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4-Dimensional Lattice Path Enumeration

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Abstract

Consider a set of vectors, L, which consists of 4-dimensional vectors whose coordinates are 0 or 1. We find explicit formulas that counts the number of lattice paths from origin to (a, b, c, d) using vectors in $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\} \cup L$ for various choices of L. In some cases we also give the recursive formulas for the same problem. Next we determine the minimum number of vectors that must be used to reach (a, b, c, d), also called the minimum distance problem, for different sets of vectors.

Keywords: Lattice path, shortest path, 4-dimensions
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1. Introduction

A lattice path can be defined as an ordered sequence of vectors, all being in a certain set, from point A to point B in the n-dimensional plane. A comprehensive study on lattice paths was made in [5], and history of lattice paths was surveyed in [3]. We adopt the notation used in [1]. Let $L(a, b, c, d)$ be the number of lattice paths from origin to (a, b, c, d) using vectors in $L \cup \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$. In this paper, we focus on different choices of L where coordinates of each vector in L is 1 or 0. Table 1 shows all possible 4-dimensional vectors satisfying this criteria (the vector $(0, 0, 0, 0)$ is removed because it can be used infinite times).

We take two different approaches to compute $L(a, b, c, d)$: direct and recursive. *Direct approach* gives the number of paths to (a, b, c, d) without examining other points in the plane. Sometimes these computations are resource intensive. In such case a *recursive formula* is an alternative

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way to compute $L(a,b,c,d)$. Recursive formula relates the point (a,b,c,d) to preceding points and it is used again to relate these preceding points further back, eventually leading to a base case.

In [1] a similar problem was studied in 2 and 3 dimensions. However, the 4-dimensional case was briefly mentioned and solution to the case $L=\emptyset$ was given which is $L(a,b,c,d)=\binom{a+b+c+d}{a,b,c,d}$. We explore this problem in 4-dimensional space.

The minimum distance is the number of vectors in the lattice path from origin to the point P, which has the least amount of vectors. Evoniuk et al. [2] and Iwanojko et al. [4] studied the minimum distance problem in 2-dimensional space. For example when $L=\{(1,1)\}$, the set of allowed vectors is $L\cup\{(1,0),(0,1)\}$. The minimum distance to (a,b) is $\max(a,b)$, meaning that there has to be at least $\max(a,b)$ vectors to reach (a,b). We take this problem to 4-dimensional space and solve it for various sets of vectors.

Table 1. 4-dimensional vectors whose coordinates are 1 or 0

Number	Vector
1	(1,0,0,0)
2	(0,1,0,0)
3	(0,0,1,0)
4	(0,0,0,1)
5	(1,1,0,0)
6	(1,0,1,0)
7	(1,0,0,1)
8	(0,1,1,0)
9	(0,1,0,1)
10	(0,0,1,1)
11	(1,1,1,0)
12	(1,1,0,1)
13	(1,0,1,1)
14	(0,1,1,1)
15	(1,1,1,1)

2. $|L|=1$

In this section, we consider various choices of the set L that contain only one vector.

2.1. $L=\{(1,1,0,0)\}$

In this case we are only allowed to use a diagonal vector that moves 1 unit on x and y axes as well as the standard basis vectors.

Proposition 2.1. *Let $L=\{(1,1,0,0)\}$. Summing up the number of paths for all valid values of k,* *that are* $1, 2, \ldots, \min(a, b)$*, we see that*

$$L(a,b,c,d) = \sum_{k=0}^{\min(a,b)} \binom{a+b+c+d-k}{a-k, b-k, c, d, k}. \tag{1}$$

Proof. We first explore the case where only one $(1,1,0,0)$ vector is used. The equation $(a,b,c,d) - (1,1,0,0) = (a-1, b-1, c, d)$ tells us if only one $(1,1,0,0)$ vector is used, there must be $a-1$ amount of $(1,0,0,0)$ vectors, $b-1$ amount of $(0,1,0,0)$ vectors, c amount of $(0,0,1,0)$ vectors and d amount of $(0,0,0,1)$ vectors. The number of different arrangements of such objects is

$$\binom{(a-1)+(b-1)+c+d+1}{a-1, b-1, c, d, 1} = \binom{a+b+c+d-1}{a-1, b-1, c, d, 1}.$$

Now assume there are k $(1,1,0,0)$ vectors, $(a,b,c,d) - k(1,1,0,0) = (a-k, b-k, c, d)$. In order to reach (a,b,c,d), we must use $a-k$ amount of $(1,0,0,0)$ vectors, $b-k$ amount of $(0,1,0,0)$ vectors, c amount of $(0,0,1,0)$ vectors and d amount of $(0,0,0,1)$ vectors. This tells that there are $\binom{a+b+c+d-k}{a-k,b-k,c,d,k}$ ways of reaching point (a,b,c,d) using only k amount of $(1,1,0,0)$ vectors.

$\square$

Example 2.1. We find $L(1,3,2,2)$.

1. $k = 0$. This is the case where we do not use any $(1,1,0,0)$ vectors. In this case, the number of lattice paths from origin to $(1,3,2,2)$ is $\frac{8!}{1!3!2!2!} = 1680$.
2. $k = 1$. We use 1 $(1,1,0,0)$ vector and there are $\frac{7!}{1!2!2!2!1!} = 630$ paths.

$L(1,3,2,2) = 2310$.

Next, we find a recursive relation. We consider the set of points, call T, where we can get to (a,b,c,d) using only one vector. For $P_1, P_2 \in T$ and $P_1 \neq P_2$, a lattice path to (a,b,c,d) passing from P_1 can not be equal to a lattice path passing from P_2 as their last vectors are different. Therefore sum of number of paths to all points in T will result in $L(a,b,c,d)$. The recursive relation in the case $L = \{(1,1,0,0)\}$ is

$$L(a,b,c,d) = L(a-1,b,c,d) + L(a,b-1,c,d) + L(a,b,c-1,d) + L(a,b,c,d-1) + L(a-1,b-1,c,d).$$

2.2. $L = \{(0,1,0,1)\}$

This case is similar to the previous case, which is $L = \{(1,1,0,0)\}$. However when $L = \{(0,1,0,1)\}$ the maximum amount of $(0,1,0,1)$ vectors is $\min(b,d)$ instead of $\min(a,b)$. We go through a combinatorial process similar to the case where $L = \{(1,1,0,0)\}$.

Corollary 2.1. *For* $L = \{(0,1,0,1)\}$

$$L(a,b,c,d) = \sum_{k=0}^{\min(b,d)} \binom{a+b+c+d-k}{a, b-k, c, d-k, k}. \tag{2}$$

The recursive relation for the sequence where $L = \{(0,1,0,1)\}$ is

$$L(a,b,c,d) = L(a-1,b,c,d)+L(a,b-1,c,d)+L(a,b,c-1,d)+L(a,b,c,d-1)+L(a,b-1,c,d-1).$$

We found further results on this type of sets.

Corollary 2.2.

$$L(a,b,c,d) = \sum_{k=0}^{\min(a,c)} \binom{a+b+c+d-k}{a-k,b,c-k,d,k} \textit{ for } L = \{(1,0,1,0)\} \tag{3}$$

$$L(a,b,c,d) = \sum_{k=0}^{\min(a,d)} \binom{a+b+c+d-k}{a-k,b,c,d-k,k} \textit{ for } L = \{(1,0,0,1)\} \tag{4}$$

$$L(a,b,c,d) = \sum_{k=0}^{\min(b,c)} \binom{a+b+c+d-k}{a,b-k,c-k,d,k} \textit{ for } L = \{(0,1,1,0)\} \tag{5}$$

$$L(a,b,c,d) = \sum_{k=0}^{\min(c,d)} \binom{a+b+c+d-k}{a,b,c-k,d-k,k} \textit{ for } L = \{(0,0,1,1)\} \tag{6}$$

2.3. $L = \{(1,1,1,0)\}$

We considered the case when only two of the coordinates of the vector in L are 1. Now we look at the case when 3 of those coordinates are 1. First we consider $L = \{(1,1,1,0)\}$.

Proposition 2.2. *Let* $L = \{(1,1,1,0)\}$. *Then,*

$$L(a,b,c,d) = \sum_{k=0}^{\min(a,b,c)} \binom{(a-k)+(b-k)+(c-k)+d+k}{a-k,b-k,c-k,d,k} = \sum_{k=0}^{\min(a,b,c)} \binom{a+b+c+d-2k}{a-k,b-k,c-k,d,k}. \tag{7}$$

Proof. Assume k amount of $(1,1,1,0)$ vectors are used. By the equation $(a,b,c,d)-k(1,1,1,0) = (a-k,b-k,c-k,d)$, we can see the number of each standard basis vector that must be used. The valid values of k are $0,1,\ldots,\min(a,b,c)$ since we dont have any vectors with negative coordinates. □

For $L = \{(1,1,1,0)\}$, $L(a,b,c,d)$ can be found with the recursive formula

$$L(a,b,c,d) = L(a-1,b,c,d)+L(a,b-1,c,d)+L(a,b,c-1,d)+L(a,b,c,d-1)+L(a-1,b-1,c-1,d).$$

Using a similar approach, we can find $L(a,b,c,d)$ for other cases.

Corollary 2.3.

$$L(a,b,c,d) = \sum_{k=0}^{\min(a,b,d)} \binom{a+b+c+d-2k}{a-k, b-k, c, d-k, k} for L = \{(1,1,0,1)\} \tag{8}$$

$$L(a,b,c,d) = \sum_{k=0}^{\min(a,c,d)} \binom{a+b+c+d-2k}{a-k, b, c-k, d-k, k} for L = \{(1,0,1,1)\} \tag{9}$$

$$L(a,b,c,d) = \sum_{k=0}^{\min(b,c,d)} \binom{a+b+c+d-2k}{a, b-k, c-k, d-k, k} for L = \{(0,1,1,1)\} \tag{10}$$

3. $|L| = 2$

It is interesting what happens when there are 2 vectors in L. We begin with exploring the case $L = \{(1,1,0,0),(0,0,1,1)\}$.

3.1. $L = \{(1,1,0,0),(0,0,1,1)\}$

Theorem 3.1. *For* $L = \{(1,1,0,0),(0,0,1,1)\}$*, we have*

$$L(a,b,c,d) = \sum_{k=0}^{\min(a,b)} \sum_{m=0}^{\min(c,d)} \binom{a+b+c+d-k-m}{a-k, b-k, c-m, d-m, k, m}. \tag{11}$$

Proof. Let a lattice path use k vectors in the direction $(1,1,0,0)$ and m vectors in the $(0,0,1,1)$ direction. It is easy to see that $0 \le k \le \min(a,b)$. By $(a,b,c,d)-k(1,1,0,0) = (a-k,b-k,c,d)$, we see that $0 \le m \le \min(c,d)$ with respect to k. The equation $(a-k,b-k,c,d)-m(0,0,1,1) = (a-k,b-k,c-m,d-m)$ implies that a path must use $a-k$ vectors in the $(1,0,0,0)$ direction, $b-k$ vectors in the $(0,1,0,0)$ direction, $c-m$ vectors in the $(0,0,1,0)$ direction, and $d-m$ vectors in the $(0,0,0,1)$ direction. The number of different arrangements of such vectors (including $(1,1,0,0)$ and $(0,0,1,1)$) is $\binom{a+b+c+d-k-m}{a-k,b-k,c-m,d-m,k,m}$. Summing the results for all valid values of k and m gives the number of paths from origin to (a,b,c,d).

□

3.2. $L = \{(1,0,1,1),(1,1,1,1)\}$

Proposition 3.1. *For* $L = \{(1,0,1,1),(1,1,1,1)\}$

$$L(a,b,c,d) = \sum_{k=0}^{\min(a,c,d)} \sum_{m=0}^{\min(a-k,b,c-k,d-k)} \binom{a+b+c+d-2k-3m}{a-k-m, b-m, c-k-m, d-k-m, k, m}. \tag{12}$$

Proof. Assume a lattice path uses d_{xzt} amount of $(1,0,1,1)$ vectors and d_{xyzt} amount of $(1,1,1,1)$ vectors. By the equation $(a,b,c,d) - d_{xzt}(1,0,1,1) = (a - d_{xzt}, b, c - d_{xzt}, d - d_{xzt})$ the valid interval for d_{xyzt} with respect to the number of $(1,0,1,1)$ vectors used is $0 \leq d_{xyzt} \leq \min(a - d_{xzt}, b, c - d_{xzt}, d - d_{xzt})$. Then, we subtract the $(1,1,1,1)$ vectors and get $(a - d_{xzt}, b, c - d_{xzt}, d - d_{xzt}) - d_{xyzt}(1,1,1,1) = (a - d_{xzt} - d_{xyzt}, b - d_{xyzt}, c - d_{xzt} - d_{xyzt}, d - d_{xzt} - d_{xyzt})$. Lastly we replace d_{xzt} with k, and d_{xyzt} with m. □

We found further results for the case $|L| = 2$.

Corollary 3.1. *For $L = \{(0,1,1,0),(0,1,1,1)\}$ we have*

$$L(a,b,c,d) = \sum_{k=0}^{\min(b,c)} \sum_{m=0}^{\min(b-k,c-k,d)} \binom{a+b+c+d-k-2m}{a, b-k-m, c-k-m, d-m, k, m}. \tag{13}$$

Corollary 3.2. *For $L = \{(0,1,1,1),(1,1,1,0)\}$ we have*

$$L(a,b,c,d) = \sum_{k=0}^{\min(b,c,d)} \sum_{m=0}^{\min(a,b-k,c-k)} \binom{a+b+c+d-2k-2m}{a-m, b-k-m, c-k-m, d-k, k, m}. \tag{14}$$

Corollary 3.3. *For $L = \{(0,1,0,1),(1,1,0,0)\}$ we have*

$$L(a,b,c,d) = \sum_{k=0}^{\min(b,d)} \sum_{m=0}^{\min(a,b-k)} \binom{a+b+c+d-k-m}{a-m, b-k-m, c, d-k, k, m}. \tag{15}$$

Using a similar approach, $L(a,b,c,d)$ can be derived for the remaining vector sets.

4. $|L| > 2$

As a result of having more vectors in L, the formulas get longer and more complicated. We begin with $L = \{(1,0,1,0),(0,1,1,0),(0,1,1,1)\}$.

4.1. $L = \{(1,0,1,0),(0,1,1,0),(0,1,1,1)\}$

Proposition 4.1. *Let $L = \{(1,0,1,0),(0,1,1,0),(0,1,1,1)\}$. Then,*

$$L(a,b,c,d) = \sum_{k=0}^{A} \sum_{m=0}^{B} \sum_{n=0}^{C} \binom{a+b+c+d-k-m-2n}{a-k, b-m-n, c-k-m-n, d-n, k, m, n} \tag{16}$$

with $A = \min(a,c)$*,*
$B = \min(b, c-k)$*,*
$C = \min(b-m, c-k-m, d)$*.*

Proof. Assume the number of $(1,0,1,0)$ vectors used in a lattice path is d_{xz}, the number of $(0,1,1,0)$ vectors is d_{yz} and the number of $(0,1,1,1)$ vectors is d_{yzt}. We find the valid intervals for the vectors in L.

1) $0 \leq d_{xz} \leq \min(a, c)$

2) After using d_{xz} amount of $(1, 0, 1, 0)$ vectors, the distance needed to travel to (a, b, c, d) is $(a, b, c, d) - d_{xz}(1, 0, 1, 0) = (a - d_{xz}, b, c - d_{xz}, d)$. From here, we can tell that the number of $(0, 1, 1, 0)$ vectors can have values $0 \leq d_{yz} \leq \min(b, c - d_{xz})$.

3) By the equation $(a - d_{xz}, b, c - d_{xz}, d) - d_{yz}(0, 1, 1, 0) = (a - d_{xz}, b - d_{yz}, c - d_{xz} - d_{yz}, d)$, the valid interval for d_{yzt} is $0 \leq d_{yzt} \leq \min(b - d_{yz}, c - d_{xz} - d_{yz}, d)$.

Next, we find the number of each standard basis vector that must be used in the path by subtracting $(0, 1, 1, 1)$ vectors from the last result,

$$(a-d_{xz}, b-d_{yz}, c-d_{xz}-d_{yz}, d)-d_{yzt}(0, 1, 1, 1) = (a-d_{xz}, b-d_{yz}-d_{yzt}, c-d_{xz}-d_{yz}-d_{yzt}, d-d_{yzt}).$$

Lastly we replace d_{xz} with k, d_{yz} with m, and d_{yzt} with n. □

4.2. $L = \{(1, 1, 1, 0), (1, 1, 0, 1), (1, 0, 1, 1), (0, 1, 1, 1)\}$

This is the case where exactly 3 coordinates of each vector in L are 1.

Theorem 4.1. *Let* $L = \{(1, 1, 1, 0), (1, 1, 0, 1), (1, 0, 1, 1), (0, 1, 1, 1)\}$.

$$L(a, b, c, d) = \sum_{k=0}^{A} \sum_{m=0}^{B} \sum_{n=0}^{C} \sum_{p=0}^{D} \binom{a + b + c + d - 2(k + m + n + p)}{a - k - m - n, b - k - m - p, c - k - n - p, d - m - n - p, k, m, n, p} \tag{17}$$

with $A = \min(a, b, c)$,
$B = \min(a - k, b - k, d)$,
$C = \min(a - k - m, c - k, d - m)$,
$D = \min(b - k - m, c - k - n, d - m - n)$.

For other combinations of vector sets, $L(a, b, c, d)$ can be found using similar algorithms.

5. Minimum distance problem

In this section, we explore the problem of determining the minimum number of vectors needed in order to reach (a, b, c, d). We denote this number by $s(a, b, c, d)$ and it is also called the *minimum distance* from origin to (a, b, c, d). The general idea behind determining $s(a, b, c, d)$ is to maximize the number of vectors with the greatest coordinate sum.

For $L_1 \subset L_2$, let $s_1 = s(a, b, c, d)$ for L_1 and $s_2 = s(a, b, c, d)$ for L_2. Compared to L_1, usage of additional vectors in L_2 is optional and may or may not create a shorter path.Thus, $s_2 \leq s_1$. A precise formula for a quantitatively large L is complicated. Thus, we only consider relatively small sets which can be used to find an upper bound for $s(a, b, c, d)$ for larger sets.

5.1. Minimum distance for $L = \{(1, 1, 1, 1)\}$

Proposition 5.1. *Let* $L = \{(1, 1, 1, 1)\}$. *Then* $s(a, b, c, d) = a + b + c + d - 3\min(a, b, c, d)$.

Proof. For the case $L = \{(1, 1, 1, 1)\}$, maximum number of $(1, 1, 1, 1)$ vectors is $\min(a, b, c, d)$. Then, the total number of standard basis vectors is $a + b + c + d - 4\min(a, b, c, d)$. Lastly, we add the number of $(1, 1, 1, 1)$ vectors to this quantity. □

5.2. Minimum distance for $L = \{(1,1,0,0),(0,0,1,1)\}$

Theorem 5.1. *Let $L = \{(1,1,0,0),(0,0,1,1)\}$. Then the minumum number of vectors from $L \cup \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$ that must be used in order to reach (a,b,c,d) is $\max(a,b)+\max(c,d)$. In other words, $s(a,b,c,d) = \max(a,b)+\max(c,d)$.*

Proof. The vectors which travel on x or y axes are $(1,0,0,0)$, $(0,1,0,0)$ and $(1,1,0,0)$. The vectors which travel on z or t axes are $(0,0,1,0)$, $(0,0,0,1)$ and $(0,0,1,1)$. As the two sets do not share a vector, we must find the minimum distance from origin to $(a,b,0,0)$ and the minimum distance from origin to $(0,0,c,d)$ and add these quantities.

To find $s(a,b,0,0)$, it is necessary to maximize the number of $(1,1,0,0)$ vectors used in a path since its coordinate sum is greater than the other two. We can use at most $\min(a,b)$ amount of $(1,1,0,0)$ vectors. $(a,b,0,0) - \min(a,b)(1,1,0,0) = (a-\min(a,b), b-\min(a,b), 0, 0)$ implies that in total there are $a+b-2\min(a,b)+\min(a,b)$ vectors which is equavilent to $\max(a,b)$. Similary, $s(0,0,c,d) = \max(c,d)$. Thus, our result that $s(a,b,c,d) = \max(a,b)+\max(c,d)$ follows. □

5.3. Minimum distance for $L = \{(0,1,1,1),(0,1,1,0)\}$

Theorem 5.2. *For $L = \{(0,1,1,1),(0,1,1,0)\}$,we have*

$$s(a,b,c,d) = \begin{cases} a+\max(b,c), & \textit{if } \min(b,c,d) = d \\ a+|b-c|+d, & \textit{if } \min(b,c,d) \neq d. \end{cases} \tag{18}$$

Proof. We first consider the case $\min(b,c,d) = d$. We can use at most d amount of $(0,1,1,1)$ vectors. By the equation $(a,b,c,d) - s(0,1,1,1) = (a, b-d, c-d, 0)$ we see that the maximum amount of $(0,1,1,0)$ vectors is $\min(b-d, c-d)$. The remaining $b+c+d-3d-2\min(b-d,c-d)+a$ vectors will be standart basis vectors which is equavilent to $a+\max(b,c)-\min(b,c)-d$. Finally adding the number of $(0,1,1,1)$ and $(0,1,1,0)$ vectors we get $s(a,b,c,d) = a+\max(b,c)$.

Next, assume $\min(b,c,d) \neq d$. There can be at most $\min(b,c)$ number of $(0,1,1,1)$ vectors used in a path. We have $(a,b,c,d) - \min(b,c)(0,1,1,1) = (a, b-\min(b,c), c-\min(b,c), d-\min(b,c))$. At least one of $b-\min(b,c)$ or $c-\min(b,c)$ is zero. This implies that there are not any $(0,1,1,0)$ vectors. There are a vectors in the direction $(1,0,0,0)$, $b-\min(b,c)$ vectors in the direction $(0,1,0,0)$, $c-\min(b,c)$ vectors in the $(0,0,1,0)$ direction and $d-\min(b,c)$ vectors in the $(0,0,0,1)$ direction. By the former statement that at least one of $b-\min(b,c)$ or $c-\min(b,c)$ is zero, there are total of $\max(b,c)-\min(b,c)$ vectors in the directions $(0,1,0,0)$ and $(0,0,1,0)$. Finally adding up the results we get $s(a,b,c,d) = a+\max(b,c)-\min(b,c)+d$. Note that $\max(b,c)-\min(b,c) = |b-c|$ □

Example 5.3. 1) We find $s(4,7,6,3)$. By Eq. (18), we get $s(a,b,c,d) = 11$. We can use at most 3 $(0,1,1,1)$ vectors, after that we can use at most 3 $(0,1,1,0)$ vectors, the rest must be standard basis vectros: $3(0,1,1,1)+3(0,1,1,0)+4(1,0,0,0)+1(0,1,0,0) = (4,7,6,3)$.

2) Next we show that $s(2,4,8,6) = 12$. The sum of the amounts of $(0,1,1,1)$ vectors and $(0,0,0,1)$ vectors must be 6. There is no distance left to travel either on y or z axes, since $\min(4,8,6) = 4 \neq 6$. Next we use $|4-8|$ amount of $(0,0,1,0)$ vectors. Finally using 2 $(1,0,0,0)$ vectors we get $6(0,1,1,1)+4(0,0,1,0)+2(1,0,0,0) = (2,4,8,6)$.

5.4. $L = \{(0,1,1,1),(0,1,1,0,),(r_1,r_2,r_3,r_4)\}$

So far in this paper, we considered vector sets L whose vector coordinates are 0 or 1. For the rest of this section we do not align with this rule.

We now consider a set where each coordinate of a vector in L does not have to be 1 or 0. We solve the minimum distance problem for $L = \{(0,1,1,1),(0,1,1,0,),(r_1,r_2,r_3,r_4)\}$ with $r_i \neq 0$ for all i.

Theorem 5.4. *Let* $L = \{(0,1,1,1),(0,1,1,0,),(r_1,r_2,r_3,r_4)\}$ *with* $r_1, r_2, r_3, r_4 \neq 0$*, and write* $k = \lfloor \min(\frac{a}{r_1}, \frac{b}{r_2}, \frac{c}{r_3}, \frac{d}{r_4}) \rfloor$*. Then*

$$s(a,b,c,d) = \begin{cases} k(1-r_1) + a + \max(b-kr_2, c-kr_3), \\ \qquad\qquad \textit{if} \min(b-kr_2, c-kr_3, d-kr_4) = d-kr_4 \\ k(1-r_1-r_4) + a + |b-c+k(r_3-r_2)| + d, \\ \qquad\qquad \textit{if} \min(b-kr_2, c-kr_3, d-kr_4) \neq d-kr_4. \end{cases} \tag{19}$$

Proof. As (r_1,r_2,r_3,r_4) vector does not travel less than other vectors on each direction, the number of this vector must be maximized to find $s(a,b,c,d)$. It is easy to see that the maximum number of this vector that can be used is $k = \min(\lfloor \frac{a}{r_1} \rfloor, \lfloor \frac{b}{r_2} \rfloor, \lfloor \frac{c}{r_3} \rfloor, \lfloor \frac{d}{r_4} \rfloor) = \lfloor \min(\frac{a}{r_1}, \frac{b}{r_2}, \frac{c}{r_3}, \frac{d}{r_4}) \rfloor$. After this, there is still $(a-kr_1, b-kr_2, c-kr_3, d-kr_4)$ distance left to travel using the vectros $(0,1,1,1)$ and $(0,1,1,0)$. The solution to this is given in Eq. (18). □

Example 5.5. Let $L = \{(0,1,1,1),(0,1,1,0,),(2,1,2,4)\}$. Then Eq. (19) gives

$$s(a,b,c,d) = \begin{cases} -k + a + \max(b-k, c-2k), & \text{if } \min(b-k, c-2k, d-4k) = d-4k \\ -5k + a + |b-c+k| + d, & \text{if } \min(b-k, c-2k, d-4k) \neq d-4k \end{cases}$$

with $k = \lfloor \min(\frac{a}{2}, b, \frac{c}{2}, \frac{d}{4}) \rfloor$. Now we find $s(4,2,9,11)$. We get $k = 2$ and $s(4,2,9,11) = 10$. Actually, we have $2(2,1,2,4) + 5(0,0,1,0) + 3(0,0,0,1) = (4,2,9,11)$. If we tried finding $s(4,3,9,11)$, we would see that $s(4,3,9,11) = 9$ which is smaller than $s(4,2,9,11)$. It is true since by adding up 1 to the distance on the y-axis, we can now use a $(0,1,1,1)$ vector instead of a $(0,0,1,0)$ and a $(0,0,0,1)$ vector.

5.5. $L = \{(-1,0,1,0),(1,1,1,1)\}$

The coordinate sum of $(-1,0,1,0)$ is 0. The idea behind using this vector is to be able to use $(1,1,1,1)$ vector.

Theorem 5.6. *For* $L = \{(-1,0,1,0),(1,1,1,1)\}$*, the minimum distance to point* (a,b,c,d) *is*

$$s(a,b,c,d) = \begin{cases} b+c+d-2a-2\min(b-a, d-a, \lfloor \frac{c-a}{2} \rfloor), & \textit{if} \min(a,b,c,d) = a \\ a+b+c+d-3\min(a,b,c,d), & \textit{if} \min(a,b,c,d) \neq a. \end{cases}$$

Proof. Assume $a \leq b \leq c \leq d$. It is clear that shortest path to (a, b, c, d) contains at least a $(1, 1, 1, 1)$ vectors. Next we use $(-1, 0, 1, 0)$ vectors to make space for more $(1, 1, 1, 1)$ vectors. There are no vectors going to a negative direction on y and t axes (2nd and 4th axes respectivly). As $(-1, 0, 1, 0)$ vectors travels on the z axis, the difference between traveled distances on x and z axes must be at least 2 to be able to use a $(-1, 0, 1, 0)$ vector. Hence, at most $\min(b-a, d-a, \lfloor \frac{c-a}{2} \rfloor)$ amount of $(-1, 0, 1, 0)$ vetors can be used which is also the number of additional $(1, 1, 1, 1)$ vectors.. The number of standard basis vectors in the shortest path is $a + b + c + d - 4a - 4\min(b - a, d - a, \lfloor \frac{c-a}{2} \rfloor) = b + c + d - 3a - 4\min(b - a, d - a, \lfloor \frac{c-a}{2} \rfloor)$. Finally adding up the number of $(1, 1, 1, 1)$ and $(-1, 0, 1, 0)$ vectors we get $b + c + d - 2a - 2\min(b - a, d - a, \lfloor \frac{c-a}{2} \rfloor)$.

If $\min(a, b, c, d) \neq a$, the shortest path does not contain any $(-1, 0, 1, 0)$ vectors. It is easy to see that the shortest path uses exactly $\min(a, b, c, d)$ amount of $(1, 1, 1, 1)$ vectors, resulting in $a + b + c + d - 3\min(a, b, c, d)$ total vectors. □

6. Lattice paths with vectors of fixed length

For the vector set $V = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 01)\} \cup \{(i, j, p, n-i-j-p) : 0 \leq i \leq n,\ 0 \leq j \leq n-i\ 0 \leq p \leq n-i-j\}$, we find the number of lattice paths to (a, b, c, d). $V(a, b, c, d)$ stands for the number of such paths. Notice that the coordinate sum of the vectors besides the standard basis vectors are allways n.

Theorem 6.1. *The number of lattice paths to (a, b, c, d) using vectors in V is*

$$\sum_{(a,b,c,d)\in\mathbb{N}^4} V(a, b, c, d)x^a y^b z^c t^d = \sum_{\ell=0}^{\lfloor \frac{a+b+c+d}{n} \rfloor} k \left(\sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{p=0}^{n-i-j} x^i y^j z^p t^{n-i-j-p} \right)^{\ell} (x+y+z+t)^{a+b+c+d-n\ell} \tag{20}$$

with $k = \binom{a+b+c+d-(n-1)\ell}{\ell}$ and $n \geq 2$.

Proof. We can find all possible ways to make an ordered list of ℓ steps with a length of n by expanding $\left(\sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{p=0}^{n-i-j} x^i y^j z^p t^{n-i-j-p} \right)^{\ell}$. The number of standard basis vectors required to reach (a, b, c, d) is $a+b+c+d-n\ell$. All possible ways to make an ordered list of $a+b+c+d-n\ell$ vectors is given by expanding $(x + y + z)^{a+b+c-n\ell}$. Multiplying these gives all possible ways to make an ordered list of ℓ steps with a length of n followed by an ordered list of $a + b + c + d - n\ell$ standard basis vectors. Lastly, we choose ℓ positions from $a + b + c + d - (n - 1)\ell$ total positions to be the vectors with length n, $\binom{a+b+c-(n-1)\ell}{\ell}$. □

7. Conclusion

In this paper, we studied lattice paths in 4-dimensions. We found explicit formulas which gives the number of paths from origin to (a, b, c, d) for various vector sets. Then we explored the minimum distance problem. We found piecevise functions that give the number of minimal length latice paths for an arbitrary choice of vectors.

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