Prime ideal graphs of commutative rings

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Abstract

Let R be a finite commutative ring with identity and P be a prime ideal of R. The vertex set is $R \setminus \{0\}$ and two distinct vertices are adjacent if their product in P. This graph is called the prime ideal graph of R and denoted by Γ_P. The relationship among prime ideal, zero-divisor, nilpotent and unit graphs are studied. Also, we show that Γ_P is simple connected graph with diameter less than or equal to two and both the clique number and the chromatic number of the graph are equal. Furthermore, it has girth 3 if it contains a cycle. In addition, we compute the number of edges of this graph and investigate some properties of Γ_P.

1. Introduction

In [5] Beck defined a zero divisor graph for commutative ring R with vertex set R and two vertices are adjacent if their product is zero. It is denoted by $\Gamma(R)$. After this time, the study of algebraic structures using the properties of graphs has become an exciting research topic. It is leading to many fascinating results and questions. There are many papers on assigning a graph to a ring, for example see [7, 1, 2]. In [3], Anderson and Livingston did a little change with definition of zero diversor graph $\Gamma(R)$ by replacing its vertex set with $Z(R) \setminus \{0\}$. In [7, 4, 1, 2] several other type of graphs such as the zero divisor graph $\Gamma(R)$, unit graph $G(R)$ and nilpotent graph $\Gamma_N(R)$ has been defined for a commutative ring R.

A graph is connected if there exists a path connecting any two distinct vertices. The distance between vertices x and y is the length of the shortest path connecting them, and the diameter of a graph Γ, $diam(\Gamma)$, is the supremum of these distances. The girth of a graph, denoted $g(\Gamma)$ is the length of the shortest cycle in the graph G. The eccentricity of a vertex v in Γ, denoted by $ecc(v)$, is the largest distance between v and any other vertex u in Γ. The eccentric connectivity polynomial of a graph Γ is defined by $E(\Gamma, x) = \sum_{v \in V(\Gamma)} deg(v)x^{ecc(v)}$. Similarly, the total eccentricity polynomial of Γ is defined as $\theta(\Gamma, x) = \sum_{v \in V(\Gamma)} x^{ecc(v)}$. A transmission of a vertex v in Γ is $\sigma(v, \Gamma) = \sum_{v \in \Gamma} d(u, v)$. The transmission of a graph Γ is $\sigma(\Gamma) = \sum_{v \in \Gamma} \sigma(v, \Gamma)$. The mean distance of a graph Γ is $\mu(\Gamma) = \frac{\sigma(\Gamma)}{n(n-1)}$, where n is the order of Γ.

Let R be a commutative ring with unity. The nilpotent graph of R denoted by $\Gamma_N(R)$, is a graph with vertex set $R \setminus \{0\}$ and two distinct vertices x and y are connected if and only if $xy \in N$, where N is the set of all nilpotent elements of R [7]. The unit graph of R, denoted by $G(R)$, has its set of vertices equal to the set of all elements of R; distinct vertices x and y are adjacent if and only if $x + y$ is a unit of R [4]. The zero-divisor graph of a ring R is defined as the graph $\Gamma(R)$ that its vertices are all non-zero zero-divisors of R in which for any two distinct vertices x and y, are adjacent if and only if $xy = 0$ [1].

In this paper, we introduce a new type of graph structure on a commutative ring R as follows: we take $R \setminus \{0\}$ as a vertex set and two distinct vertices x and y are adjacent if $xy \in P$, where P is a prime (primary) ideal of R. It is called the prime (primary) ideal graph that associated to (primary) ideal of a ring R and denoted Γ_P. Our definition is the generalization of the zero-divisor graph, that is the zero-divisor graph and nilpotent graph of R are the subgraphs of this a new graph.

In section 2, we give some basic results and concepts which will be used in our study. Section 3, devoted to study the relationship among prime ideal graph and other type of graphs. Furthermore, we show that the prime ideal graph Γ_P is simply connected with diameter at most 2 and girth 3 if it contains cycle. Also, we prove that it has equal number of chromatic and clique. Finally, we compute the eccentric connectivity polynomial (the total eccentricity polynomial), the transmission and the mean distance of Γ_P.

2. Preliminaries

The following result and concept are well known in [8].

Definition 2.1. *Let R be a ring. An element $a \in R$ is said to be nilpotent if $a^n = 0$ for some positive integer n. The set of nilpotent elements in a ring R is called the nilradical of R. We denote the nilradical by N.*

Theorem 2.1. *Let N be the nilradical of a ring R. Then N is the intersection of all prime ideals in R that is,* $\bigcap_{P \text{ prime ideal}} P = N$.

Definition 2.2. *A subset X of the vertices of G is called a clique if the induced subgraph on X is a complete graph. The maximum size of a clique in a graph G is called the clique number of G. It is denoted by $\omega(G)$.*

Definition 2.3. *Let S be a subset of $V(G)$ and $N_r[S]$ be the set of vertices in G which are in S or adjacent to a vertex in S. If $N_r[S] = V(G)$, then S is called a dominating set for G. The dominating number $\gamma(G)$ of G is the minimum size of a dominating set of the vertices of G.*

Definition 2.4. *A subset X of the vertices of G is called an independent set if the induced subgraph on X has no edges. The maximum size of an independent set in a graph G, denoted by $\alpha(G)$ is called the independence number of G.*

Definition 2.5. *A proper coloring of a graph G is a coloring of the vertices of G so that no two adjacent vertices receive the same color. The chromatic number $\chi(G)$ is the least integer k such that there is a proper coloring of G using k colors.*

3. Some Properties of Prime Ideal Graphs

Throughout this paper all rings are finite commutative with identity. Now, we study the relationship among prime ideal graphs and other kind of graphs on the same ring. We will show that other type of graphs such as zero-divisor graph, nilpotent graph and primary ideal graph are subgraphs of our graph.

Definition 3.1. *Let R be a commutative ring with unity. The prime ideal graph of R denoted by $\Gamma_P(R)$, is a graph with vertex set $R \setminus \{0\}$ and two distinct vertices x and y are connected if and only if $xy \in P$, where P is prime ideal of R.*

Example 3.1. *Consider the ring $R = \mathbb{Z}_2[x]/\langle x^3 + 1 \rangle = \{0, 1, x^2, x, x+x^2, 1+x^2, 1+x, 1+x+x^2\}$. The prime ideals of R are $P_1 = \langle x + 1 \rangle$ and $P_2 = \langle x^2 + x + 1 \rangle$. The prime ideal graphs of them are the following: $\Gamma_{P_2} = K_{1,6}$ and $\Gamma_{P_1} =$*
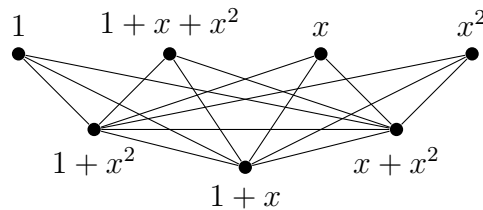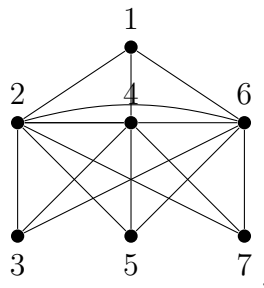

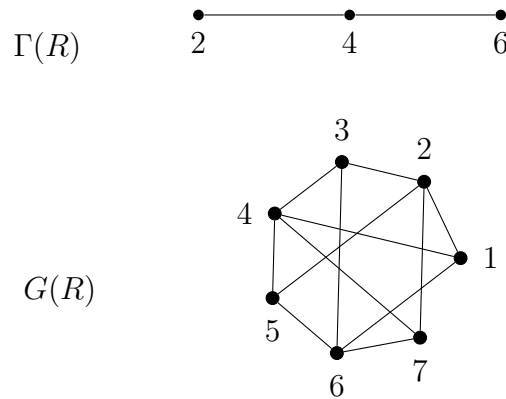**Lemma 3.1.** *Every zero divisor graph is subgraph of prime ideal graph.*

Proof. Clear. □

The converse of Lemma 3.1 is not true for the following example;

Example 3.2. *Consider $R = \mathbb{Z}_8$ and $\langle 2 \rangle$ is prime ideal of R. Its prime ideal graph is*

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It is the same as the nilpotent graph of R. In addition the zero divisors are the set $\{2, 4, 6\}$ and the unit set is $\{1, 3, 5, 7\}$ and their related graphs are

$\Gamma(R)$

$G(R)$

Hence $\Gamma(R)$ and $G(R)$ are subgraphs of $\Gamma_{\langle 2 \rangle}$

In general, there is no relationship between unit graph and prime ideal graph. For instance in $\mathbb{Z}_{77}$, there is an edge between 3 and 5 in $G(\mathbb{Z}_{77})$ because 3+5=8 which is unit in $\mathbb{Z}_{77}$. However $3 * 5 = 15$ which is also unit in $\mathbb{Z}_{77}$. This edge does not belong to prime ideal graph because P does not contain unit elements.

Lemma 3.2. *Every primary ideal graph is a subgraph of prime ideal graph.*

Proof. Clear. □

Lemma 3.3. *Every nilpotent graph is subgraph of prime ideal graph.*

Proof. By definitions both graph have the same number of vertices. Let $xy \in E(\Gamma_N(R))$, then by Theorem 2.1, we have $xy \in \bigcap_{P \text{ prime ideal}} P$, so $xy \in P$ for all prime ideal P of R. Therefore $E(\Gamma_N(R)) \subseteq E(\Gamma_P)$. We are done. □

Theorem 3.1. *Let P be a prime ideal of R. Then $\bigcap_{P \text{ is prime ideal}} \Gamma_P(R) = \Gamma_N(R)$.*

Proof. It well known that we have the same number of vertices in $\bigcap \Gamma_P(R)$ and $\Gamma_N(R)$. Let

$$e \in \bigcap \Gamma_P(R) \iff e \in \Gamma_P(R) \quad \text{for all prime ideal } P$$
$$\iff e \in P, \quad \text{for all prime ideal } P$$
$$\iff e \in \bigcap_{P \text{ is prime}} P$$
$$\iff e \in N(R)$$
$$\iff e \in \Gamma_N(R)$$

Hence, $\bigcap_{P \text{ is prime ideal}} \Gamma_P(R) = \Gamma_N(R)$. □

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Lemma 3.4. *If R is a local ring, then the prime ideal graph and nilpotent graph are identical.*

Proof. Clear. $\square$

Now we present our main results which are related to Γ_P. The following lemma tell us our graph is always bi-regular. Throughout this paper, we assume that $P^* = P \setminus \{0\}$ and $|P| = h$.

Lemma 3.5. *Let P be a prime ideal of a finite ring R and $x \in \Gamma_P$. Then*

$$
deg(x) = \begin{cases} |R| - 2 & \text{if } x \in P \\ |P^*| & \text{otherwise} \end{cases}
$$

Proof. It is obvious from the definition of ideal $rp \in P$ for any $r \in R$ and $p \in P$. Then all elements in $R \setminus \{0\}$ connected to p. As p is not connected to itself, therefore degree of each element in P is $|R| - 2$.

Moreover, for $x \notin P$, there are two cases, case one if x is connected to all elements in $P \setminus \{0\}$. Thus we have $h - 1$ edges and the other case come from definition of prime ideal graph. $\square$

Theorem 3.2. *Let R be a ring and P be a prime ideal of R with h elements. Then the prime ideal graph has $|R| - 1$ vertices and $(h-1)(|R| - 1 - \frac{h}{2})$ edges.*

Proof. We know that each element either in P or not. By Lemma 3.5, we have
$deg(x) = \begin{cases} |R| - 2 & \text{if } x \in P^* \\ |P^*| & \text{otherwise} \end{cases}$.
Now there are $h - 1$ elements in $P^* = P \setminus \{0\}$. Also there are $(|R| - 1) - (h - 1) = |R| - h$ elements which are not in P^*. From the handshaking lemma, we have $\sum_{x \in V(\Gamma_P)} deg(x) = 2|E(\Gamma_P)|$, that is

$$
\sum_{x \notin P^*} deg(x) + \sum_{x \in P^*} deg(x) = 2|E(\Gamma_P)|
$$
$$
(h-1)(|R| - 2) + (|R| - h)(h - 1) = 2|E(\Gamma_P)|
$$
$$
(h-1)\big[2|R| - h - 2\big] = 2|E(\Gamma_P)|.
$$

Therefore $|E(\Gamma_P)| = (h-1)[|R| - 1 - \frac{h}{2}]$. $\square$

Lemma 3.6. *Let P be a prime ideal of a finite ring R and Γ_P be the prime ideal graph. Then*
$d(x, y) = \begin{cases} 1 & \text{if } x \in P \text{ or } y \in P \\ 2 & \text{if } x \notin P \text{ and } y \notin P \end{cases}$.

Proof. Let $x, y \in \Gamma_P$. For $x, y \in R \setminus \{0\}$, if x or y in P, then $xy \in P$. So there is an edge between of them, that is $d(x, y) = 1$. Let $x \notin P$ and $y \notin P$, then $xp \in P$ and $yp \in P$, for all $p \in P$, that is there are edges between x and p (y and p). Thus $d(x, y) \leq d(x, p) + d(p, y) = 1 + 1 = 2$ and the result follows. $\square$

Theorem 3.3. *For any ring R, Γ_P is a simple, connected graph with diameter two.*

Proof. The connectedness and simplicity follow from Definition 3.1. The rest follows from Lemma 3.6

$\square$

Proposition 3.1. *Let R be a ring and P be a prime ideal of R. The following hold:*

1. *If $|P| = 2$, then $gr(\Gamma_P) = \infty$.*
2. *If $|P| \geq 2$, then $gr(\Gamma_P) = 3$.*

Proof.   1. It is clear that x is adjacent to y and x or y is adjacent to all elements in R. Furthermore, far any r_1 in R is not adjacent to r_2 because $r_1, r_2 \notin P$. Hence Γ_P has no cycle and the result follows.
2. Since $|P| > 2$, then we can assume that without loss of generality $P = \{x, y, z\}$. All elements in P are adjacent together. Therefore Γ_P contains a cycle of length 3, as required.

$\square$

Lemma 3.7. *Let R be a ring and let P be a primary ideal of R with h elements such that it is not prime ideal, then degree of each vertex of the primary ideal graph is defined by*

$$
deg(x) = \begin{cases} |R| - 2 & \text{if } x \in P \\ 2^{h-1} & \text{if } xy \in P, \text{ then neither } x \text{ nor } y \text{ in } P \\ h - 1 & \text{otherwise} \end{cases}
$$

Proof. Clear. $\square$

Example 3.3. *Consider $R = \mathbb{Z}_8$ and $P = \langle 4 \rangle$ is primary ideal of R. The primary ideal graph Γ_P is:*
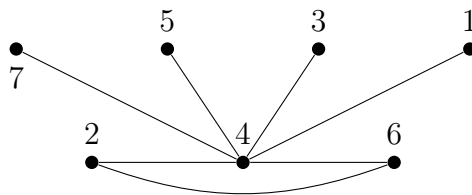
We observe that Γ_P is a subgraph of $\Gamma_{<2>}$.

Theorem 3.4. *Let R be a ring and P be a primary ideal of R with h elements, which is not prime ideal, then the primary ideal graph has order $|R| - 1$ and size $\frac{h-1}{2}(\frac{3|R|}{2} - 2) + \frac{2^h}{4}(\frac{|R|}{2} - h)$.*

Proof. There are $h-1$ non-zero elements in P of degree $|R|-2$ and there are $\frac{|R|}{2}$ non-zero elements in R of degree $h - 1$. In addition, there are $\frac{|R|}{2} - h$ non-zero elements in R of degree 2^{h-1}. From the hand shaking lemma, we have $2|E(\Gamma_P)| = \sum_{x \in V} deg(x)$, then

$$
2|E(\Gamma_P)| = (|R| - 2)(h - 1) + (h - 1)\frac{|R|}{2} + 2^{h-1}(\frac{|R|}{2} - h).
$$

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Thus,

$$|E(\Gamma_P)| = \frac{h-1}{2}(\frac{3|R|}{2} - 2) + \frac{2^h}{4}(\frac{|R|}{2} - h).$$

$\square$

Lemma 3.8. *Let R be a ring and Γ_P be the prime ideal graph. Then the following hold;*

1. *The chromatic number, $\chi(\Gamma_P) = |P|$.*
2. *The dominating number, $\gamma(\Gamma_P) = 1$.*
3. *The clique number $\omega(\Gamma_P) = |P|$.*
4. *The independent number $\omega(\Gamma_P) = |R| - |P^*|$.*

Proof.   1. Since all elements in P^* are connected to each other then they must have different colors and other vertices that are not in P^*, they are not connected to each other so they can have the same color. Thus we have $|P| = |P^*| + 1$ colors.

2. Since any element in P^* is connected to all other elements in the ring, then the domination number is 1.

3. By Lemma 3.5, the degree set of vertices of Γ_P is $\{m - 2, |P^*|\}$. By Kapoor Polimeni and Wall Theorem, we have $\Gamma_P \cong K_{|P^*|} + K^c_{|R|-1-|P^*|}$. We deduce that it contains the complete graph $K_{|P^*|+1}$. This completes the proof.

4. It follows from part 3.

$\square$

Theorem 3.5. *Let R be a ring and Γ_P be the prime ideal graph, where $|R| = n$ and $|P| = m$. Then the following hold;*

1. $E(\Gamma_P, x) = (m - 1)[(n - 2)x + (n - m)x^2]$.
2. $\theta(\Gamma_P, x) = (m - 1)x + (n - m)x^2$.
3. $\sigma(x, \Gamma_P) = m - 1$ *if* $x \in P^*$.
4. $\sigma(x, \Gamma_P) = 2(n - m)$ *if* $x \notin P$.
5. $\sigma(\Gamma_P) = 2n^2 + 3m^2 - 2m(1 + n) + 1$.
6. $\mu(\Gamma_P) = \frac{2n^2+3m^2-2m(n+1)+1}{n^2-3n+2}$.

Proof. The proof follows from Lemma 3.5 and Lemma 3.6. $\square$

Acknowledgement

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