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The local metric dimension of split and unicyclic graphs

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Abstract

A set W is called a *local resolving set* of G if the distance of u and v to some elements of W are distinct for every two adjacent vertices u, v in G. The *local metric dimension* of G is the minimum cardinality of a local resolving set of G. A connected graph G is called a *split graph* if $V(G)$ can be partitioned into two subsets V_1 and V_2 where an induced subgraph of G by V_1 and V_2 is a complete graph and an independent set, respectively. We also consider a graph, namely the *unicyclic graph* which is a connected graph containing exactly one cycle. In this paper, we provide a general sharp bounds of local metric dimension of split graph. We also determine an exact value of local metric dimension of any unicyclic graphs.

Keywords: local basis, local metric dimension, local resolving set, split graph, unicyclic graph
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1. Introduction

All graphs in this paper are finite, simple, and connected. Let $G = (V(G), E(G))$ be a graph with $V(G)$ and $E(G)$ are vertex and edge set of G, respectively. The *distance* between two vertices u and v in a graph G is the length of a shortest path from u to v in G, denoted by $d(u, v)$. Let

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$W = \{w_1, w_2, ..., w_k\}$ be a subset of $V(G)$. The *representation* of v with respect to W is defined as the k-tuple $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$. The set W then is called as a *resolving set* of G if $r(u|W) \neq r(v|W)$ for every two distinct vertices u and v in G. A *basis* of G is a resolving set of G with minimum cardinality and we call its cardinality by the *metric dimension* of G, denoted by $dim(G)$.

The metric dimension was first introduced by Slater in 1975 [27] and independently by Harary and Melter in 1976 [12]. Generally, determining the metric dimension of any graphs is an *NP*-complete problem. However, the metric dimension of certain particular graphs have been determined, such as cycles [6], trees [6, 12, 14], wheels [3, 4, 26], fans [4], complete n-partite graphs [6], unicyclic graphs [18], honeycomb networks [16], regular graph [23], Cayley graphs [10], Jahangir graphs [28], and Sierpiński graphs [15]. Moreover, Chartrand et al. [6] have characterized all graphs of order $n \geq 3$ with metric dimension 1, $n-1$, and $n-2$. The metric dimension of graph obtained from a graph operation also has been studied such as Cartesian product graphs [5, 8, 14], join product graphs [3, 4, 26], corona product graphs [13, 29], strong product graphs [21], lexicographic product graphs [25], and comb product graphs [24]. This concept also has an application in many diverse areas, including robotic navigation [7, 14], chemistry [6], strategy in mastermind [11], and network discovery and verification [2].

Another version on metric dimension is the local metric dimension. In this concept, a subset W of $V(G)$ is called as a *local resolving set* of G if $r(u|W) \neq r(v|W)$ for every two adjacent vertices u, v of G. A *local basis* of G is a local resolving set of G with minimum cardinality and we call its cardinality by the *local metric dimension* of G, denoted by $lmd(G)$.

The local metric dimension problems were first studied by Okamoto et al. [17]. They have characterized all graphs of order n with local metric dimension 1, $n-2$, and $n-1$, which can be seen in the following theorem.

Theorem 1.1. *[17]Let G be a connected graph of order $n \geq 2$. Then*

1. $lmd\,(G) = 1$ *if and only if G is bipartite.*
2. $lmd\,(G) = n-1$ *if and only if $G = K_n$.*
3. $lmd\,(G) = n-2$ *if and only if $\omega(G) = n-2$ where $\omega(G)$ is the order of the biggest clique in G.*

Determining a local metric dimensions between a graph obtained by a graph operation with the original graphs is also an interesting problem. Okamoto et al. also have determined the local metric dimension of Cartesian product graphs. The local metric dimension of corona product graphs, rooted product graphs, block graphs, bouquet graphs, and chain of graphs have been investigated by Rodríguez-Velázquez et al. [19, 20]. Meanwhile, some results for certain class of graphs can be seen in [1, 9, 22]

In this paper, we obtain two main results. The first result is related to split graph. We provide sharp lower and upper bound for the local metric dimension of any split graphs. We also give an existence of a split graph whose local metric dimension is in between those bounds. The second result is related to unicyclic graph. In this paper, we determine the local metric dimension of any unicyclic graphs.

2. Split Graph

In this section, we define $Sp(m,n)$ as a split graph where $V_1 = \{a_i | 1 \leq i \leq m\}$ and $V_2 = \{b_j | 1 \leq j \leq n\}$. For $1 \leq i \leq m$, let $A_i = \{b \in V_2 | a_i b \in E(Sp(m,n))\}$.

We obtain the general bound for the local metric dimension of any split graphs, which can be seen in the following theorem.

Theorem 2.1. *For $n, m \in \mathbb{N}$, let $Sp(m,n)$ be a split graph. Then*

$$\lceil \log_2 m \rceil \leq lmd(Sp(m,n)) \leq m.$$

Proof. For the upper bound, let $W = V_1$. So, it is clear that $V(Sp(m,n)) \setminus W = V_2$. If every vertex in V_2 has different representation with respect to W, then W is a local resolving set of $Sp(m,n)$. Otherwise, let b_i and b_j be two distinct vertices in V_2 satisfying $r(b_i|W) = r(b_j|W)$. Note that $b_i b_j \notin E(Sp(m,n))$ for $i \neq j$. It implies that W is still a local resolving set of $Sp(m,n)$. Since $|W| = m$, we obtain $lmd(Sp(m,n)) \leq m$.

Now, suppose that W is a local basis of $Sp(m,n)$ satisfying $|W| \leq \lceil \log_2 m \rceil - 1$. Note that for $m \in \{1,2\}$, we have a contradiction since $\lceil \log_2 m \rceil - 1 = 0$. Now, we assume that $m \geq 3$.

First, we will show that $\lceil \log_2 m \rceil - 1 \leq m - 2$ by mathematical induction. In the other hand, $\lceil \log_2 m \rceil \leq m - 1$. For $m = 3$, it is true that $\lceil \log_2 m \rceil = 2 \leq m - 1$. We assume that $\lceil \log_2 k \rceil \leq k - 1$ for a natural number $k \geq 3$. For $m = k + 1$, we obtain $\lceil \log_2 m \rceil = \lceil \log_2 (k+1) \rceil \leq \lceil \log_2 k \rceil + 1 \leq k - 1 + 1 = k = m - 1$. Therefore, $\lceil \log_2 m \rceil \leq m - 1$ which implies $\lceil \log_2 m \rceil - 1 \leq m - 2$.

Since we have $|W| \leq \lceil \log_2 m \rceil - 1 \leq m - 2$ and $|V_1| = m$, there exist two distinct vertices a_i and a_j in V_1 such that W does not contain $\{a_i, a_j\} \cup A_i \cup A_j$. Note that every vertex $x \in E(Sp(m,n)) \backslash (\{a_i, a_j\} \cup A_i \cup A_j)$ satisfies $d(x, a_i) = d(x, a_j)$. It implies that $r(a_i|W) = r(a_j|W)$. Since $a_i a_j \in E(Sp(m,n))$, it follows that we have a contradiction. □

In the next two theorems, we give an existence of split graph whose local metric dimension satisfies either the lower bound or the upper bound in Theorem 2.1.

Theorem 2.2. *For $n, m \in \mathbb{N}$, there exists a split graph $Sp(m,n)$ where $lmd(Sp(m,n)) = \lceil \log_2 m \rceil$.*

Proof. Let $m \geq 3$ and $n = \lceil \log_2 m \rceil$. Let us consider a split graph $Sp(m,n)$ when the edge set of the split graph is constructed as follows:

1. An induced subgraph of $Sp(m,n)$ by $V_1 = \{a_i | 1 \leq i \leq m\}$ is a complete graph.
2. Let a_i be represented by a binary number of $i - 1$ with length $\lceil \log_2 m \rceil$.
3. If the j-th position of binary number of a_i is 1, then connect a_i to b_j. Otherwise, a_i and b_j are not adjacent.

Now, we will show that the split graph $Sp(m,n)$ defined above has $lmd(Sp(m,n)) = \lceil \log_2 m \rceil = n$. By Theorem 2.1, we only need to show that $lmd(Sp(m,n)) \leq \lceil \log_2 m \rceil$. We define $W = V_2$. Note that $V(Sp(m,n)) \setminus W = V_1$. Since two distinct vertices a_i and a_j have different binary number, there exists a vertex b in V_2 such that $ba_i \in E(Sp(m,n))$ but $ba_j \notin E(Sp(m,n))$. It follows that $r(a_i|W) \neq r(a_j|W)$, which implies W is a local resolving set of $Sp(m,n)$. □

An illustration of the split graph $Sp(m,n)$ as defined on proof of Theorem 2.2 can be seen in figure below. In Figure 1, we have a split graph $Sp(7,3)$. The binary number of every vertex $v \in V_1$ represents the connection of vertex v to some vertices in V_2. However, in figure below, the connection between two distinct vertices in V_1 is not given since it is clear that an induced subgraph of $Sp(7,3)$ by V_1 is a complete graph.

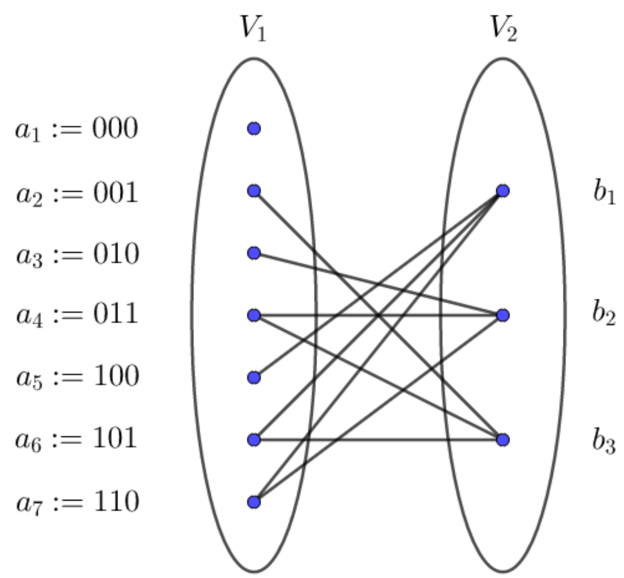

Figure 1. Graph $Sp(7,3)$ as defined on proof of Theorem 2.2

Theorem 2.3. *For $n, m \in \mathbb{N}$, there exists a split graph $Sp(m,n)$ where $lmd(Sp(m,n)) = m$.*

Proof. For $n, m \in \mathbb{N}$, let $Sp(m,n)$ be a split graph where $a_ib_j \in E(Sp(m,n))$ for $1 \le i \le m$ and $1 \le j \le n$. We will show that $lmd(Sp(m,n)) = m$. By Theorem 2.1, we only need to show that $lmd(Sp(m,n)) \ge m$.

Suppose that $lmd(Sp(m,n)) \le m-1$ and W be a local basis of $Sp(m,n)$. We distinguish two cases.

1. $W \subset V_1$
   Since $|V_1| = m$, there exists $a_i \in V_1$ where $i \in \{1,2,\dots,m\}$ such that $a_i \notin W$. We also consider a vertex b_1 of V_2. Note that a_i and b_1 are adjacent to every vertex of $V_1 \setminus \{a_i\} = W$. Therefore, $r(a_i|W) = (1,1,\dots,1) = r(b_1|W)$. Since a_i and b_1 are adjacent in $Sp(m,n)$, we obtain a contradiction.
2. $W \cap V_2 \neq \emptyset$
   Then $|W \cap V_1| \le m-2$. So, there exist two distinct vertices a_i and a_j in V_1 such that W does not contain $\{a_i, a_j\} \cup A_i \cup A_j$. Since a_i and a_j are adjacent to every other vertices in $Sp(m,n)$, we obtain that $r(a_i|W) = (1,1,\dots,1) = r(a_j|W)$, a contradiction.

□

In the theorem below, we give an existence of a split graph whose local metric dimension is in between the lower and upper bound in Theorem 2.1.

Theorem 2.4. *There exist* $c, n, m \in \mathbb{N}$ *with* $m \geq 3$ *and* $c \in \{\lceil \log_2 m \rceil + 1, \lceil \log_2 m \rceil + 2, \ldots, m-1\}$, *such that* $lmd(Sp(m,n)) = c$.

Proof. For $m \geq 4$ and $n \geq 3$, let $Sp(m,n)$ be a split graph where the degree of every vertex in V_2 is 1. We will show that $lmd(Sp(m,n)) = m-1$. Note that $\lceil \log_2 m \rceil < m-1 < m$.

For the lower bound, suppose that $lmd(Sp(m,n)) \leq m-2$ and W is a local basis of $Sp(m,n)$. We define $Q = \{a \in V_1 \mid a \in W$ or $(b \in V_2 \cap W$ and $ab \in E(Sp(m,n)))\}$. So, $|Q| \leq |W| \leq m-2$. Therefore, there exist two distinct vertices a_i and a_j in $V_1 \setminus Q$ such that $r(a_i|Q) = r(a_j|Q)$. It follows that $r(a_i|W) = r(a_j|W)$, a contradiction.

For the upper bound, we define $W = \{a_i | 1 \leq i \leq m-1\}$. Therefore, we obtain $r(a_n|W) = (1, 1, \ldots, 1)$. Now, we consider the representation of vertex b_j of V_2 for $1 \leq j \leq n$. Since b_j is adjacent to only one vertex of V_1, there exists $a \in V_1 \cap W$ such that $ab_j \notin E(Sp(m,n))$. Therefore, we obtain $d(a, b_j) = 2 \neq 1 = d(a, a_n)$ which implies $r(b_j|W) \neq r(a_n|W)$. So, W is a local resolving set of $Sp(m,n)$. □

3. Unicyclic Graph

In this section, let G be an unicyclic graph. Note that, the graph G can be obtained from a tree T by adding an edge $e = xy$ to two non-adjacent vertices $x, y \in V(T)$. Now, let us consider the cycle of G. If the cycle is even, then G is bipartite graph, which implies $lmd(G) = 1$ [17]. In lemma below, we investigate a property if G contains an odd cycle.

Lemma 3.1. *Let G be a unicyclic graph containing odd cycle* C. *For any vertex* $v \in V(G)$, *there exists exactly one pair of adjacent vertices* x *and* y *of G satisfying* $d(x,v) = d(y,v)$. *Moreover,* x *and* y *must be in* C.

Proof. Let G be a unicyclic graph containing odd cycle C_n where $n \geq 3$. Let $V(C_n) = \{c_0, c_1, \ldots, c_{n-1}\}$ with $E(C_n) = \{c_0c_1, c_1c_2, \ldots, c_{n-2}c_{n-1}, c_{n-1}c_0\}$ and $n = 2k+1$ where $k \geq 1$. Let G' is a subgraph of G such that $G' = G \setminus E(C_n)$. So, G' is a disconnected graph containing n components where every component is a tree. For $i \in \{1, 2, \ldots, n\}$, we define T_i as a component of G' containing vertex c_i.

Let v be a vertex of G. So, there exists $i \in \{1, 2, \ldots, n\}$ such that $v \in V(T_i)$. Let x and y be an adjacent vertices in G. If there exists $j \in \{1, 2, \ldots, n\}$ such that $x, y \in V(T_j)$, then we have $d(v, z) = d(v, c_i) + d(c_i, c_j) + d(c_j, z)$ for $z \in \{x, y\}$. Since T_j is a tree and every two distinct vertices in a tree has a unique path between them, we obtain that either $d(c_j, x) < d(c_j, y)$ or $d(c_j, x) > d(c_j, y)$, which implies $d(v, x) \neq d(v, y)$. Therefore, x and y must be from two different components of G'. It follows that x and y must be in C_n.

Now, let x and y be two adjacent vertices in C_n and $v \in V(T_i)$. If $d(c_i, x) < diam(C_n) = k$, then we have either $d(c_i, x) < d(c_i, y)$ or $d(c_i, x) > d(c_i, y)$, which implies $d(v, x) \neq d(v, y)$. So, it must be $d(c_i, x) = k = d(c_i, y)$. Since n is odd, we obtain the two adjacent vertices are $x = c_{i+k}$ and $y = c_{i+k+1}$ where both indexes are on modulo n.

Now, suppose that there are two distinct pairs of adjacent vertices x_1, y_1 and x_2, y_2 of G such that for any vertex $v \in V(G)$, $d(x_1, v) = d(y_1, v)$ and $d(x_2, v) = d(y_2, v)$. By the similar argument above, we will obtain that x_1, y_1 and x_2, y_2 are from different cycle of G. Therefore, G contains at least two cycles, a contradiction. □

Now, we are ready to prove the local metric dimension of unicyclic graph.

Theorem 3.1. *Let G be a unicyclic graph of order at least three. If G contains a cycle with p vertices, then*

$$lmd(G) = \begin{cases} 1, & \text{if } p \text{ is even;} \\ 2, & \text{if } p \text{ is odd.} \end{cases}$$

Proof. We distinguish two cases.

Case 1. p is even.
Then the unicyclic graph G is a bipartite graph. According to [17], we have $lmd(G) = 1$.

Case 2. p is odd.
Then the unicyclic graph G is not bipartite graph. Consequently, $lmd(G) \geq 2$. Now we will prove that $lmd(G) \leq 2$ by construct a local resolving set of G. Let C be a cycle contained in G. By Lemma 3.1, for vertex $v \in V(G)$, there exist exactly one pair of adjacent vertices x and y in C such that $d(x, v) = d(y, v)$. Now, we define $W = \{v, x\}$. Since there is no two adjacent vertices having the same representation with respect to W, we obtain that W is a local resolving set of G. Therefore $lmd(G) \leq 2$.

□

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