant vertices among all the interior vertices. Then,

$$c(n,d,k) = \frac{1}{2}\left[\sigma_0\left(f, \frac{d}{2}\right)\sigma_0\left(g, \frac{d-2}{2}\right) + \rho_0\left(f, \frac{d}{2}\right)\rho_0\left(g, \frac{d-2}{2}\right)\right]. \tag{13}$$

The parity of n plays an important part in the number of solutions of equations (11) and (12). Suppose first that $f < n-d$. When f or $n-d-f-1$ pendent vertices are attached to the internal vertices with odd index, the resulting graph has stable sets of cardinality $n - \frac{d}{2} - f - 1$ and $\frac{d}{2} + f$. The only exception to this fact occurs when n is odd and $f = \frac{n-d-1}{2}$, where both stable sets have cardinality $k = \frac{n+1}{2}$.

Theorem 5.2. *Let $d \geq 4$ be an even integer and $k = \frac{n+1}{2}$ where $n \geq d$ is an odd integer. Then, the number of nonisomorphic caterpillars of size n, diameter d, and smaller stable set of cardinality k is:*

i. when $n \equiv 1(\mathit{mod}\ 4)$ and $d \equiv 0(\mathit{mod}\ 4)$
$c(n,d,k) = \frac{1}{2}\left[C\left(\frac{n-3}{2}, \frac{d-2}{2}\right)C\left(\frac{n-3}{2}, \frac{d-4}{2}\right) + C\left(\frac{n-5}{4}, \frac{d-4}{4}\right)C\left(\frac{n-5}{4}, \frac{d-4}{4}\right)\right].$

ii. when $n \equiv 1(\mathit{mod}\ 4)$ and $d \equiv 2(\mathit{mod}\ 4)$
$c(n,d,k) = \frac{1}{2}\left[C\left(\frac{n-3}{2}, \frac{d-2}{2}\right)C\left(\frac{n-3}{2}, \frac{d-4}{2}\right) + C\left(\frac{n-5}{4}, \frac{d-4}{4}\right)C\left(\frac{n-5}{4}, \frac{d-6}{4}\right)\right].$

iii. when $n \equiv 3(\mathit{mod}\ 4)$
$c(n,d,k) = \frac{1}{2}C\left(\frac{n-3}{2}, \frac{d-2}{2}\right)C\left(\frac{n-3}{2}, \frac{d-4}{2}\right).$

Proof. Since $k = \frac{n+1}{2}$, the value of f in (11) is $\frac{n-d-1}{2}$ and the value of g in (12) is $\frac{n-d+1}{2}$. Thus,

$$\sigma_0\left(f, \tfrac{d}{2}\right) = \sigma_0\left(\tfrac{n-d-1}{2}, \tfrac{d}{2}\right) = C\left(\tfrac{n-d-1}{2} + \tfrac{d}{2} - 1, \tfrac{d}{2} - 1\right) = C\left(\tfrac{n-3}{2}, \tfrac{d-2}{2}\right)$$

and

$$\sigma_0\left(g, \tfrac{d-2}{2}\right) = \sigma_0\left(\tfrac{n-d+1}{2}, \tfrac{d-2}{2}\right) = C\left(\tfrac{n-d+1}{2} + \tfrac{d-2}{2} - 1, \tfrac{d-2}{2} - 1\right) = C\left(\tfrac{n-3}{2}, \tfrac{d-4}{2}\right).$$

If $n \equiv 1(\text{mod}\ 4)$ and $d \equiv 0(\text{mod}\ 4)$, then

$$\rho_0\left(f, \tfrac{d}{2}\right) = \rho_0\left(\tfrac{n-d-1}{2}, \tfrac{d}{2}\right) = C\left(\tfrac{n-d-1}{4} + \tfrac{d}{4} - 1, \tfrac{d}{4} - 1\right) = C\left(\tfrac{n-5}{4}, \tfrac{d-4}{4}\right)$$

and

$$\rho_0\left(g,\tfrac{d-2}{2}\right)=\rho_0\left(\tfrac{n-d+1}{2},\tfrac{d-2}{2}\right)=C\left(\tfrac{n-d-1}{4}+\tfrac{d-4}{4},\tfrac{d-4}{4}\right)=C\left(\tfrac{n-5}{4},\tfrac{d-4}{4}\right).$$

Based on these calculations and equation (13), we get that in this case,

$$c(n,d,k)=\tfrac{1}{2}\left[C\left(\tfrac{n-3}{2},\tfrac{d-2}{2}\right)C\left(\tfrac{n-3}{2},\tfrac{d-4}{2}\right)+C\left(\tfrac{n-5}{4},\tfrac{d-4}{4}\right)C\left(\tfrac{n-5}{4},\tfrac{d-4}{4}\right)\right].$$

The remaining cases can be proven in a similar way, so we omit their proofs. We just need to mention that in case (iii), the associated values of ρ_0 are equal to zero. □

Suppose now that $f=n-d$ and $g=0$, the graphs obtained have stable sets of cardinalities $\frac{d}{2}$ and $n+1-\frac{d}{2}$, .i.e., they are caterpillars of size n, diameter d with smaller stable set of cardinality $k=\frac{d}{2}$. Therefore, the value of $c(n,d,k)$ can be easily found using equation (13) and the fact that $\sigma_0\left(0,\frac{d-2}{2}\right)=\rho_0\left(0,\frac{d-2}{2}\right)=1$, so we omit the proof.

Theorem 5.3. *Let $d\geq 4$ be an even integer, and $n\geq d$. If $k=\frac{d}{2}$, then the number of nonisomorphic caterpillars of size n, diameter d, and smaller stable set of cardinality k is:*

i. $c(n,d,k)=\frac{1}{2}\left[C\left(n-\frac{d}{2}-1,\frac{d}{2}-1\right)+C\left(\frac{n}{2}-\frac{d+4}{4},\frac{d-4}{4}\right)\right]$ *when n is even and $d\equiv 0(\mathit{mod}\ 4)$.*

ii. $c(n,d,k)=\frac{1}{2}\left[C\left(n-\frac{d}{2}-1,\frac{d}{2}-1\right)+C\left(\frac{n}{2}-\frac{d+2}{4},\frac{d-2}{4}\right)\right]$ *when n is even and $d\equiv 2(\mathit{mod}\ 4)$.*

iii. $c(n,d,k)=\frac{1}{2}C\left(n-\frac{d}{2}-1,\frac{d}{2}-1\right)$ *when n is odd and $d\equiv 0(\mathit{mod}\ 4)$.*

iv. $c(n,d,k)=\frac{1}{2}\left[C\left(n-\frac{d}{2}-1,\frac{d}{2}-1\right)+C\left(\frac{n-1}{2}-\frac{d+2}{4},\frac{d-2}{4}\right)\right]$ *when n is odd and $d\equiv 2(\mathit{mod}\ 4)$.*

Now we analyze the general case. As we mentioned before, a caterpillar of size n, diameter d even, with stable sets of cardinalities $\frac{d}{2}+f$ and $n-\frac{d}{2}-f-1$ for every $f\in\{0,1,\dots,n-d-1\}$, except in the case where n is odd and $f=\frac{n-d-1}{2}$, that was analyzed, independently, in Theorem 5.2.

In view of the fact that d is even, we know that the path P of maximum length has $\frac{d+2}{2}$ vertices with even index and $\frac{d}{2}$ vertices with odd index. Let $f\in\{0,1,\dots,\lfloor\frac{n-d-2}{2}\rfloor\}$, if f vertices are attached to the interior vertices with odd index, then the stable sets have cardinalities $\frac{d+2}{2}+f$ and $n-\frac{d}{2}-f$; if instead of attaching f we attach $n-d-f-1$ pendent vertices, the stable sets have cardinalities $n-\frac{d}{2}-f$ and $\frac{d+2}{2}+f$, respectively. Since $f\leq\lfloor\frac{n-d-2}{2}\rfloor$, we know for sure that $\frac{d+2}{2}+f<n-\frac{d}{2}-f$. This implies that for each $k\geq\frac{d+2}{2}$, it is enough to consider the values of f in the set $\{0,1,\dots,\lfloor\frac{n-d-2}{2}\rfloor\}$. Consequently, if $k=\frac{d+2}{2}+f$, with $f\in\{0,1,\dots,\lfloor\frac{n-d-2}{2}\rfloor\}$, we get that

$$\begin{aligned}c(n,d,k)=\frac{1}{2}\Bigg[&\sigma_0\left(f,\frac{d}{2}\right)\sigma_0\left(n-d-f,\frac{d-2}{2}\right)+\rho_0\left(f,\frac{d}{2}\right)\rho_0\left(n-d-f,\frac{d-2}{2}\right)\\&+\sigma_0\left(n-d-f-1,\frac{d}{2}\right)\sigma_0\left(f+1,\frac{d-2}{2}\right)+\rho_0\left(n-d-f-1,\frac{d}{2}\right)\rho_0\left(f+1,\frac{d-2}{2}\right)\Bigg]\end{aligned}$$

Since $f=k-\frac{d+2}{2}$, the factors in this formula can be written in terms of n, d, and k.

$$\sigma_0\left(f, \frac{d}{2}\right) = \sigma_0\left(k - \frac{d}{2} - 1, \frac{d}{2}\right) = C\left(k-2, \frac{d-2}{2}\right)$$
$$\sigma_0\left(n-d-f, \frac{d-2}{2}\right) = \sigma_0\left(n - \frac{d}{2} - k + 1, \frac{d}{2} - 1\right) = C\left(n-k-1, \frac{d-4}{2}\right)$$
$$\sigma_0\left(n-d-f-1, \frac{d}{2}\right) = \sigma_0\left(n - \frac{d}{2} - k, \frac{d}{2}\right) = C\left(n-k-1, \frac{d-2}{2}\right)$$
$$\sigma_0\left(f+1, \frac{d-2}{2}\right) = \sigma_0\left(k - \frac{d}{2}, \frac{d}{2} - 1\right) = C\left(k-2, \frac{d-4}{2}\right)$$
$$\rho_0\left(f, \frac{d}{2}\right) = \rho_0\left(\frac{2k-d-2}{2}, \frac{d}{2}\right)$$
$$\rho_0\left(n-d-f, \frac{d-2}{2}\right) = \rho_0\left(\frac{2n-d-2k+}{2}, \frac{d-2}{2}\right) = \rho_0\left(\frac{2n-d-2k+2}{2}, \frac{d-2}{2}\right)$$
$$\rho_0\left(n-d-f-1, \frac{d}{2}\right) = \rho_0\left(n-d-k+\frac{d}{2}+1-1, \frac{d}{2}\right) = \rho_0\left(\frac{2n-d-2k}{2}, \frac{d}{2}\right)$$
$$\rho_0\left(f+1, \frac{d-2}{2}\right) = \rho_0\left(k - \frac{d}{2} - 1 + 1, \frac{d}{2} - 1\right) = \rho_0\left(\frac{2k-d}{2}, \frac{d-2}{2}\right)$$

Unfortunately the value of ρ_0 cannot be calculated directly as the value of σ_0, in this case we need to analyze the different possibilities for the parity of the parameters n, k, and $\frac{d}{2}$.

1. When n is even, k is even, and $\frac{d}{2}$ is even:

$\rho_0\left(\frac{2k-d-2}{2}, \frac{d}{2}\right) = 0$

$\rho_0\left(\frac{2n-d-2k+2}{2}, \frac{d-2}{2}\right) = C(\frac{n-k-2}{2}, \frac{d-4}{4})$

$\rho_0\left(\frac{2n-d-2k}{2}, \frac{d}{2}\right) = C(\frac{n-k-2}{2}, \frac{d-4}{4})$

$\rho_0\left(\frac{2k-d}{2}, \frac{d-2}{2}\right) = C(\frac{k-2}{2}, \frac{d-4}{4})$

2. When n is even, k is even, and $\frac{d}{2}$ is odd:

$\rho_0\left(\frac{2k-d-2}{2}, \frac{d}{2}\right) = C(\frac{k-2}{2}, \frac{d-2}{4})$

$\rho_0\left(\frac{2n-d-2k+2}{2}, \frac{d-2}{2}\right) = C(\frac{n-k-2}{2}, \frac{d-6}{4})$

$\rho_0\left(\frac{2n-d-2k}{2}, \frac{d}{2}\right) = C(\frac{n-k-2}{2}, \frac{d-2}{4})$

$\rho_0\left(\frac{2k-d}{2}, \frac{d-2}{2}\right) = 0$

3. When n is even, k is odd, and $\frac{d}{2}$ is even:

$\rho_0\left(\frac{2k-d-2}{2}, \frac{d}{2}\right) = C(\frac{k-3}{2}, \frac{d-4}{4})$

$\rho_0\left(\frac{2n-d-2k+2}{2}, \frac{d-2}{2}\right) = C(\frac{n-k-1}{2}, \frac{d-4}{4})$

$\rho_0\left(\frac{2n-d-2k}{2}, \frac{d}{2}\right) = 0$

$\rho_0\left(\frac{2k-d}{2}, \frac{d-2}{2}\right) = C(\frac{k-3}{2}, \frac{d-4}{4})$

4. When n is even, k is odd, and $\frac{d}{2}$ is odd:

$\rho_0\left(\frac{2k-d-2}{2}, \frac{d}{2}\right) = C(\frac{k-3}{2}, \frac{d-2}{4})$ $\quad$ $\rho_0\left(\frac{2n-d-2k}{2}, \frac{d}{2}\right) = C(\frac{n-k-1}{2}, \frac{d-2}{4})$

$\rho_0\left(\frac{2n-d-2k+2}{2}, \frac{d-2}{2}\right) = 0$ $\quad$ $\rho_0\left(\frac{2k-d}{2}, \frac{d-2}{2}\right) = C(\frac{k-3}{2}, \frac{d-6}{4})$

5. When n is odd, k is even, and $\frac{d}{2}$ is even:

$\rho_0\left(\frac{2k-d-2}{2}, \frac{d}{2}\right) = 0$ $\quad$ $\rho_0\left(\frac{2n-d-2k}{2}, \frac{d}{2}\right) = 0$

$\rho_0\left(\frac{2n-d-2k+2}{2}, \frac{d-2}{2}\right) = C(\frac{n-k-1}{2}, \frac{d-4}{4})$ $\quad$ $\rho_0\left(\frac{2k-d}{2}, \frac{d-2}{2}\right) = C(\frac{k-2}{2}, \frac{d-4}{4})$

6. When n is odd, k is even, and $\frac{d}{2}$ is odd:

$\rho_0\left(\frac{2k-d-2}{2}, \frac{d}{2}\right) = C(\frac{k-2}{2}, \frac{d-2}{4})$ $\quad$ $\rho_0\left(\frac{2n-d-2k}{2}, \frac{d}{2}\right) = C(\frac{n-k-1}{2}, \frac{d-2}{4})$

$\rho_0\left(\frac{2n-d-2k+2}{2}, \frac{d-2}{2}\right) = 0$ $\quad$ $\rho_0\left(\frac{2k-d}{2}, \frac{d-2}{2}\right) = 0$

7. When n is odd, k is odd, and $\frac{d}{2}$ is even:

$\rho_0\left(\frac{2k-d-2}{2}, \frac{d}{2}\right) = C(\frac{k-3}{2}, \frac{d-4}{4})$ $\quad$ $\rho_0\left(\frac{2n-d-2k}{2}, \frac{d}{2}\right) = C(\frac{n-k-2}{2}, \frac{d-4}{4})$

$\rho_0\left(\frac{2n-d-2k+2}{2}, \frac{d-2}{2}\right) = C(\frac{n-k-2}{2}, \frac{d-4}{4})$ $\quad$ $\rho_0\left(\frac{2k-d}{2}, \frac{d-2}{2}\right) = C(\frac{k-3}{2}, \frac{d-4}{4})$

8. When n is odd, k is odd, and $\frac{d}{2}$ is odd:

$\rho_0\left(\frac{2k-d-2}{2}, \frac{d}{2}\right) = C(\frac{k-3}{2}, \frac{d-2}{4})$ $\quad$ $\rho_0\left(\frac{2n-d-2k}{2}, \frac{d}{2}\right) = C(\frac{n-k-2}{2}, \frac{d-2}{4})$

$\rho_0\left(\frac{2n-d-2k+2}{2}, \frac{d-2}{2}\right) = C(\frac{n-k-2}{2}, \frac{d-6}{4})$ $\quad$ $\rho_0\left(\frac{2k-d}{2}, \frac{d-2}{2}\right) = C(\frac{k-3}{2}, \frac{d-6}{4})$

We summarize this result in the following theorem.

Theorem 5.4. *Let n and d be positive integers, where $n \geq d$, $d \geq 4$ is even and $k \geq \frac{d+2}{2}$. The number of nonisomorphic caterpillars of size n, diameter d, and smaller stable set of cardinality k is:*

$$\begin{aligned} 2c(n,d,k) = {} & C\left(k-2, \frac{d-2}{2}\right) C\left(n-k-1, \frac{d-4}{2}\right) \\ & + \rho_0\left(\frac{2k-d-2}{2}, \frac{d}{2}\right) \rho_0\left(\frac{2n-d-2k+2}{2}, \frac{d-2}{2}\right) \\ & + C\left(n-k-1, \frac{d-2}{2}\right) C\left(k-2, \frac{d-4}{2}\right) \\ & + \rho_0\left(\frac{2n-d-2k}{2}, \frac{d}{2}\right) \rho_0\left(\frac{2k-d}{2}, \frac{d-2}{2}\right). \end{aligned}$$

We conclude this work with Table 4 where the initial values of $c(n,d,k)$ are shown.

References

[1] R. B. J. T. Allenby and A. Slomson, *How to Count, an Introduction to Combinatoircs*, CRC Press, Boca Raton, 2011.

[2] F. Harary and A. J. Schwenk, The number of caterpillars, *Discrete Math.*, **6(4)** (1973), 359–365.

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Table 4. Number of nonisomorphic caterpillars of size n, diameter d, and smaller stable set of cardinality k

n	d	k				Total
		1	2	3	4	
2	2	1				1
Total		1				1
3	2	1				1
	3	0	1			1
Total		1	1			2
4	2	1				1
	3	0	1			1
	4	0	1			1
Total		1	2			3
5	2	1				1
	3	0	1	1		2
	4	0	1	1		2
	5	0	0	1		1
Total		1	2	3		6
6	2	1				1
	3	0	1	1		2
	4	0	2	2		4
	5	0	0	2		2
	6	0	0	1		1
Total		1	3	6		10
7	2	1				1
	3	0	1	1	1	3
	4	0	2	3	1	6
	5	0	0	3	3	6
	6	0	0	2	1	3
	7	0	0	0	1	1
Total		1	3	9	7	20

n	d	k					Total
		1	2	3	4	5	
8	2	1					1
	3	0	1	1	1		3
	4	0	3	3	3		9
	5	0	0	4	6		10
	6	0	0	4	5		9
	7	0	0	0	3		3
	8	0	0	0	1		1
Total		1	4	12	19		36
9	2	1					1
	3	0	1	1	1	1	4
	4	0	1	4	5	2	12
	5	0	0	5	8	6	19
	6	0	0	6	8	5	19
	7	0	0	0	6	6	12
	8	0	0	0	2	2	4
	9	0	0	0	0	1	1
Total		1	2	16	30	23	72
10	2	1					1
	3	0	1	1	1	1	4
	4	0	4	4	4	4	16
	5	0	0	6	10	12	28
	6	0	0	9	13	16	38
	7	0	0	0	10	18	28
	8	0	0	0	6	10	16
	9	0	0	0	0	4	4
	10	0	0	0	0	1	1
Total		1	5	20	44	66	136