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Eigenvalues of Antiadjacency Matrix of Cayley Graph of $\mathbb{Z}_n$

Juan Daniel, Kiki Ariyanti Sugeng, Nora Hariadi

Department of Mathematics, Faculty of Mathematics and Natural Science, Universitas Indonesia, Depok, 16424, Indonesia

juan.daniel@sci.ui.ac.id, kiki@sci.ui.ac.id, nora.hariadi@sci.ui.ac.id

Abstract

In this paper, we give a relation between the eigenvalues of the antiadjacency matrix of Cay$(\mathbb{Z}_n, S)$ and the eigenvalues of the antiadjacency matrix of Cay$(\mathbb{Z}_n, (\mathbb{Z}_n - \{0\}) - S)$, as well as the eigenvalues of the adjacency matrix of Cay$(\mathbb{Z}_n, S)$. Then, we give the characterization of connection set S where the eigenvalues of the antiadjacency matrix of Cay$(\mathbb{Z}_n, S)$ are all integers.

Keywords: Antiadjacency matrix, Cayley graph, group $\mathbb{Z}_n$, eigenvalues, adjacency matrix, circulant matrix
Mathematics Subject Classification : 05C50

1. Introduction

Cayley graph has always been an interesting subject in mathematics, since it allows us to understand group theory by using graph theory and vice versa. Let G be a group with 0_G as its identity element and S be an inverse-closed (that is $S = S^{-1}$, where $S^{-1} = \{s^{-1} : s \in S\}$) subset of $G - \{0_G\}$. Cayley graph on group G with connection set S, denoted by Cay(G, S) is defined as graph with G as its vertex set and arcs (x, y) for every pair (x, y) which satisfies $xy^{-1} \in S$ [9]. One of the most active research area in Cayley graph is considering the eigenvalues of the adjacency matrix of Cayley graphs.

There are many intriguing research problems related to eigenvalues research of the adjacency matrix, such as the problems of finding integral Cayley graphs. Integral graph itself is defined as graphs whose eigenvalues of adjacency matrix are all integer. The search for integral graphs began

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from the research done by Harary and Schwenk [5] in 1974. There are many research about Cayley integral graphs on various groups, such as permutation group ([4], [6]), dihedral group ([1], [10]), and $\mathbb{Z}_n$ group ([7], [14]).

In his research, So [14] has found a necessary and sufficient condition for Cayley graph of $\mathbb{Z}_n$ to be integral, which is stated in the following theorem

Theorem 1.1. [14] *Let n be an integer greater than 1 and d be a positive factor of n. Define $S_n(d) = \{a : 0 < a \leq n, gcd(a, n) = d\}$. Then Cay$(\mathbb{Z}_n, S)$ is an integral graph if and only if S is the union of some $S_n(d)$.*

Klotz and Sander [7] in 2007 also found the sufficient condition in Theorem 1.1 through different method. They introduced the concept of gcd graph, which is graph with vertex set $\mathbb{Z}_n$ and arcs (x, y) if and only if gcd $(x - y, n)$ is a divisor of n. He then proved that gcd graphs are all integral graphs. It can be seen that the graphs mentioned in [7] are equivalent with the graphs mentioned in [14]. In 2017, Mirafzal et. al. [11] proved that Cay$(\mathbb{Z}_{2n}, (\mathbb{Z}_{2n} - \{0\}) - \{n\})$ is an integral graph for all integers $n \geq 2$. Moreover, they found that all the eigenvalues of the adjacency matrix of the graph are $2n - 2, 0,$ and -2 and their multiplicities are $1, n, n - 1$, respectively.

There are many matrix representations of graphs, such as antiadjacency matrix. Antiadjacency matrix of graph G, usually denoted by B, is a matrix $B = J - A$ with J equals to $n \times n$ matrix with all of its entries are 1 and A is the adjacency matrix of graph G [3]. Stin et al. [15] studied about the eigenvalues of the antiadjacency matrix of cyclic directed prism graph. Murni et al. [12] also researched about the antiadjacency matrix of inverse graph of $\mathbb{Z}_n$. They found several spectra of inverse graph of $\mathbb{Z}_n$ for few n. Oktradifa et al. [13] investigated the eigenvalues of antiadjacency matrix of directed unicyclic helm graph.

There are not many research on the eigenvalues of the antiadjacency matrix of Cayley graphs of groups. Thus, in this research, we are interested to explore the properties of the eigenvalues of the antiadjacency matrix of Cayley graphs.

2. Preliminary Results

Matrix C is said to be a circulant matrix if C is a square matrix of size $n \times n$ which can be represented as follows

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix} \quad (1)$$

[8].

Below we present some well known results about circulant matrix.

Lemma 2.1. [2] [16] *Adjacency matrix of Cay$(\mathbb{Z}_n, S)$ is a circulant matrix.*

Theorem 2.1. [8] *The eigenvalues of circulant matrix* C *given in Equation 1 are*

$$\lambda_t = \sum_{j=0}^{n-1} c_j \xi^{tj},$$

where $\xi = \exp(2\pi i/n)$ *and* $t = 0, 1, 2, ..., n-1$.

We make the following observation now, which we will apply frequently in the next section.

Observation 2.1. If $\xi = \exp(\frac{2\pi i}{n})$, then

$$\sum_{j=0}^{n-1} \xi^{tj} = \begin{cases} n, & \text{if } t = 0 \mod n, \\ 0, & \text{if } t = 1 \mod n, 2 \mod n, ..., (n-1) \mod n \end{cases}$$

Proof. For $t = 0 \mod n$, we can write t as kn with $k \in \mathbb{Z}$ and so

$$\sum_{j=0}^{n-1} \xi^{kn\cdot j} = \sum_{j=0}^{n-1} \exp(\frac{2\pi i}{n} knj) = \sum_{j=0}^{n-1} 1^{kj} = n.$$

For other values of t, $t = kn + c$, with k an integer and $c \in \{1, ..., n-1\}$. Observe that $\xi^{kn+c} = \xi^c$ and so $\sum_{j=0}^{n-1} \xi^{tj}$ is a geometric series with ratio ξ^c, so

$$\begin{aligned} \sum_{j=0}^{n-1} \xi^{tj} &= \frac{\xi^{t\cdot 0}(1-\xi^{cn})}{1-\xi^c} \\ &= \frac{1(1-\exp(\frac{2\pi i}{n} cn))}{1-\xi^c} \\ &= 0. \end{aligned}$$

□

3. Main Results

In this section, we present our results about the eigenvalues of the antiadjacency matrix of Cay$(\mathbb{Z}_n, S)$. Not only that, we also present the relation between eigenvalues of the antiadjacency matrix of Cay$(\mathbb{Z}_n, S)$ and eigenvalues of two other matrix representations of Cayley graph of $\mathbb{Z}_n$, which are the antiadjacency matrix of Cay$(\mathbb{Z}_n, (\mathbb{Z}_n - \{0\}) - S)$ and the adjacency matrix of Cay$(\mathbb{Z}_n, S)$.

Before we go into the results, we introduce the following simple lemma, which is very essential for our proof in the next two theorems.

Lemma 3.1. *Antiadjacency matrix of Cay*$(\mathbb{Z}_n, S)$ *is a circulant matrix.*

Proof. By Lemma 2.1, the adjacency matrix A of Cay$(\mathbb{Z}_n, S)$ is a circulant matrix. Hence, matrix A can be written in the following form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \dots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{pmatrix}.$$

By the definition of antiadjacency matrix, we have B can be written as

$$B = J - A = \begin{pmatrix} 1-a_0 & 1-a_1 & 1-a_2 & \dots & 1-a_{n-1} \\ 1-a_{n-1} & 1-a_0 & 1-a_1 & \dots & 1-a_{n-2} \\ 1-a_{n-2} & 1-a_{n-1} & 1-a_0 & \dots & 1-a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-a_1 & 1-a_2 & 1-a_3 & \dots & 1-a_0 \end{pmatrix}.$$

Rewrite $1 - a_i$ as b_i then

$$B = \begin{pmatrix} b_0 & b_1 & b_2 & \dots & b_{n-1} \\ b_{n-1} & b_0 & b_1 & \dots & b_{n-2} \\ b_{n-2} & b_{n-1} & b_0 & \dots & b_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_1 & b_2 & b_3 & \dots & b_0 \end{pmatrix}.$$

Conclusively, B is a circulant matrix. □

Now, with the aid of Lemma 3.1, we shall present our first theorem which gives the connection between the eigenvalues of the adjacency matrix of Cay$(\mathbb{Z}_n, S)$ and the eigenvalues of the antiadjacency matrix of Cay$(\mathbb{Z}_n, S)$.

Theorem 3.1. *If the spectrum of the adjacency matrix A of Cay$(\mathbb{Z}_n, S)$ is*

$$\mathit{Spec}(A) = \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \dots & \lambda_k \\ m_0 & m_1 & m_2 & \dots & m_k \end{pmatrix},$$

then the spectrum of the antiadjacency matrix B of Cay$(\mathbb{Z}_n, S)$ is

$$\mathit{Spec}(B) = \begin{pmatrix} n-\lambda_0 & -\lambda_k & -\lambda_{k-1} & \dots & -\lambda_1 & -\lambda_0 \\ 1 & m_k & m_{k-1} & \dots & m_1 & m_0 - 1 \end{pmatrix}.$$

Proof. From Corollary 2.1, matrix A is a circulant matrix, so it can be written as

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \dots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{pmatrix}.$$

Thus, from Theorem 2.1, the eigenvalues of A are

$$\lambda_t = \sum_{j=0}^{n-1} a_j \xi^{tj},$$

with $\xi = \exp(\frac{2\pi i}{n})$ and $t = 0, 1, 2, .., n-1$.

From the definition of antiadjacency matrix B of Cay$(\mathbb{Z}_n, S)$, matrix B can be written as

$$B = J - A = \begin{pmatrix} 1-a_0 & 1-a_1 & ... & 1-a_{n-1} \\ 1-a_{n-1} & 1-a_0 & ... & 1-a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1-a_1 & 1-a_2 & ... & 1-a_0 \end{pmatrix}.$$

Since matrix B is circulant by Corollary 3.1, then by Theorem 2.1, the eigenvalues of B are

$$\lambda'_t = \sum_{j=0}^{n-1}(1-a_j)\xi^{tj} = \sum_{j=0}^{n-1}\xi^{tj} - \sum_{j=0}^{n-1} a_j\xi^{tj}$$

Therefore, we have

$$\lambda'_t = \sum_{j=0}^{n-1}\xi^{tj} - \lambda'_t, \tag{2}$$

with $\xi = \exp(\frac{2\pi i}{n})$ and $t = 0, 1, 2, \ldots, n-1$. For $t = 0$, by using Lemma 2.1, Equation 2 becomes

$$\lambda'_0 = \sum_{j=0}^{n-1}\xi^{0\cdot j} - \lambda_0 = n - \lambda_0.$$

On the other hand, for nonzero t, by Lemma 2.1, Equation 2 becomes

$$\lambda'_t = \sum_{j=0}^{n-1}\xi^{t\cdot j} - \lambda_t = 0 - \lambda_t = -\lambda_t.$$

Hence, we obtain the following:

$$\lambda'_t = \begin{cases} -\lambda_t, & \text{for } t = 1, 2, ..., n-1, \\ n - \lambda_t, & \text{for } t = 0. \end{cases}$$

Therefore, an eigenvalue λ_0 of matrix A corresponds to an eigenvalue $n - \lambda_0$ of matrix B. For other $n-1$ eigenvalues of matrix A, λ_i, B has $-\lambda_i$ as its eigenvalues. Thus,

$$\text{Spec}(B) = \begin{pmatrix} n-\lambda_0 & -\lambda_k & -\lambda_{k-1} & ... & -\lambda_1 & -\lambda_0 \\ 1 & m_k & m_{k-1} & ... & m_1 & m_0 - 1 \end{pmatrix}.$$

□

We present our another main result, which describes the relation between the eigenvalues of the antiadjacency matrix of Cay$(\mathbb{Z}_n, S)$ and Cay$(\mathbb{Z}_n, (\mathbb{Z}_n - \{0\}) - S)$.

Theorem 3.2. *Let S be an inverse-closed subset of $\mathbb{Z}_n - \{0\}$. If the spectrum of the antiadjacency matrix B of Cay$(\mathbb{Z}_n, S)$ is*

$$Spec(B) = \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & ... & \lambda_k \\ m_0 & m_1 & m_2 & ... & m_t \end{pmatrix},$$

then the spectrum of the antiadjacency matrix B' of Cay$(\mathbb{Z}_n, (\mathbb{Z}_n - \{0\}) - S)$ is

$$Spec(B') = \begin{pmatrix} n+1-\lambda_0 & 1-\lambda_k & 1-\lambda_{k-1} & ... & 1-\lambda_1 & 1-\lambda_0 \\ 1 & m_k & m_{k-1} & ... & m_1 & m_0 - 1 \end{pmatrix}.$$

Proof. Let A be the adjacency matrix of Cay$(\mathbb{Z}_n, S)$ where

$$A = \begin{pmatrix} a_0 & a_1 & ... & a_{n-1} \\ a_{n-1} & a_0 & ... & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & ... & a_0 \end{pmatrix}.$$

By the same method as in proof of Theorem 3.1, we can assume that the eigenvalues of B are

$$\lambda_t' = \sum_{j=0}^{n-1} (1 - a_j)\xi^{tj},$$

where $t = 0, 1, \ldots n-1$.

Let A' be the adjacency matrix of Cay$(\mathbb{Z}_n, (\mathbb{Z}_n - \{0\}) - S)$. Observe that $0 \notin (\mathbb{Z}_n - \{0\}) - S$ and $0 \notin S$, thus all entries in the main diagonal of both A' and A are 0. For $i = 1, 2, ..., n-1$, note that $i \in S$ if and only if $i \notin (\mathbb{Z}_n - \{0\}) - S$. Consequently, the adjacency matrix of Cay$(\mathbb{Z}_n, (\mathbb{Z}_n - \{0\}) - S)$ is

$$A' = \begin{pmatrix} a_0 & 1-a_1 & ... & 1-a_{n-1} \\ 1-a_{n-1} & a_0 & ... & 1-a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1-a_1 & 1-a_2 & ... & a_0 \end{pmatrix},$$

so the antiadjacency matrix of matriks Cay$(\mathbb{Z}_n, (\mathbb{Z}_n - \{0\}) - S)$ is

$$B' = J - A' = \begin{pmatrix} 1 & 1 & ... & 1 \\ 1 & 1 & ... & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & ... & 1 \end{pmatrix} - \begin{pmatrix} a_0 & 1-a_1 & ... & 1-a_{n-1} \\ 1-a_{n-1} & a_0 & ... & 1-a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1-a_1 & 1-a_2 & ... & a_0 \end{pmatrix}.$$

Thus,

$$B' = \begin{pmatrix} 1-a_0 & a_1 & ... & a_{n-1} \\ a_{n-1} & 1-a_0 & ... & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & ... & 1-a_0 \end{pmatrix}.$$

From Lemma 3.1, B' is also a circulant matrix. By Theorem 2.1, the eigenvalues of B' are

$$\lambda_t = (1-a_0)\xi^{t\cdot 0} + \sum_{j=1}^{n-1} a_j \xi^{tj} = (1-a_0) + \sum_{j=1}^{n-1} a_j \xi^{tj}$$

for $t = 0, 1, 2, \ldots, n-1$ and $\xi = \exp\left(\frac{2\pi i}{n}\right)$. Then,

$$\begin{aligned} \lambda'_t &= (1-a_0) + \sum_{j=1}^{n-1}(a_j - 1)\xi^{tj} + \sum_{j=1}^{n-1}\xi^{tj} \\ &= (1-a_0) + (1-a_0)\xi^{t\cdot 0} + \sum_{j=0}^{n-1}(a_j - 1)\xi^{tj} + \sum_{j=1}^{n-1}\xi^{tj}. \end{aligned}$$

Since a_0 is always zero, we have

$$\lambda'_t = 2 - \lambda_t + \left(\sum_{j=0}^{n-1}\xi^{tj} - \xi^{t\cdot 0}\right). \tag{3}$$

Now, we divide the proof in two cases for $t = 0$ and $t \neq 0$. For $t = 0$, from Equation 3 and Lemma 2.1, we have

$$\lambda'_0 = 2 - \lambda_0 + (n-1) = n + 1 - \lambda_0.$$

For $t \neq 0$, by Equation 3 and Lemma 2.1, we have

$$\lambda'_t = 2 - \lambda_t + (0-1) = 1 - \lambda_t.$$

Conclusively,

$$\lambda'_t = \begin{cases} 1 - \lambda_t, & \text{for } t = 1, 2, ..., n-1, \\ n + 1 - \lambda_t, & \text{for } t = 0. \end{cases}$$

Therefore, an eigenvalue λ_0 of matrix B corresponds to an eigenvalue $n+1-\lambda_0$ of matrix B'. For $n-1$ other eigenvalues λ_i of B, B' has $1-\lambda_i$ as its eigenvalue, so we obtain

$$\text{Spec}(B') = \begin{pmatrix} n+1-\lambda_0 & 1-\lambda_k & 1-\lambda_{k-1} & ... & 1-\lambda_1 & 1-\lambda_0 \\ 1 & m_k & m_{k-1} & ... & m_1 & m_0 - 1 \end{pmatrix}.$$

$\square$

By using Theorem 1.1 and Theorem 3.1, we can easily give the necessary and sufficient condition such that the eigenvalues of the antiadjacency matrix of Cay$(\mathbb{Z}_n, S)$ are integer.

Corollary 3.1. *All the eigenvalues of the antiadjacency matrix of Cay$(\mathbb{Z}_n, S)$ are integers if and only if S is the union of some $S_n(d)$.*

As another application of Theorem 3.1, we give the following result regarding the eigenvalues of the antiadjacency matrix of Cay$(\mathbb{Z}_n, S)$ for the connection set $S \subseteq \mathbb{Z}_n - \{0\}$ such that $S \cup \{0\}$ is a subgroup of $\mathbb{Z}_n$.

Corollary 3.2. *Let S be a subset of $\mathbb{Z}_n - \{0\}$ such that $S \cup \{0\}$ is a subgroup of $(\mathbb{Z}_n, +_{modn})$. The spectrum of antiadjacency matrix B of Cay$(\mathbb{Z}_n, S)$ is*

$$Spec(B) = \begin{pmatrix} n-|S| & 1 & -|S| \\ 1 & n-\frac{n}{|S|+1} & \frac{n}{|S|+1}-1 \end{pmatrix}. \tag{4}$$

Proof. Let m be the generating element of $S \cup \{0\}$, where $km = n$, then $S = \{m, 2m, ..., (k-1)m\}$. Let A be the adjacency matrix of Cay$(\mathbb{Z}_n, S)$ where

$$A = \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_{n-1} & a_0 & \dots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \dots & a_0 \end{pmatrix}.$$

By Lemma 2.1, A is a circulant matrix. Moreover, $a_i = 1$ if $i \in S$ and $a_i = 0$ otherwise. So, by Theorem 2.1, the eigenvalues of A is

$$\begin{aligned} \lambda_t &= \sum_{j=0}^{n-1} a_j \exp\left(\frac{2\pi i}{n} tj\right) \\ &= \sum_{l=1}^{k-1} \exp\left(\frac{2\pi i}{km} tlm\right) \\ &= \sum_{l=0}^{k-1} \exp\left(\frac{2\pi i}{k} tl\right) - 1 \end{aligned}$$

By Lemma 2.1, $\lambda_t = k-1$ if $t = 0, k, 2k, ..., (m-1)k$ and $\lambda_t = -1$ for other values of t. By Theorem 3.1, the spectrum of the antiadjacency matrix B Cay$(\mathbb{Z}_n, S)$ is

$$\text{Spec}(B) = \begin{pmatrix} n-k+1 & 1 & 1-k \\ 1 & n-m & m-1 \end{pmatrix}$$

Since $n = km$ and $|S| = k-1$, then the spectrum above can be written as

$$\text{Spec}(B) = \begin{pmatrix} n-|S| & 1 & -|S| \\ 1 & n-\frac{n}{|S|+1} & \frac{n}{|S|+1}-1 \end{pmatrix}.$$

□

We close this section by giving an example of how to generally find the eigenvalues of the antiadjacency matrix of Cay$(\mathbb{Z}_n, S)$.

Example 3.1. Let n be a positive integer greater than 1. We illustrate how Theorem 3.1 is applied to find the eigenvalues of the antiadjacency matrix of Cayley graphs of $\mathbb{Z}_n$. In this example, we only demonstrate it for the case of Cay$(\mathbb{Z}_{2n}, \{1, n, 2n-1\})$. Refer to Figure 1 for the case $n = 4$.

Consider Cay$(\mathbb{Z}_{2n}, S)$, where $S = \{1, n, 2n-1\}$. Thus, the adjacency matrix A of Cay$(\mathbb{Z}_{2n}, S)$ is

$$\begin{array}{cc}
\begin{matrix} 0 & 1 & 2 & \dots & n & \dots & 2n-2 & 2n-1 \end{matrix} & \\
\begin{pmatrix}
0 & 1 & 0 & \dots & 1 & \dots & 0 & 1 \\
1 & 0 & 1 & \dots & 0 & \dots & 0 & 0 \\
0 & 1 & 0 & \dots & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & \dots & 0 & 1 \\
1 & 0 & 0 & \dots & 0 & \dots & 1 & 0
\end{pmatrix} &
\begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ 2n-2 \\ 2n-1 \end{matrix}
\end{array}.$$

Since the adjacency matrix of Cay$(\mathbb{Z}_{2n}, S)$ is circulant, by Theorem 2.1, the eigenvalues are

$$\lambda_t = \xi^{t\cdot 1} + \xi^{t\cdot n} + \xi^{t\cdot(2n-1)}$$

with $\xi = \exp\left(\frac{\pi i}{n}\right)$ and $t = 0, 1, ..., 2n-1$. Then,

$$\begin{aligned}
\lambda_t &= \exp\left(\frac{\pi i t}{n}\right) + \exp\left(\frac{\pi i t n}{n}\right) + \exp\left(\frac{\pi i t(2n-1)}{n}\right) \\
&= \exp\left(\frac{\pi i t}{n}\right) + \exp\left(\pi i t\right) + \exp\left(-\frac{\pi i t}{n}\right) \\
&= 2\cos\left(\frac{\pi t}{n}\right) + \cos(\pi t)
\end{aligned}$$

Hence, the eigenvalues of A are

$$\lambda_t = \begin{cases} 3, & \text{if } t = 0 \\ 2\cos\left(\frac{\pi t}{n}\right) + \cos(\pi t) & \text{if } t = 1, 2, ..., 2n-1. \end{cases}$$

Hence, by Theorem 3.1, the eigenvalues of the antiadjacency matrix B are

$$\lambda_t' = \begin{cases} n-3, & \text{if } t = 0 \\ -\left(2\cos\left(\frac{\pi t}{n}\right) + \cos(\pi t)\right) & \text{if } t = 1, 2, ..., 2n-1. \end{cases}$$

Remark 3.1. By following the same step in the above example, we can find the formula of the eigenvalues of the antiadjacency matrix of Cay$(\mathbb{Z}_n, S)$ for any generating set S.

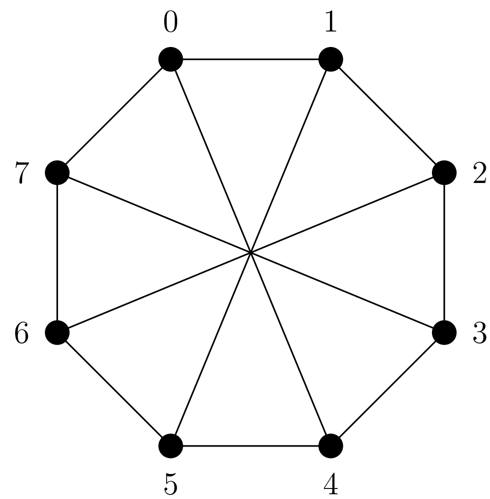

Figure 1. Graph Cay$(\mathbb{Z}_8, \{1, 4, 7\})$

4. Conclusion

In this paper, we examined the relation between eigenvalues of the antiadjacency matrix of Cay$(\mathbb{Z}_n, S)$ and Cay$(\mathbb{Z}_n, (\mathbb{Z}_n - \{0\}) - S)$. We also examined the relation between eigenvalues of the antiadjacency matrix of Cay$(\mathbb{Z}_n, S)$ and eigenvalues of the adjacency matrix of Cay$(\mathbb{Z}_n, S)$. Those results were mainly obtained by using the property of circulant matrix. We also obtained that all of the eigenvalues of the antiadjacency matrix of Cay$(\mathbb{Z}_n, S)$ are integer if and only if S is the union of some $S_n(d) = \{a : 0 < a \leq n, \text{gcd}(a, n) = d\}$.

Open Problem: What are the sufficient and necessary conditions for all the eigenvalues of the antiadjacency matrix of Cayley graphs of other groups, such as dihedral group and permutation group, to be an integer?

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