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Index graphs of finite permutation groups

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Abstract

Let G be a subgroup of S_n. For $x \in G$, the index of x in G is denoted by $ind\ x$ is the minimal number of 2-cycles needed to express x as a product.

In this paper, we define a new kind of graph on G, namely the index graph and denoted by $\Gamma^{ind}(G)$. Its vertex set the set of all conjugacy classes of G and two distinct vertices $x \in C_x$ and $y \in C_y$ are adjacent if $Gcd(ind\ x, ind\ y) \neq 1$. We study some properties of this graph for the symmetric groups S_n, the alternating group A_n, the cyclic group C_n, the dihedral group D_{2n} and the generalized quaternain group Q_{4n}. In particular, we are interested in the connectedness of them.

Keywords: Finite Permutation Group, Index Graph, Connectedness
Mathematics Subject Classification : 20B40

1. Introduction

Nowadays, graph theory has been proven to be an interesting tool for the study of finite groups. There are a lot of papers associate graphs with finite groups such as [6, 2, 7].

Throughout this paper, we assume that G is a finite permutation subgroup of S_n. A transposition (2-cycle) is a permutation which exchanges two elements and keeps all others fixed. Any permutation in G can be written as a product of transpositions. Also, the product of two even (odd) permutations in G is even. In addition, the product of an even permutation and an odd permutation G is odd. The order of x in G is the smallest $n \in \mathbb{Z}^+$ such that $x^n = e$ and it is denoted by $|x|$ [4].

A component of a graph Γ is a subgraph that is maximal with respect to the property of being connected and it is denoted by $k(\Gamma)$. A graph is connected if $k(\Gamma) = 1$. A vertex x in a connected

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graph Γ is called a cut vertex if $k(\Gamma - x) > k(\Gamma)$. The distance between two vertices x and y denoted by $d(x, y)$ is the length of the shortest $x - y$ path. The eccentricity $e(x)$ of a vertex x of a connected graph Γ is $max_{y \in V(\Gamma)} d(x, y)$. The diameter of a graph Γ, denoted by $diam(\Gamma) = max_{x \in V(\Gamma)} e(x)$. The radius of a graph Γ is $min_{x \in V(\Gamma)} e(x)$ and denoted by $rad(\Gamma)$. The girth of a graph, denoted by $g(\Gamma)$ is the length of the shortest cycle in the graph Γ [8].

The main purpose of this paper is to study the connectedness of this graph. Furthermore, we find the condition that makes this graph connected and then we are going to check which of them are Eulerian and Hamiltonian.

This paper contains three sections. The first section contains an introduction. In section 2, we provide some notation and basic results in group and graph theory. In section 3, we introduce the index graph on all conjugacy classes of a finite permutation group G and show that the index graphs are simple connected for all groups without transposition class. Furthermore, we deduce that Q_{4n},n is even, D_{2n}, n is odd prime and C_n, either n is prime or a product of odd prime numbers are complete graphs.

2. Preliminaries

We begin by setting up some notation and stating a few results which will be used.

Lemma 2.1. *[5]. Let G be a finite group acting on a finite set Ω with $|\Omega| = n$.*

1. $ind\, x = ind\, x^y$ *for all* $y \in G$.
2. $ind\, x \geq ind\, x^j$ *for all integers* j.

It is well known that in group theory conjugate elements are in the same class. So part 1. of Lemma 2.1 allow us to take just one representative element in each class C of G because all elements in the same class have the same indices.

Remark 2.1. *If $x, y \in G$ and $ind\, x = ind\, y$. It may be x, y in the same class or not. For instance in S_4, we have two elements $(12)(34)$ and (123) have the same index, but they are lie in different classes.*

Theorem 2.1. *[8] A connected graph Γ is Eulerian if and only if $deg(v)$ is even for each vertex v in $V(\Gamma)$.*

Corollary 2.1. *[8] If Γ is a simple graph with at least three vertices, and if $deg(v) \geq \frac{n}{2}$ for each vertex v in $V(\Gamma)$, then Γ is Hamiltonian.*

Lemma 2.2. *[8] Let Γ be a graph with vertex set $V(\Gamma) = \{v_1, ..., v_n\}$. Then $\sum_{i=1}^{n} deg(v_i)$ is equal to twice the number of edges.*

Theorem 2.2. *(Bertrand's postulate)[1] If $n \geq 1$, there is at least one prime p such that $n < p \leq 2n$.*

Theorem 2.3. *(Lagrange's Theorem)[4] If G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$.*

The presentation of a generalized quaternain group of order $4n$ is

$$Q_{4n} = \langle x, y | x^{2n} = e, x^n = y^2, yx = x^{-1}y \rangle,$$

the presentation of a cyclic group is

$$C_n = \langle x | x^n = e \rangle$$

and the presentation of D_{2n} is given by

$$D_{2n} = \langle x, y | x^n = e = y^2, yxy^{-1} = x^{-1} \rangle.$$

3. Some properties of index graphs

Throughout this section, we assume that G is a finite permutation group and it acts on a finite set Ω by conjugation. The multi-set of conjugacy classes $C = \{C_1, ..., C_l\}$ of G. The conjugacy classes of a finite group are well known in representation theory. According to Atlas notation, we are going to use the symbol d_jA where d_j is order of x_j of type A and so on [3]. A new type of graph is defined which is called the index graph of G.

Definition 3.1. *The index graph, $\Gamma^{ind}(G)$, is the simple undirected graph whose vertex set is the set C, and two distinct vertices are adjacent if their indices representatives are not co-prime.*

To illustrate our definition, we provide a number of detailed examples as follows:

Example 3.1. *If $G = 3^2 : D_8$, then the index subgraph $\Gamma^{ind}(G)$ present in Figure 1.*

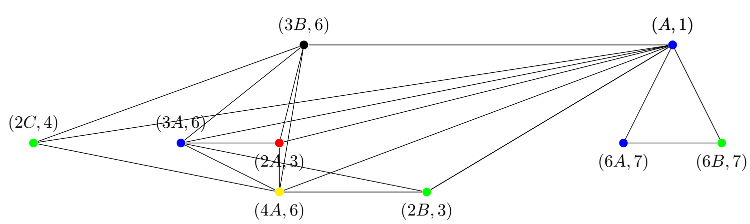

Figure 1. The index graph of $3^2 : D_8$ with the conjugacy class and indices

The first component of the pair is the conjugacy class of G and the second component is the index of the representative element in the conjugacy class. Now we present some results which are related to the index subgraph.

Example 3.2. *Let $G = C_6$ be the cyclic group generated by x. Then we can take $x = (123456), x^2 = (135)(246), x^3 = (14)(25)(36), x^4 = (153)(264)$ and $x^5 = (165432)$. Then $ind\ x = ind\ x^5 = 5, ind\ x^2 = ind\ x^4 = 4$ and $ind\ x^3 = 3$. The index graph $\Gamma^{ind}(G)$ present in Figure 2.*

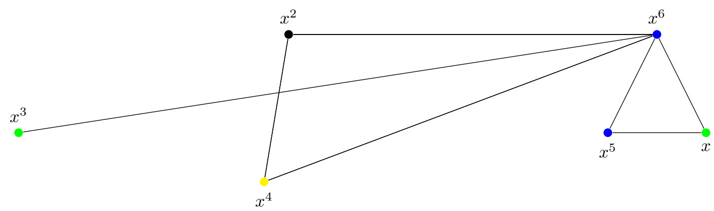

Figure 2. The index graph of C_6

Lemma 3.1. *If C_t is a class of transpositions in G, then its representative isolated vertex in $\Gamma^{ind}(G)$.*

Proof 3.1. *Let C_t be the class of transposition contains x and $ind\, x = 1$. It is clear that $Gcd(ind\, x, ind\, y) = 1$ for all $y \in C_i$ with $i \neq t$. This means that its representative not adjacent to all other vertices. We are done.*

The following useful two results are an immediate consequence of Lemma 3.1.

Corollary 3.1. *Let G be a group acting on a finite set Ω that contains the class of transposition, then $\Gamma^{ind}(G)$ is disconnected.*

Corollary 3.2. *Let G be the symmetric group S_n, where $n \geq 3$. Then $\Gamma^{ind}(G)$ is disconected.*

Example 3.3. *The index graph of $G = D_8$ is disconnected because it contains the class of transposition $C_t = (12)^G$.*

From now on, we assume that G does not contain a class of transposition. Recall that l is the number of conjugacy classes of G.

Lemma 3.2. *In $\Gamma^{ind}(G)$, $deg(\{e\}) = l - 1$.*

Proof. Since G has l conjugacy classes, then the trivial conjugacy class is adjacent to all other conjugacy classes because $Gcd(ind\, e, ind\, x) \neq 1$ for $x \neq e$. □

Lemma 3.3. *If there is a vertex in $\Gamma^{ind}(G)$ of degree one, then $\{e\}$ is a cut vertex.*

Proof. Without loss of generality we assume that the vertex C has degree one. By Lemma 3.2, C is only adjacent to $\{e\}$. So $\Gamma^{ind}(G) \setminus \{e\}$ is disconnected graph. Thus $\{e\}$ is a cut vertex. □

Theorem 3.1. *Let G be a group doesn't include transposition elements. Then $diam(\Gamma^{ind}(G)) = 2$ and $rad(\Gamma^{ind}(G)) = 1$. Furthermore, $\Gamma^{ind}(G)$ is connected and it has girth 3.*

Proof. By Lemma 3.2, we have $deg(\{e\}) = l - 1$. So the distance between any two vertices either 1 or 2. Thus $diam(\Gamma^{ind}(G)) = 2$ and $rad(\Gamma^{ind}(G)) = 1$. The connectedness follows from Lemma 3.2. One can obtain the path between any two vertices. Thus it is connected. Since there are two non identity classes that are adjacent, then we can take these classes with the identity class which give the cycle graph of length 3 and so the girth of $\Gamma^{ind}(G)$ is 3. This completes the proof. □

Proposition 3.1. *The index graph $\Gamma^{ind}(G)$ of a group G is complete graph if and only if G is a subgroup of A_n.*

Proof. To prove that G is a subset of A_n. Let x be an element of order d in G. If x is even permutation, then we are done. If x is odd permutation. Then x^{2k} is even permutation. By Lemma 2.1, we obtain $ind\ x \geq ind\ x^2 \geq ... \geq ind\ x^{d-1}$. By Theorem 2.2, there is a prime number $ind\, x^d \leq p \leq ind\, x$ such that $Gcd(p, ind\, x^i) = 1$ with $p \neq ind\ x^i$. That is x^i is not adjacent to at least one of x^j where $ind\, x^j = p$. Thus $\Gamma_s^{ind}(G)$ is not complete, a contradiction. Therefore x must be even permutation. Conversely, let $x, y \in \Gamma^{ind}(G)$, then $x \in C_x$, $y \in C_y$ and $G = \cup_{x \in G} C_x$. This implies that $x, y \in A_n$. So $gcd(ind\, x, ind\, y) \neq 1$. Hence $\Gamma^{ind}(G)$ is complete graph. □

Remark 3.1. *If H is a subgroup of G, then $\Gamma^{ind}(H)$ may or may not be a subgraph of $\Gamma^{ind}(G)$.*

Example 3.4. *Consider $V_4 = \{(), (12)(34), (13)(24), (14)(23)\}$ and $H = \{(), (123), (132)\}$ which are subgroups of S_4. So $\Gamma^{ind}(V_4) = K_4$, $\Gamma^{ind}(H) = K_3$ and $\Gamma^{ind}(S_4)$ is.*

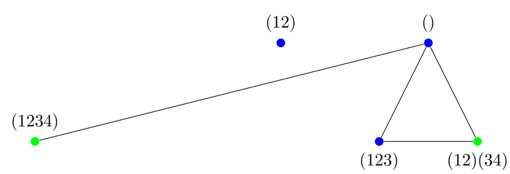

Figure 3. The index graph of S_4

Remark 3.2. *For any two groups G and H. The direct product $G \times H$ and the semi direct product $G \rtimes H$ have the following properties:*

1. $\Gamma^{ind}(G \times H) \ncong \Gamma^{ind}(G) \times \Gamma^{ind}(H)$.
2. $\Gamma^{ind}(G \rtimes H) \ncong \Gamma^{ind}(G) \rtimes \Gamma^{ind}(H)$.
   For instance $G = C_2$ and $H = C_3$.

Let $|\Omega| = n$. Now we begin to count how many of these case give the complete graph for n. Our first case is $n = p$ or $n = p^2$ where p is an odd prime number.

Proposition 3.2. *If $G = D_{2n}$ is acting on Ω, then the index graph of D_{2n} is $K_{\frac{1}{2}(n+3)}$.*

Proof. Since $n = p$ or p^2, where p is an odd prime number, then G has $\frac{1}{2}(n+3)$ conjugacy classes as follows: $\{a^i, a^{-i}\}$ for $i = 1, 2, ..., \frac{n-1}{2}$ and $\{a^j b\}$ for $j = 0, 1, 2, ..., n-1$. So $ind\, a^i = n-1$ for $i = 1, 2, ..., \frac{n-1}{2}$ and $ind a^j b = \frac{n-1}{2}$ for $j = 0, 1, 2, ..., n-1$. Therefore $Gcd(n-1, \frac{n-1}{2}) = \frac{n-1}{2} \neq 1$. Thus the index graph of D_{2n} is $K_{\frac{1}{2}(n+3)}$. □

Corollary 3.3. *If $n = p$ or p^2, where p is an odd prime number, then $\Gamma^{ind}(D_{2n})$ is Hamiltonian and Eularian.*

Table 1. All Small Abelian Groups of Order 16

Group	Degree	Number of vertices	Number of edges	Number of components
D_{12}	6	6	7	1
D_{16}	8	7	11	1
D_{20}	10	8	12	1
D_{24}	12	9	18	1
D_{28}	14	10	19	1

Remark 3.3. *If $\Gamma^{ind}(G) = K_l$, then neither $G \cong D_{2p}$ nor $G \cong D_{2p^2}$. For instance $\Gamma^{ind}(G) = K_{24}$, but $G \cong D_{90}$*

Next, consider for $n \neq p$ and $n \neq p^2$. We present a table on the index graphs of dihedral groups of some orders, as shown in Table 1.

Now we begin to count how many of these case give the complete graph for n. Our first case is n even number.

Proposition 3.3. *Let $|\Omega| = n$ be an even number and $G = Q_{4n}$ acting on Ω. Then the index graph of Q_{4n} is K_{n+3}.*

Proof. Let x be a non trivial element in Q_{4n}. By Theorem 2.3, $|x||4n$ and n is even. Thus $d = |x|$ must be even. On the other hand, $ind\ x = n - \frac{1}{d}\sum_{i=0}^{d-1} f(x^i)$. From this we deduce that $ind\ x = 4n - \frac{1}{d}(4n)$ is even because d is even. It means that the indices of all elements in conjugacy classes of Q_{4n} are even. Thus the greatest common divisor of each pair is not one. This completes the proof. $\square$

Corollary 3.4. *If $G = Q_{4n}$ acting on Ω and n is an even number, then $\Gamma^{ind}(G)$ is Hamiltonian and Eularian.*

Next, consider for n is an odd number.

Proposition 3.4. *If $G = Q_{4n}$ is acting on Ω, where $|\Omega| = 4n$, $n \geq 3$ is an odd number, then*

1. $|V(\Gamma^{ind}(G))| = n + 3$ *and*

$$deg_{\Gamma^{ind}(G)}(x) = \begin{cases} n+2 & if either |x| = 1 \text{ or } |x| = n \\ n & if |x| = 2n \\ s & if |x| = 4 \end{cases} \tag{1}$$

2. *The degree set of $\Gamma^{ind}(G)$ is $\{s, n, n+2\}$ where $s = deg(x)$ and $|x| = 4$.*
3.

$$|E(\Gamma^{ind}(G))| = \begin{cases} \frac{n^2+2n+3}{2} + s & if n_1 > n_2 \\ \frac{n^2+2n-1}{2} + s & if n_1 < n_2 \\ \frac{n^2+2n+1}{2} + s & if n_1 = n_2 \end{cases} \tag{2}$$

Proof. 1. The group Q_{4n} has $n + 3$ conjugacy classes. Thus $|V(\Gamma^{ind}(G))| = n + 3$ and the second part follows from Definition 3.1.

2. It follows from part 1.
3. Now $|V(\Gamma^{ind}(G))| = n+3 = n_1+n_2+2$. This implies that $n_1+n_2 = n+1$ where there are n_1 of order n or 1 and there are n_2 of order $2n$. Also, we have two elements of order 4. If $n_1 = n_2$, then $n_1 = \frac{n+1}{2}$ and by Lemma 2.2, we have $|E(\Gamma^{ind}(G))| = \frac{n^2+2n+1}{2} + s$. If $n_1 > n_2$, then the investegation show that $n_1 - n_2 = 2$. This imples that $n_1 = \frac{n+3}{2}$ and $n_2 = \frac{n-1}{2}$. By Lemma 2.2, we have $|E(\Gamma^{ind}(G))| = \frac{n^2+2n+3}{2} + s$. Similarly, the final case.

□

Proposition 3.5. *Suppose that $G = C_n$ acting on itself where $n = 2^k$ and k is an odd prime number. Then*

$$deg_{\Gamma^{ind}(G)}(x) = \begin{cases} 2^k - 1 & |x| = 1 \\ 2^{k-1} & |x| = 2^k \\ 2^{k-1} - 1 & \textit{otherwise} \end{cases} \tag{3}$$

Proof. It is clear that there are $\phi(n)$ of order 2^k and they are all adjacent together. So the degree each of them is $2^{k-1} - 1$. The identity element is also adjacent to them. Therefore the degree each of them is 2^{k-1}. Also there are $n - \phi(n) - 1$ elements of order 2^l where $0 < l < k$. Then the index of each of them is $2^k(1 - 2^l)$ which is even. Hence the degree each of them is $2^{k-1} - 1$. □

The following result follows from Proposition 3.5.

Corollary 3.5. *Suppose that $G = C_n$ acting on itself where $n = 2^k$ and k is an odd prime number. Then the index graph $\Gamma^{ind}(G)$ has size $\phi(n)^2$.*

Proof. The proof follows from Proposition 3.5 and Lemma 2.2. □

Proposition 3.6. *Suppose that $G = C_n$ where $n = 2p$ acting on itself. Then*

$$deg(x) = \begin{cases} 2p - 1 & |x| = 1 \\ 1 & |x| = 2 \\ p - 1 & \textit{otherwise} \end{cases} \tag{4}$$

Proof. If $n = 2p$, then G has elements of order $1, 2, p, 2p$ in classes C_1, C_2, C_p, C_{2p} with indices $0, p, 2p-2, 2p-1$ respectively. We have $Gcd(2p-1, 2p-2) = 1$ and $Gcd(2p-1, p) = Gcd(p-1, p) = 1$. There are $2p-2$ elements of order p and $2p$. In particular, there are $p-1$ elements of order p which are adjacent among them. These vertices adjacent to the trivial class. So the degree of vertices of order p is $p-1$. Similarly for elements of order $2p$. Hence

$$deg(x) = \begin{cases} 2p - 1 & |x| = 1 \\ 1 & |x| = 2 \\ p - 1 & \text{otherwise} \end{cases} \tag{5}$$

□

Corollary 3.6. *Suppose that $G = C_n$ acting on itself where $n = 2p$. Then the index graph has size $p^2 - p + 1$.*

Proof. The proof follows from Proposition 3.6 and Lemma 2.2. □

Proposition 3.7. *If $G = C_n$ where $n = p_1^{\alpha_1}, ..., p_r^{\alpha_r}$, p_i are odd prime number, then $\Gamma^{ind}(G) = K_n$.*

Proof. Since $G = C_n =< x >$, then $ind\, x = n - 1$ and $ind\, x^{(\prod_{i=1}^{k} p_i^{\alpha_i})} = n - (\prod_{i=k+1}^{r} p_i^{\alpha_i})$ is even number for all $x \in C_i$ where $i = 1, ..., n$. So the greatest common divisor between each pair of the conjugacy classes is a multiple of 2. Thus $\Gamma^{ind}(G) = K_n$. □

Corollary 3.7. *If $n = p_1^{\alpha_1}, ..., p_r^{\alpha_r}$, p_i are odd primes number, then $\Gamma^{ind}(C_n)$ is Hamiltonian and Eularian.*

Proposition 3.8. *If G is an abelian group such that not cyclic and it acts on Ω, where $|\Omega| = n$. Then $\Gamma^{ind}(G)$ is not complete graph.*

Proof. For each $1 \leq i \leq l$, let $x_i \in C_i$. Then $indx_i = m_i \leq n-1$. By Theorem 2.2 there is at least one prime p such that $\frac{m_i}{2} < p \leq m_i$. It is clear that $Gcd(p, s) = 1$ for all s where $\frac{m_i}{2} < s \leq m_i$. Claim that $Gcd(p, k) = 1$ for all $k \leq \frac{m_i}{2}$ and $p \neq k$. By contradiction, assume that there exists k such that $Gcd(p, k) > 1$. This implies that $p|k$, that is $k > p$. We deduce that $\frac{m_i}{2} \geq k > p$ which is a contradiction and the result follows. □

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