Indonesian Journal of Combinatorics **5** (2) (2021), 110–121

INDONESIAN JOURNAL OF COMBINATORICS

On local antimagic vertex coloring of corona products related to friendship and fan graph

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Abstract

Let $G = (V, E)$ be connected graph. A bijection $f : E \to \{1, 2, 3, \ldots, |E|\}$ is a local antimagic of G if any adjacent vertices $u, v \in V$ satisfies $w(u) \neq w(v)$, where $w(u) = \Sigma_{e \in E(u)} f(e)$, $E(u)$ is the set of edges incident to u. When vertex u is assigned the color $w(u)$, we called it a local antimagic vertex coloring of G. A local antimagic chromatic number of G, denoted by $\chi_{la}(G)$, is the minimum number of colors taken over all colorings induced by the local antimagic labeling of G. In this paper, we determine the local antimagic chromatic number of corona product of friendship and fan with null graph on m vertices, namely, $\chi_{la}(F_n \odot \overline{K_m})$ and $\chi_{la}(f_{(1,n)} \odot \overline{K_m})$.

Keywords: Corona product, fan, friendship, local antimagic vertex coloring, local antimagic chromatic number
Mathematics Subject Classification: 05C15, 05C76, 05C78
DOI: 10.19184/ijc.2021.5.2.7

1. Introduction

All graphs $G = (V, E)$ considered in this paper are simple and finite. A vertex coloring of a graph G is an assignment of color to vertices of G such that every two adjacent vertices have a different color. A k-coloring of G is defined as a map $h : V \to \{1, 2, \ldots, k\}$ such that $h(u) \neq h(v)$ for any adjacent vertices $u, v \in V$. The chromatic number of G, denoted by $\chi(G)$, is the smallest positive integer k assigned to G.

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Received: 24 July 2021, Revised: 29 October 2021, Accepted: 15 December 2021.

Hartsfield and Ringel [6] introduced the principle of antimagic labeling, and then Gallian [4] surveyed the researches conducted on graph labeling and its variation, including antimagic labeling. The antimagic on a graph is defined as follows. Let $f : E \to \{1, 2, 3, \dots, |E|\}$ be a bijection. The weight of vertex u, denoted $w(u)$, is defined as $w(u) = \Sigma_{e \in E(u)} f(e)$, where $E(u)$ is the set of edges incident to u. The graph G is called antimagic if $w(u) \neq w(v)$, for every two vertices $u, v \in V$. Arumugam et al. [1] introduced the term of local antimagic as follows. A graph G is called local antimagic if $w(u) \neq w(v)$, for any adjacent vertices $u, v \in V$. If for every distinct weight we assign distinct color, then it is called local antimagic vertex coloring. The local chromatic number of G, denoted by $\chi_{la}(G)$, is the minimum number of colors taken over all colorings induced by local antimagic labeling of G. The local chromatic number of some graphs has been discovered, such as a tree, path, cycle, friendship, complete bipartite, an amalgamation of paths, wheel [3], kite, and cycle with two pendants [9].

Putri et al. [11] initiated a variation of local antimagic coloring named local antimagic total vertex labeling where the label is assigned to the vertices and edges of G. The weight of vertex $u \in V$, w(u), is the sum of labels of all edges incident with u and the label of u itself. A local antimagic total chromatic number of G, denoted by $\chi_{lat}(G)$, is the minimum number of colors induced by local vertex antimagic total labeling of G. The local antimagic total chromatic number of some graphs have been discovered such as star, a double star, banana tree, centipede, amalgamation of graphs [11] and the corona product of some graphs with K_2 [7].

If the vertices of G received the smaller label in the local antimagic total labeling, then it is called the super local antimagic total. While, when the smaller labels are assigned to edges of G, it is called super edge local antimagic total labeling. The super local antimagic chromatic number and the super edge local antimagic chromatic number of G is denoted by $\chi_{slat}(G)$ and $\chi_{selat}(G)$ respectively. The super local antimagic chromatic number of some graphs have been discovered such as fan, gear, sunflower [10], star, double star, cycle, path, cubic bipartite, wheel, amalgamation of graph, and several joint product graphs [12]. On the other hand, the super edge local antimagic total chromatic number has been discovered for path and its derivation, hedge, hedgerow, star, and an amalgamation of graphs [5].

A corona product of H and G, denoted by $G \odot H$, is a graph obtained by taking one copy of G along with $|V(G)|$ copies of H and putting extra edges making the i-th vertex of G adjacent to every vertex of the i-th copy of H [3]. A null graph on m vertices, denoted by $\overline{K_m}$, as a graph that has m isolated vertices [2].

In this paper, we study the local antimagic chromatic number of corona products of friendship and fan with a null graph on m vertices. Arumugam et al. [1] proved a sharp lower bound for any tree, and Lau et al. [8] generalized the theorem as follows.

Theorem 1.1. *[8] Let G be a graph having k pendants. If G is not K_2, the $\chi_{la}(G) \geq k + 1$ and the bound is sharp.*

2. Main Results

2.1. Corona Products of Friendship and Null Graphs

A friendship graph F_n can be constructed by joining n copies of C_3 with a common vertex. Figure 1 illustrates the graph $F_n \odot \overline{K_m}$. Since $F_1 \cong C_3$ and Arumugam et al. [2] already give

$\chi_{la}(C_n \odot \overline{K_m})$, here we consider $F_n \odot \overline{K_m}$ for $n \geq 2$ and $m \geq 1$.

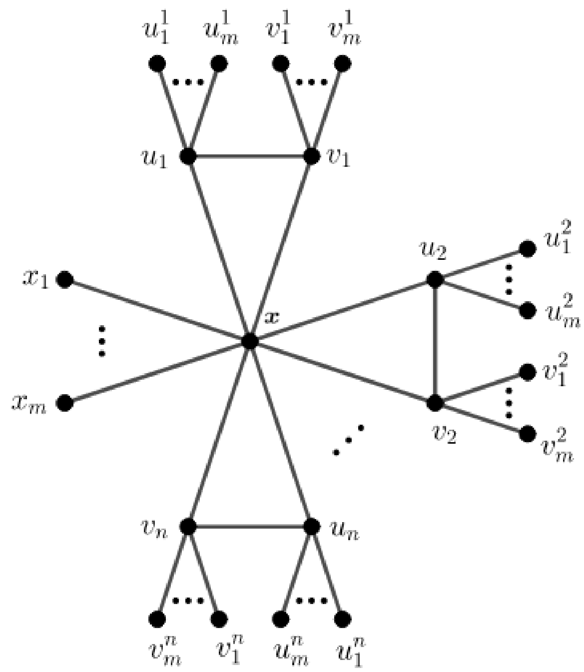

Figure 1. The graf $F_n \odot \overline{K_m}$

Theorem 2.1. *Let F_n be fan graph on n cycles and $\overline{K_m}$ null graph on m vertices. For $n \geq 2$ and $m \geq 1$, $\chi_{la}(F_n \odot \overline{K_m}) = m(2n+1)+3$.*

Proof. Let $V(F_n \odot \overline{K_m}) = \{x, v_i, u_i, v_j^i, u_j^i, x_j | 1 \leq i \leq n$ and $1 \leq j \leq m\}$ and $E(F_n \odot \overline{K_m}) = \{xu_i, xv_i, xx_j, u_iv_i, u_iu_j^i, v_iv_j^i | 1 \leq i \leq n$ and $1 \leq j \leq m\}$.

For the upper bound, we show that $\chi_{la}(F_n \odot \overline{K_m}) \leq m(2n+1)+3$. Define $f : E \rightarrow \{1, 2, \ldots, m(2n+1)+3n\}$. Label u_iv_i, xu_i, xv_i, and xx_j for $1 \leq i \leq n$ and $1 \leq j \leq m$ as follows.

$$\begin{aligned} f(u_iv_i) &= i, \\ f(xu_i) &= 3n+2-2i, \\ f(xv_i) &= 3n+1-2i, \\ f(xx_j) &= 2mn+3n+j. \end{aligned}$$

Then, to label $u_iu_j^i$ and $v_iv_j^i$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, we divide into two cases according to parity of m.

Case 1. m is odd

For $1 \leq i \leq n$ and $1 \leq j \leq m$, label the edges as follows.

$$f(u_i u_j^i) = \begin{cases} n(2j+1)+i, & j \text{ is odd}, \\ 2n(j+1)+1-i, & j \text{ is even}, \end{cases}$$
$$f(v_i v_j^i) = \begin{cases} 2n(j+1)+i, & j \text{ is odd}, \\ n(2j+3)+1-i, & j \text{ is even}. \end{cases}$$

The labeling f is obviously a local antimagic with weights as follows.

$$\begin{aligned} w(u_i) &= m^2n + mn + 4n + 2 + \tfrac{3mn+m-3n-1}{2}, \\ w(v_i) &= m^2n + 2mn + 2n + 1 + \tfrac{3mn+m+n-1}{2}, \\ w(x) &= 4n^2 + n + 2m^2n + 3mn + \tfrac{m^2+m}{2}, \\ w(x_j) &= 2mn + 3n + j, \\ w(u_j^i) &= \begin{cases} n(2j+1)+i, & j \text{ is odd}, \\ 2n(j+1)+1-i, & j \text{ is even}, \end{cases} \\ w(v_j^i) &= \begin{cases} 2n(j+1)+i, & j \text{ is odd}, \\ n(2j+3)+1-i, & j \text{ is even}. \end{cases} \end{aligned}$$

Case 2. m is even
For $1 \leq i \leq n$ and $1 \leq j \leq m$, label the edges as follows.

$$f(u_i u_j^i) = \begin{cases} 3n+2i-1, & j = 1, \\ n(2j+1)+i, & j \neq 1 \text{ and } j \text{ is odd}, \\ 2n(j+1)+1-i, & j \text{ is even}, \end{cases}$$
$$f(v_i v_j^i) = \begin{cases} 3n+2i, & j = 1, \\ 2n(j+1)+i, & j \neq 1 \text{ and } j \text{ is odd}, \\ n(2j+3)+1-i, & j \text{ is even}. \end{cases}$$

The labeling f is obviously a local antimagic with weights as follows.

$$\begin{aligned} w(u_i) &= m^2n + 2mn + 3n + 1 + \tfrac{mn+m}{2}, \\ w(v_i) &= m^2n + mn + 2n + 1 + \tfrac{5mn+m}{2}, \\ w(x) &= 4n^2 + n + 2m^2n + 3mn + \tfrac{m^2+m}{2}, \\ w(x_j) &= 2mn + 3n + j, \\ w(u_j^i) &= \begin{cases} 3n+2i-1, & j = 1, \\ n(2j+1)+i, & j \neq 1 \text{ and } j \text{ is odd}, \\ 2n(j+1)+1-i, & j \text{ is even}, \end{cases} \\ w(v_j^i) &= \begin{cases} 3n+2i, & j = 1, \\ 2n(j+1)+i, & j \neq 1 \text{ and } j \text{ is odd}, \\ n(2j+3)+1-i, & j \text{ is even}. \end{cases} \end{aligned}$$

Note that for $1 \leq i \leq n, 1 \leq j \leq m$, the weights of $u_j^i,\ v_j^i$, and x_j depend on i and j while the weight of u_i, v_i, and x are constant. Hence, we have $2mn + m + 3$ different weights in total.

Therefore, $\chi_{la}(F_n \odot \overline{K_m}) \le m(2n+1)+3$.

For the lower bound, we show that $\chi_{la}(F_n \odot \overline{K_m}) \ge m(2n+1)+3$. Since $F_n \odot \overline{K_m}$ has $2mn+m$ pendants, by using Theorem 1.1, we have $\chi_{la}(F_n \odot \overline{K_m}) \ge 2mn+m+1$. Suppose $\chi_{la}(F_n \odot \overline{K_m}) \ge 2mn+m+1$. Then, $w(x)$ will equal to either $w(u_j^i)$ or $w(v_j^i)$ for some i and j. Since $d(x) = 2n+m$, we obtain $w(x) \ge \sum_{k=1}^{2n+m} k = \frac{(2n+m)(2n+m+1)}{2}$, where $d(x)$ is degree of vertex x. On the other hand, the weights of either $w(u_j^i)$ or $w(v_j^i) \le |E| = 2mn+m+3n$ which implies $w(x) \ge \frac{(2n+m)(2n+m+1)}{2} = 2n^2+2mn+n+\frac{m^2+m}{2} \ge 2mn+(2n+1)n+m > 2mn+3n+m$. It is a contradiction. Therefore, the color of $w(x)$ must be different from all pendants and now we have extended the lower bound to $\chi_{la}(F_n \odot \overline{K_m}) \ge 2mn+m+2$.

Suppose $\chi_{la} \ge 2mn+m+2$. Then, either $w(u_i) = w(v_j^i)$ or $w(v_i) = w(u_j^i)$ must be satisfy for some j. Suppose $w(u_i) = w(v_j^i)$. Notice that $w(u_i) \ge \frac{\sum_{k=1}^{2n+mn} k}{n} = \frac{(2n+mn)(2n+mn+1)}{2n}$, while $w(v_j^i) \le 2mn+m+3n$. It is not hard to verify that $\frac{(2n+mn)(2n+mn+1)}{2n} = 2mn+(\frac{mn+1}{2})m+4n+2 > 2mn+m+3n$ for $n \ge 2$ and $m \ge 1$. It is a contradiction since $w(v_j^i) < w(u_i)$. We can construct the same argument to show a contradiction for the case $w(u_i) = w(x_j)$ or $w(v_i) = w(x_j)$ for some j. Therefore, the color of $w(v_i)$ must be different from all pendants and now we have extended the lower bound to $\chi_{la}(F_n \odot \overline{K_m}) \ge 2mn+m+3$.

Since both inequalities $\chi_{la}(F_n \odot \overline{K_m}) \le 2mn+m+3$ and $\chi_{la}(F_n \odot \overline{K_m}) \ge 2mn+m+3$ hold, then $\chi_{la}(F_n \odot \overline{K_m}) = 2mn+m+3$. □

We give the local antimagic vertex coloring for $F_5 \odot \overline{K_3}$ with $\chi_{la}(F_5 \odot \overline{K_3}) = 36$ in Figure 2.

2.2. Corona Products of Fan and Null Graphs

A fan graph $f_{(1,n)}$ is defined as the graph $K_1 + P_n$ where K_1 is the null graph on one vertex and P_n is the path graph on n vertices. Figure 3 illustrates the graph $f_{(1,n)} \odot \overline{K_m}$. Since $f_{(1,2)} \cong C_3$ and Arumugam et al. [2] already give $\chi_{la}(C_n \odot \overline{K_m})$, here we consider $f_{(1,n)} \odot \overline{K_m}$ for $n \ge 3$ and $m \ge 1$.

Theorem 2.2. *Let $f_{(1,n)}$ be friendship of $n+1$ vertices and $\overline{K_m}$ null graph on m vertices. For $n \ge 3$ and $m \ge 1$, $\chi_{la}(f_{(1,n)} \odot \overline{K_m}) = m(n+1)+3$.*

Proof. Let $V(f_{(1,n)} \odot \overline{K_m}) = \{x, v_i, v_j^i, x_j | 1 \le i \le n \text{ and } 1 \le j \le m\}$ and $E(f_{(1,n)} \odot \overline{K_m}) = \{xx_j, xv_i, v_iv_{i+1}, v_iv_j^i | 1 \le i \le n \text{ and } 1 \le j \le m\}$.

For the upper bound, we show that $\chi_{la}(f_{(1,n)} \odot \overline{K_m}) \le m(n+1)+3$. Define $f : E \to \{1, 2, \ldots, m(n+1)+2n-1\}$. We divide into two cases depend on the parity of n.

Case 1. n is odd

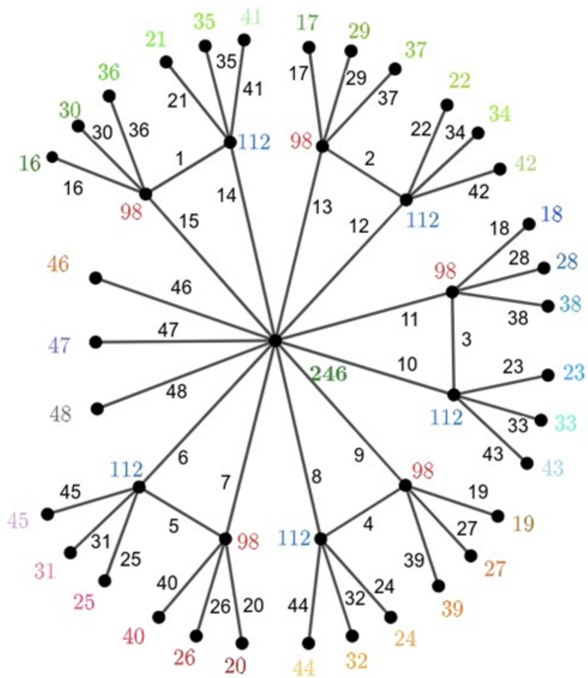

Figure 2. The local antimagic vertex coloring of $F_5 \odot \overline{K_3}$

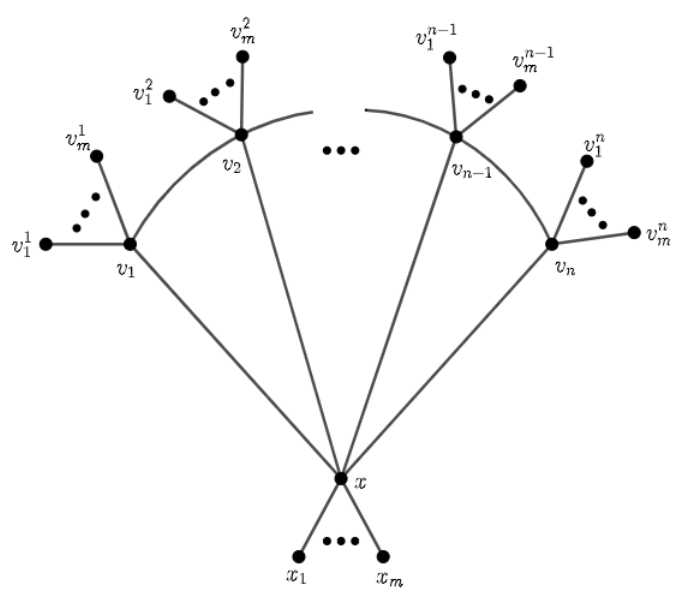

Figure 3. The graph $f_{(1,n)} \odot \overline{K_m}$

Label the edges $v_iv_{i+1}, xv_i,$ and xx_j for $1 \le i \le n$ and $1 \le j \le m$ as follows.

$$
\begin{aligned}
f(v_iv_{i+1}) &= \begin{cases} \frac{i+1}{2}, & i \text{ is odd},\\ n-1-\frac{i-2}{2}, & i \text{ is even},\end{cases}\\
f(xv_i) &= \begin{cases} 2n-1, & i=1,\\ n+\frac{i-3}{2}, & i\neq 1 \text{ and } i \text{ is odd},\\ n-1+\frac{n-1+i}{2}, & i \text{ is even},\end{cases}\\
f(xx_j) &= mn+m+2n-j.
\end{aligned}
$$

Then, to label $v_iv_j^i$ for $1 \le i \le n$ and $1 \le j \le m$, we divide into two subcases according to parity of m.

Subcase 1. m is odd

Label $v_iv_j^i$ for $1 \le i \le n$ and $1 \le j \le m$ as follows.

$$
\begin{aligned}
f(v_iv_1^i) &= \begin{cases} 2n+1+\frac{n-3}{2}, & i=n,\\ 2n+\frac{n-i-2}{2}, & i\neq n \text{ and } i \text{ is odd},\\ 3n-1-\frac{i-2}{2}, & i \text{ is even},\end{cases}\\
f(v_iv_j^i) &= \begin{cases} (j+1)n-1+i, & j\neq 1 \text{ and } j \text{ is odd},\\ (j+2)n-i, & j \text{ is even}.\end{cases}
\end{aligned}
$$

The labeling f is obviously a local antimagic with weights

$$
\begin{aligned}
w(v_i) &= \begin{cases} 2mn+2n-1+\frac{m^2-m}{2}, & i \text{ is odd},\\ 2mn+3n-1+\frac{m^2-m}{2}, & i \text{ is even}.\end{cases}\\
w(x) &= m^2n+2mn+\frac{m^2-m+3n^2-n}{2},\\
w(x_j) &= mn+m+2n-j,\\
w(v_j^i) &= \begin{cases} 2n+1+\frac{n-3}{2}, & j=1; i=n,\\ 2n+\frac{n-i-2}{2}, & j=1; i\neq n \text{ and } i \text{ is odd},\\ 3n-1-\frac{i-2}{2}, & j=1; i \text{ is even},\\ (j+1)n-1+i, & j\neq 1 \text{ and } j \text{ is odd},\\ (j+2)n-i, & j \text{ is even}.\end{cases}
\end{aligned}
$$

Subcase 2. m is even

Label $v_iv_j^i$ for $1 \le i \le n$ and $1 \le j \le m$ as follows.

$$
\begin{aligned}
f(v_iv_1^i) &= \begin{cases} 3n-1, & i=n,\\ 3n-2-i, & i\neq n \text{ and } i \text{ is odd},\\ 3n-i, & i \text{ is even},\end{cases}\\
f(v_iv_2^i) &= \begin{cases} 3n, & i=n,\\ 3n+\frac{i+1}{2}, & i\neq n \text{ and } i \text{ is odd},\\ 3n+\frac{n-1+i}{2}, & i \text{ is even},\end{cases}\\
f(v_iv_j^i) &= \begin{cases} (j+1)n-1+i, & j\neq 1 \text{ and } j \text{ is odd},\\ (j+2)n-i, & j\neq 2 \text{ and is even}.\end{cases}
\end{aligned}
$$

The labeling f is obviously a local antimagic with weights

$$
\begin{aligned}
w(v_i) &= \begin{cases} 2mn + 2n + 1 + \frac{m^2n-m}{2}, & i \text{ is odd}, \\ 2mn + 3n - 1 + \frac{m^2n-m}{2}, & i \text{ is even}, \end{cases} \\
w(x) &= m^2n + 2mn + \frac{m^2-m+3n^2-n}{2}, \\
w(x_j) &= mn + m + 2n - j, \\
w(v_j^i) &= \begin{cases} 3n-1, & j=1; i=n, \\ 3n-2-i, & j=1; i \neq n \text{ and } i \text{ is odd}, \\ 3n-i, & j=1; i \text{ is even}, \\ 3n, & j=2; i=n, \\ 3n+\frac{i+1}{2}, & j=2; i \neq n \text{ and } i \text{ is odd}, \\ 3n+\frac{n-1+i}{2}, & j=2; i \text{ is even}, \\ (j+1)n-1+i, & j \neq 1 \text{ and } j \text{ is odd}, \\ (j+2)n-i, & j \neq 2 \text{ and is even}. \end{cases}
\end{aligned}
$$

Case 2. n is even
Label the edges $\{v_iv_{i+1}, xv_i,$ and $xx_j\}$ $1 \leq i \leq n$ and $1 \leq j \leq m$ as follows.

$$
\begin{aligned}
f(v_iv_{i+1}) &= \begin{cases} \frac{i+1}{2}, & i \text{ is odd}, \\ n-1-\frac{i-2}{2}, & i \text{ is even}, \end{cases} \\
f(xv_i) &= \begin{cases} 2n-1, & i=1, \\ 2n-2, & i=n, \\ n+\frac{i-3}{2}, & i \neq 1 \text{ and } i \text{ is odd}, \\ n-2+\frac{n+i}{2}, & i \neq n \text{ and } i \text{ is even}, \end{cases} \\
f(xx_j) &= mn + m + 2n - j.
\end{aligned}
$$

Then, to label $v_iv_j^i$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, we divide into two subcases according to parity of m.

Subcase 1. m is odd
Label $v_iv_j^i$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ as follows.

$$
\begin{aligned}
f(v_iv_1^i) &= \begin{cases} 3n-1, & i=n, \\ 3n-1-\frac{i}{2}, & i \neq n \text{ and } i \text{ is even}, \\ 2n+\frac{n-i-1}{2}, & i \text{ is odd}, \end{cases} \\
f(v_iv_j^i) &= \begin{cases} (j+1)n-1+i, & j \neq 1 \text{ and } j \text{ is odd}, \\ (j+2)n-i, & j \text{ is even}. \end{cases}
\end{aligned}
$$

The labeling f is obviously a local antimagic with weights

$$\begin{aligned}
w(v_i) &= \begin{cases} 2mn+2n-1+\frac{m^2n-m+1}{2}, & i \text{ is odd,} \\ 2mn+3n-3+\frac{m^2n-m+1}{2}, & i \text{ is even,} \end{cases} \\
w(x) &= m^2n+2mn+\tfrac{m^2-m+3n^2-n}{2}, \\
w(x_j) &= mn+m+2n-j, \\
w(v_j^i) &= \begin{cases} 3n-1, & j=1; i=n, \\ 3n-1-\frac{i}{2}, & j=1; i\neq n \text{ and } i \text{ is even,} \\ 2n+\frac{n-i-1}{2}, & j=1; i \text{ is odd,} \\ (j+1)n-1+i, & j\neq 1 \text{ and } j \text{ is odd,} \\ (j+2)n-i, & j \text{ is even.} \end{cases}
\end{aligned}$$

Subcase 2. m is even
Label $v_iv_j^i$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ as follows

$$\begin{aligned}
f(v_iv_1^i) &= \begin{cases} 3n-1, & i=n, \\ 3n-1-i, & i\neq n, \end{cases} \\
f(v_iv_2^i) &= \begin{cases} 3n+1+\frac{n-2}{2}, & i=n, \\ 3n+1+\frac{n+i-2}{2}, & i\neq n \text{ and } i \text{ is even,} \\ 3n+\frac{i-1}{2}, & i \text{ is odd,} \end{cases} \\
f(v_iv_j^i) &= \begin{cases} (j+1)n-1+i, & j\neq 1 \text{ and } j \text{ is odd,} \\ (j+2)n-i, & j\neq 2 \text{ and } j \text{ is even.} \end{cases}
\end{aligned}$$

The labeling f is obviously a local antimagic with weights

$$\begin{aligned}
w(v_i) &= \begin{cases} 2mn+2n-1+\frac{m^2n-m}{2}, & i \text{ is odd,} \\ 2mn+3n-2+\frac{m^2n-m}{2}, & i \text{ is even,} \end{cases} \\
w(x) &= m^2n+2mn+\tfrac{m^2-m+3n^2-n}{2}, \\
w(x_j) &= mn+m+2n-j, \\
w(v_j^i) &= \begin{cases} 3n-1, & j=1; i=n, \\ 3n-1-i, & j=1; i\neq n, \\ 3n+1+\frac{n-2}{2}, & j=2; i=n, \\ 3n+1+\frac{n+i-2}{2}, & j=2; i\neq n \text{ and } i \text{ is even,} \\ 3n+\frac{i-1}{2}, & j=2; i \text{ is odd,} \\ (j+1)n-1+i, & j\neq 1 \text{ and } j \text{ is odd,} \\ (j+2)n-i, & j\neq 2 \text{ and } j \text{ is even.} \end{cases}
\end{aligned}$$

Since for $1 \leq i \leq n$ and $1 \leq j \leq m$, the weights of v_j^i and xx_j depend on i and j, while the weight of v_i and x are constant, we have $mn+m+3$ different weights in total. Therefore, $\chi_{la}(f_{(1,n)} \odot \overline{K_m}) \leq mn+m+3$.

For the lower bound, we show that $\chi_{la}(f_{(1,n)} \odot \overline{K_m}) \geq mn+m+3$. Since $f_{(1,n)} \odot \overline{K_m}$ has $mn+m$ pendants, by using Theorem 1.1, we have $\chi_{la}(f_{(1,n)} \odot \overline{K_m}) \geq mn+m+1$. Suppose

$\chi_{la}(f_{(1,n)} \odot \overline{K_m}) \geq mn + m + 1$. Then, $w(x)$ must equal to $w(v_j^i)$ for some i and j. Note that $w(x) \geq \sum_{k=1}^{m+n} k = \frac{(m+n)(m+n+1)}{2}$, while $w(v_j^i) \leq mn + m + 2n - 1$ for any i and j. It is not hard to verify that $\frac{(m+n)(m+n+1)}{2} = mn + (\frac{m+1}{2})m + (\frac{n+1}{2})n > mn + m + 2n - 1$, if $n \geq 3$. Hence, we get a contradiction. Therefore, $\chi_{la}(f_{(1,n)} \odot \overline{K_m}) \geq mn + m + 2$.

Now, suppose $\chi_{la}(f_{(1,n)} \odot \overline{K_m}) \geq mn + m + 2$. Since $w(x)$ is unique, there must be at least $\lfloor \frac{n}{2} \rfloor$ pairs of vertices such that $w(v_i) = w(v_j^i)$ for some i and j. We will show that it is impossible by considering the parity of n. First, if n is even, $w(v_i) \geq \frac{\sum_{k=1}^{\frac{mn+3n+2}{2}} k}{\frac{n}{2}} = \frac{(\frac{mn+3n+2}{2})(\frac{mn+3n+4}{2})}{n}$ for all i, while $w(v_j^i) \leq (n+1)m + 2n - 1$ for all i and j. It is not hard to verify that $\frac{(\frac{mn+3n+2}{2})(\frac{mn+3n+4}{2})}{n} = (\frac{3n+3}{2})m + (\frac{m^2+9}{4})n + (\frac{9}{2} + \frac{2}{n}) > (n+1)m + 2n - 1$. Second, if n is odd, $w(v_i) \geq \frac{\sum_{k=1}^{\frac{3n+mn-m-3}{2}} k}{\frac{n-1}{2}} = \frac{(\frac{3n+mn-m-3}{2})(\frac{3n+mn-m-1}{2})}{(n-1)}$ for all i, while $w(v_j^i) \leq (n+1)m + 2n - 1$ for all i and j. It is not hard to verify that $\frac{(\frac{3n+mn-m-3}{2})(\frac{3n+mn-m-1}{2})}{(n-1)} = \frac{m^2n^2 - 2m^2n + 6mn^2 - 10mn + 9n^2 + m^2 + 4m - 12n + 3}{4n-4} = (1 + \frac{2n-2m-6}{4n-4})mn + (\frac{mn^2+m+4}{4n-4})m + (2 + \frac{n-4}{4n-4})n + \frac{3}{4n-4} > mn + m + 2n - 1$. We have a contradiction. We can construct the same argument to show a contradiction for the case $w(v_i) = w(x_j)$ for some j. Therefore, $\chi_{la}(f_{(1,n)} \odot \overline{K_m}) \geq mn + m + 3$.

Since both $\chi_{la}(f_{(1,n)} \odot \overline{K_m}) \leq mn + m + 3$ and $\chi_{la}(f_{(1,n)} \odot \overline{K_m}) \geq mn + m + 3$ hold, then $\chi_{la}(f_{(1,n)} \odot \overline{K_m}) = mn + m + 3$. □

We give the local antimagic vertex coloring for $f_{(1,6)} \odot \overline{K_3}$ with $\chi_{la}(f_{(1,6)} \odot \overline{K_3}) = 24$ in Figure 4.

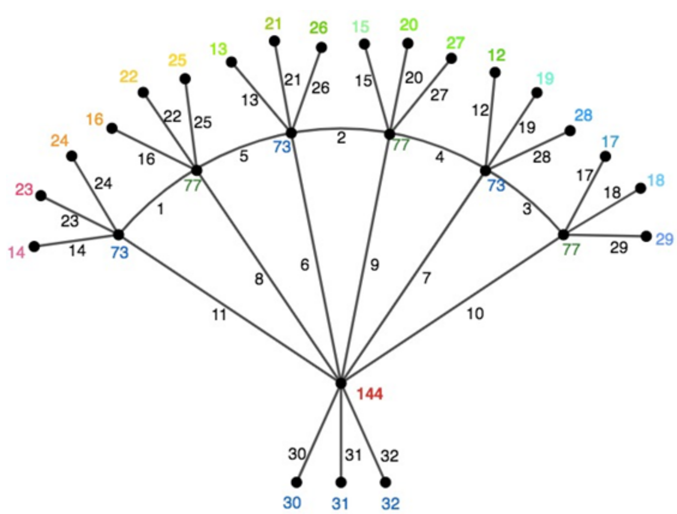

Figure 4. The local antimagic vertex coloring of $f_{(1,6)} \odot \overline{K_3}$

3. Conclusion

We summarise the results in Table 1.

Table 1. Summary

Corona Products of	Notation	χ_{la}	Condition
Friendship with Null graph on m vertices	$F_n \odot \overline{K_m}$	$m(2n+1)+3$	$n \geq 2$ and $m \geq 1$
Fan with Null graph on m vertices	$f_{(1,n)} \odot \overline{K_m}$	$m(n+1)+3$	$n \geq 3$ and $m \geq 1$

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