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Local Strong Rainbow Connection Number of Corona Product Between Cycle Graphs

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Abstract

A rainbow geodesic is the shortest path between two vertices where all edges are colored differently. An edge coloring in which any pair of vertices with distance up to d, where d is a positive integer that can be connected by a rainbow geodesic is called d-local strong rainbow coloring. The d-local strong rainbow connection number, denoted by $lsrc_d(G)$, is the least number of colors used in d-local strong rainbow coloring. Suppose that G and H are graphs of order m and n, respectively. The corona product of G and H, $G \odot H$, is defined as a graph obtained by taking a copy of G and m copies of H, then connecting every vertex in the i-th copy of H to the i-th vertex of G. In this paper, we will determine the $lsrc_d(C_m \odot C_n)$ for $d = 2$ and $d = 3$.

Keywords: local strong rainbow coloring, local strong rainbow connection number, corona product, cycle graph
Mathematics Subject Classification : 05C75

1. Introduction

Rainbow coloring was first introduced by Chartrand *et al.* in 2008. Let G be a nontrivial connected graph with vertex set $V(G)$ and edge set $E(G)$. Defined a coloring $c : E(G) \to \{1, 2, ..., k\}, k \in \mathbb{N}$, where adjacent edges may be colored the same. A path which has no color repetition is called rainbow path. If G contains a rainbow path for every pair of vertices of G then G is called rainbow connected. The coloring c is called a rainbow coloring of G. The smallest number of colors used to make G rainbow connected is called rainbow connection number, denoted

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by $rc(G)$. A rainbow geodesic is the shortest rainbow path between a pair of vertices. The graph G is said to be strongly rainbow connected if every pair of vertices of G has a rainbow geodesic. The strong rainbow connection number of G, denoted by $src(G)$, is the smallest number of colors used to make G strongly rainbow connected.

In 2022, Septyanto and Sugeng proposed a generalization of rainbow coloring, that is called d-local rainbow coloring. In this coloring, they only consider a path with distance at most d. The d-local rainbow coloring is an edge coloring in which every distinct vertices of G with distance at most d is connected by a rainbow path. The smallest number of colors used in d-local rainbow coloring is called d-local rainbow connection number and denoted by $lrc_d(G)$. Whereas, if every distinct vertices of G with distance at most d is connected by a rainbow geodesic, then it is called the d-local strong rainbow coloring. The smallest number of colors used in such coloring is called d-local strong rainbow connection number, denoted by $lsrc_d(G)$. In this paper, we determined the $lsrc_d$ of $C_m \odot C_n$ graph for $d = 2$ and $d = 3$.

2. Known Results

The following are the definition of several families of graphs.

Definition 2.1 (Chartrand, Lesniak, and Zhang, 2010). *For $n \geq 3$, a cycle graph C_n is a graph of order n and size n whose vertices can be labeled as $v_1, v_2, ..., v_n$ and whose edges are v_1v_n and v_iv_{i+1} for $i \in \{1, 2, ..., n-1\}$.*

Definition 2.2 (Bondy and Murty, 1976). *A wheel graph W_n is a graph obtained from a cycle by adding a new vertex and edges joining the new vertex to every vertex of the cycle. The new edges are called the spokes of the wheel.*

Definition 2.3 (Frucht and Harary, 1970). *Let G and H be a graph with m and n vertices, respectively. The corona $G \odot H$ is a graph obtained by taking one copy of G and m copies of H, and then joining the i-th vertex of G to every vertex in the i-th copy of H by an edge.*

The next theorems show the src and rc of several families of graphs.

Theorem 2.1 (Chartrand *et al.*, 2008). *Let G be a nontrivial connected graph of size m then*
(a) $src(G) = 1$ if and only if G is a complete graph,
(b) $rc(G) = 2$ if and only if $src(G) = 2$,
(c) $rc(G) = m$ if and only if G is a tree.

Theorem 2.2 (Chartrand *et al.*, 2008). *For each integer $n \geq 4$, $rc(C_n) = src(C_n) = \lceil n/2 \rceil$.*

Theorem 2.3 (Chartrand *et al.*, 2008). *For $n \geq 3$, the strong rainbow connection number of the wheel W_n is $src(W_n) = \lceil n/3 \rceil$*

Theorem 2.4 (Maulani *et al.*, 2020). *Rainbow connection number of corona graph $C_m \odot C_n$ is*

$$rc(C_m \odot C_n) = \begin{cases} 4, & \text{for } m = 3, n \geq 3 \\ \lceil m/2 \rceil + 3, & \text{for } m > 3, n \geq 3 \end{cases}$$

The next theorem shows the value of $lrc_d(C_n)$ which is different with $rc(C_n)$ in Theorem 2.2.

Theorem 2.5 (Septyanto and Sugeng, 2022)**.** *If* $n \geq 3$ *and* $d \leq n/2$*, then*

$$lrc_d(C_n) = lsrc_d(C_n) = \left\lceil \frac{n}{\lfloor n/d \rfloor} \right\rceil.$$

3. Main Results

The graph $C_m \odot C_n$ consists of an inner cycle graph C_m and outer cycle subgraphs $C_n^i, i = 1, 2, ..., m$. Let $G = C_m \odot C_n$. The vertex set and edge set of G can be defined as follows:
$V(G) = \{v_1, v_2, ..., v_m\} \cup \{v_{1,1}, v_{1,2}, ..., v_{1,n}\} \cup \{v_{2,1}, v_{2,2}, ..., v_{2,n}\} \cup ... \cup \{v_{m,1}, v_{m,2}, ..., v_{m,n}\}$,
$E(G) = \{v_1v_2, v_2v_3, ..., v_{m-1}v_m, v_mv_1\} \cup \{v_1v_{1,1}, v_1v_{1,2}, ..., v_1v_{1,n}\} \cup \{v_2v_{2,1}, v_2v_{2,2}, ..., v_2v_{2,n}\} \cup ... \cup \{v_mv_{m,1}, v_mv_{m,2}, ..., v_mv_{m,n}\} \cup \{v_{1,1}v_{1,2}, v_{1,2}v_{1,3}, ..., v_{1,n}v_{1,1}\} \cup \{v_{2,1}v_{2,2}, v_{2,2}v_{2,3}, ..., v_{2,n}v_{2,1}\} \cup ... \cup \{v_{m,1}v_{m,2}, v_{m,2}v_{m,3}, ..., v_{m,n}v_{m,1}\}$.

The value of $rc(C_m \odot C_n)$ is known as stated in Theorem 2.4. Thus, we have interest to find the $lsrc_d(C_m \odot C_n)$ that we present in the following two theorems for the cases $d = 2$ and $d = 3$.

Theorem 3.1. *For* $m \geq 3$ *and* $n \geq 3$*, the* d*-local strong rainbow connection number of* $C_m \odot C_n$ *for* $d = 2$ *is*

$$lsrc_2(C_m \odot C_n) = \begin{cases} \lceil n/2 \rceil, & \text{for } m = 3, n \in \{3,4,5\}, \\ \lceil n/3 \rceil + 1, & \text{for } m = 3, n \geq 6, \\ 3, & \text{for } m > 3, n \in \{3,4,5\}, \\ \lceil n/3 \rceil + 2, & \text{for } m > 3, n \geq 6. \end{cases}$$

Proof. We will prove the $lsrc_2$ of $C_m \odot C_n$ in four cases. Take any two vertices of $C_m \odot C_n$. Notice that if $v_{i_1,j} \in V(C_n^{i_1})$ and $v_{i_2,k} \in V(C_n^{i_2})$, where $i_1, i_2 \in \{1, 2, ..., m\}$, $j, k \in \{1, 2, ..., n\}$, $i_1 \neq i_2$, then there will be a $v_{i_1,j} - v_{i_1} - v_{i_2} - v_{i_2,k}$ path of length 3. Thus, we did not consider this case.

Case 1: For $m = 3$ and $n \in \{3, 4, 5\}$

Defined the coloring $c_1 : E(C_3 \odot C_n) \to \{1, ..., \lceil n/2 \rceil\}$ as follows:
For $i \in \{1, 2, 3\}$,

- $c_1(v_1v_2) = c_1(v_2v_3) = c_1(v_3v_1) = 1$,
- $c_1(v_iv_{i,j}) = 2$, where $j \in \{1, 2, ..., n\}$,
- for $n = 3$: $c_1(v_{i,1}v_{i,2}) = c_1(v_{i,2}v_{i,3}) = c_1(v_{i,3}v_{i,1}) = 1$,
- for $n = 4$: $c_1(v_{i,j}v_{i,j+1}) = \begin{cases} 1, & j \in \{1,3\}, \\ 2, & j \in \{2,4\}, \end{cases}$
- for $n = 5$: $c_1(v_{i,j}v_{i,j+1}) = \begin{cases} 1, & j \in \{1,4\}, \\ 2, & j \in \{2,5\}, \\ 3, & j = 3. \end{cases}$

Figure 1 shows an example of the 2-local strong rainbow coloring of $C_3 \odot C_4$ and it can be seen that the $lsrc_2(C_3 \odot C_4) = 2$.

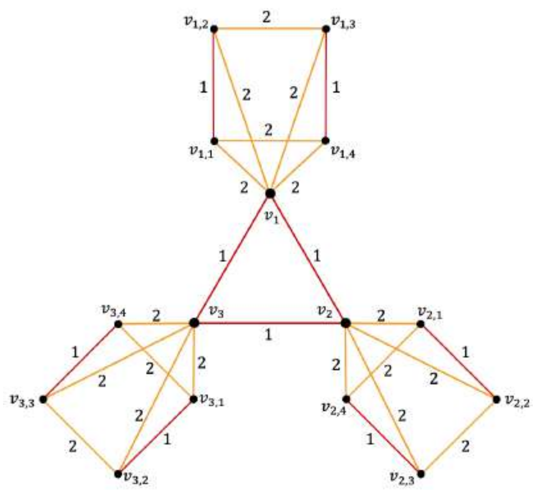

Figure 1. 2-local strong rainbow coloring illustration of $C_3 \odot C_4$

- **Subcase 1.1** If $v_{i_1} \in V(C_3), v_{i_2} \in V(C_3); i_1, i_2 \in \{1, 2, 3\}, i_1 \neq i_2$

  We knew that $C_3 = K_3$ and based on Theorem 2.1, one color is enough to color the edges of C_3.

- **Subcase 1.2** If $v_{i_1} \in V(C_3), v_{i_2,j} \in V(C_n^{i_2}); i_1, i_2 \in \{1, 2, 3\}, j \in \{1, 2, ..., n\}, i_1 \neq i_2$

  We knew that the inner graph is a complete graph, so its vertices are adjacent to each other and there is a path $P : v_{i_1} - v_{i_2} - v_{i_2,j}$, which is a $v_{i_1} - v_{i_2,j}$ geodesic for every j, of length 2. A rainbow geodesic requires that the number of colors used be equal to its length. Because $v_{i_1}v_{i_2} \in E(C_3)$, then the coloring refers to Subcase 1.1. For $v_{i_2}v_{i_2,j}$, one additional color is needed.

- **Subcase 1.3** If $v_{i,j} \in V(C_n^i), v_{i,k} \in V(C_n^i); i \in \{1, 2, 3\}, j, k \in \{1, 2, ..., n\}, j \neq k$

  Let v_i be a vertex of C_3 that is connected to every vertex of the outer cycle C_n^i. Note that $d(v_{i,j}, v_{i,k}) \leq 2$, so the path does not need to go through the vertex v_i. Therefore, colors that were already used for the edges of C_3 and connecting edges $v_iv_{i,j}$ can be used for the edges of C_n^i. For $n \in \{4, 5\}$, the coloring refers to the 2-local strong rainbow coloring of cycle graph. Based on Theorem 2.5, for $n \geq 4$, $lsrc_2(C_n) = \lceil \frac{n}{\lfloor n/2 \rfloor} \rceil$. Hence, for $n \in \{4, 5\}$, $lsrc_2(C_n) = \lceil n/2 \rceil$.

Thus, it is proved that the $lsrc_2(C_3 \odot C_n) = \lceil n/2 \rceil$ for $n \in \{3, 4, 5\}$.

Case 2: For $m = 3$ and $n \geq 6$

Defined the coloring $c_2 : E(C_3 \odot C_n) \to \{1, 2, ..., \lceil n/3 \rceil + 1\}$ as follows:

For $i \in \{1, 2, 3\}$,

- $c_2(v_1v_2) = c_2(v_2v_3) = c_2(v_3v_1) = 1$,
- $c_2(v_iv_{i,j}) = p + 2$, where $j \in \{3p+1, 3p+2, 3p+3\}, 0 \le p \le \lceil n/3 \rceil - 1$,
- $c_2(v_{i,j}v_{i,j+1}) = \begin{cases} 2, & \text{if } j \text{ is odd,} \\ 3, & \text{if } j \text{ is even,} \end{cases}$ where $j \in \{1, 2, ..., n-1\}$,
- $c_2(v_{i,n}v_{i,1}) = \begin{cases} 2, & \text{if } n \text{ is even,} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$

Consider the outer cycle subgraphs as wheel subgraphs with v_i as its vertex center, where $v_i \in \{v_1, v_2, v_3\}$. The 2-local strong rainbow coloring illustration of $C_3 \odot C_n$ can be seen in Figure 2.

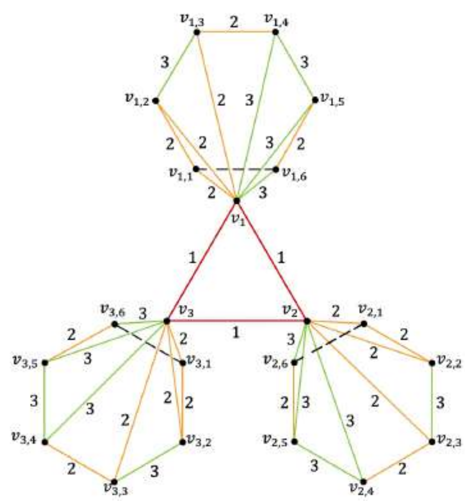

Figure 2. 2-local strong rainbow coloring illustration of $C_3 \odot C_n$

- **Subcase 2.1** If $v_{i_1} \in V(C_3), v_{i_2} \in V(C_3); i_1, i_2 \in \{1, 2, 3\}, i_1 \neq i_2$

  The coloring refers to Subcase 1.1.

- **Subcase 2.2** If $v_{i,j} \in V(C_n^i), v_{i,k} \in V(C_n^i); i \in \{1, 2, 3\}, j, k \in \{1, 2, ..., n\}, j \neq k$

  The geodesic length of a wheel graph is 2. Hence, for a pair of vertices whose length is more than 2, the path must go through its vertex center v_i. Based on Theorem 2.3, then $\lceil n/3 \rceil$ additional colors are needed to color the wheel. The colors used for the spokes of the wheel need to be distinct from the color of edges that are incident with v_i.

- **Subcase 2.3** If $v_{i_1} \in V(C_3), v_{i_2,j} \in V(C_n^{i_2}); i_1, i_2 \in \{1, 2, 3\}, j \in \{1, 2, ..., n\}, i_1 \neq i_2$

  A path $P : v_{i_1} - v_{i_2} - v_{i_2,j}$ is a $v_{i_1} - v_{i_2,j}$ geodesic of length 2. For $v_{i_1}v_{i_2}$, because $v_{i_1}v_{i_2} \in E(C_3)$, then the coloring refers to Case 2.1 and for $v_{i_2}v_{i_2,j}$, the coloring refers to Subcase 2.2.

Thus, it is proved that the $lsrc_2(C_3 \odot C_n) = \lceil n/3 \rceil + 1$ for $n \geq 6$.

Case 3: For $m > 3$ and $n \in \{3, 4, 5\}$

Defined the coloring $c_3 : E(C_m \odot C_n) \to \{1, 2, 3\}$ as follows:

For $i \in \{1, 2, ..., m\}$,

- for m even: $c_3(v_i v_{i+1}) = \begin{cases} 1, & \text{if } i \text{ is odd,} \\ 2, & \text{if } i \text{ is even,} \end{cases}$

- for m odd: $c_3(v_i v_{i+1}) = \begin{cases} 1, & \text{if } i = 1 \text{ and } i \text{ is even } (i \neq 2), \\ 2, & \text{if } i = 2 \text{ and } i \text{ is odd } (i \notin \{1, 3\}), \\ 3, & \text{if } i = 3, \end{cases}$

- for m even: $c_3(v_i v_{i,j}) = 3$, where $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., n\}$,

- for m odd: $c_3(v_i v_{i,j}) = \begin{cases} 3, & \text{if } i \in \{1, 2, 5, 6, 7, 8, ..., m\}, \\ 1, & \text{if } i = 3, \\ 2, & \text{if } i = 4, \end{cases}$ where $j \in \{1, 2, ..., n\}$,

- for $n = 3$: $c_3(v_{i,1} v_{i,2}) = c_3(v_{i,2} v_{i,3}) = c_3(v_{i,3} v_{i,1}) = 1$,

- for $n = 4$: $c_3(v_{i,j} v_{i,j+1}) = \begin{cases} 1, & j \in \{1, 3\}, \\ 2, & j \in \{2, 4\}, \end{cases}$

- for $n = 5$: $c_3(v_{i,j} v_{i,j+1}) = \begin{cases} 1, & j \in \{1, 4\}, \\ 2, & j \in \{2, 5\}, \\ 3, & j = 3. \end{cases}$

From the coloring c_3 above, we got three colors. Now, we divided the proof into subcases.

- **Subcase 3.1** If $v_{i_1} \in V(C_m), v_{i_2} \in V(C_m); i_1, i_2 \in \{1, 2, ..., m\}, i_1 \neq i_2$

  The coloring refer to the 2-local strong rainbow coloring of cycle graph. Based on Theorem 2.5, for $m \geq 4$, $lsrc_2(C_m) = 2$ if m is even and $lsrc_2(C_m) = 3$ if m is odd. If m is even, put color $1, 2, 1, ..., 2$ consecutively on the edges of C_m. If m is odd, put color $1, 2, 3$ on the first three edges of C_m and color $1, 2, 1, ..., 2$ consecutively for the remaining edges.

- **Subcase 3.2** If $v_{i_1} \in V(C_m), v_{i_2,j} \in V(C_n^{i_2}); i_1, i_2 \in \{1, 2, ..., m\}, j \in \{1, 2, ..., n\}, i_1 \neq i_2$

  If v_{i_1} and v_{i_2} are adjacent then there is a path $P : v_{i_1} - v_{i_2} - v_{i_2,j}$, which is a $v_{i_1} - v_{i_2,j}$ geodesic, of length 2. The coloring of $v_{i_1} v_{i_2}$ refers to Subcase 3.1 since it is an edge of C_m. For $v_{i_2} v_{i_2,j}$, put distinct color from the colors of edges incident with v_{i_2}. If v_{i_1} and v_{i_2} are not adjacent, then there is a path $P' : v_{i_1} - ... - v_{i_2} - v_{i_2,j}$ of length more than 2. Hence, we can ignore a pair of vertices that are not adjacent.

- **Subcase 3.3** If $v_{i,j} \in V(C_n^i), v_{i,k} \in V(C_n^i); i \in \{1, 2, ..., m\}, j, k \in \{1, 2, ..., n\}, j \neq k$
  The coloring refers to Subcase 1.3.

Thus, it is proved that the $lsrc_2(C_m \odot C_n) = 3$ for $m > 3$ and $n \in \{3, 4, 5\}$. Figure 3 shows an illustration of the 2-local strong rainbow coloring of $C_m \odot C_4$, where m is even, and it can be seen that the $lsrc_2(C_m \odot C_4) = 3$.

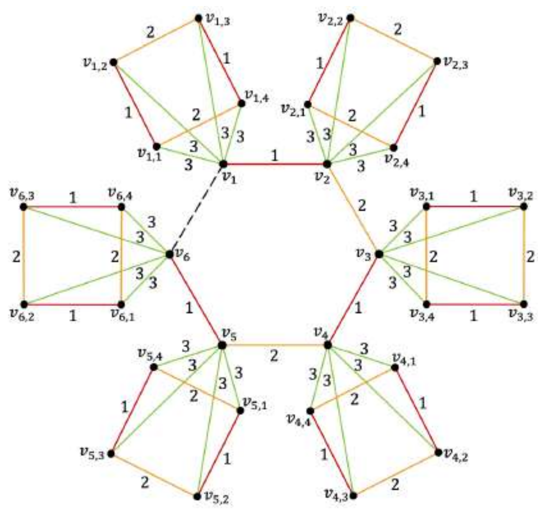

Figure 3. 2-local strong rainbow coloring illustration of $C_m \odot C_4$

Case 4: For $m > 3$ and $n \geq 6$
Defined the coloring $c_4 : E(C_m \odot C_n) \rightarrow \{1, 2, ..., \lceil n/3 \rceil + 2\}$ as follows:
For $i \in \{1, 2, ..., m\}$,

- the coloring of $v_i v_{i+1}$ is the same as $c_3(v_i v_{i+1})$,
- for m even: $c_4(v_i v_{i,j}) = p + 3$ where $j \in \{3p + 1, 3p + 2, 3p + 3\}, 0 \leq p \leq \lceil n/3 \rceil - 1$,
- for m odd:

$$c_4(v_i v_{i,j}) = \begin{cases} p+3, & i \in \{1, 2, 5, ..., m\}, j \in \{3p+1, 3p+2, 3p+3\}, 0 \leq p \leq \lceil \frac{n}{3} \rceil - 1, \\ 1, & i = 3 \text{ and } j \in \{1, 2, 3\}, \\ 2, & i = 4 \text{ and } j \in \{1, 2, 3\}, \\ p+3, & i \in \{3, 4\}, j \in \{3p+1, 3p+2, 3p+3\}, 1 \leq p \leq \lceil \frac{n}{3} \rceil - 1, \end{cases}$$

- for m even: $c_4(v_{i,j} v_{i,j+1}) = \begin{cases} 3, & \text{if } j \text{ is odd,} \\ 4, & \text{if } j \text{ is even,} \end{cases}$ where $j \in \{1, 2, ..., n-1\}$,

- for m odd: $c_4(v_{i,j}v_{i,j+1}) = \begin{cases} 3, & i \neq 3,4; j \text{ odd}, \\ 4, & i \in \{1,2,...,m\}; j \text{ even}, \\ 1, & i = 3; j \text{ odd}, \\ 2, & i = 4; j \text{ odd} \end{cases}$ where $j \in \{1,2,...,n-1\}$.

Consider the outer cycle subgraphs as wheel subgraphs with v_i as its center vertex, where $v_i \in \{v_1, v_2, ..., v_m\}$.

- **Subcase 4.1** If $v_{i_1} \in V(C_m), v_{i_2} \in V(C_m); i_1, i_2 \in \{1,2,...,m\}, i_1 \neq i_2$

  The coloring refers to Subcase 3.1.

- **Subcase 4.2** If $v_{i,j} \in V(C_n^i), v_{i,k} \in V(C_n^i); i \in \{1,2,...,m\}, j,k \in \{1,2,...,n\}, j \neq k$

  The coloring refers to Subcase 2.2.

- **Subcase 4.3** If $v_{i_1} \in V(C_m), v_{i_2,j} \in V(C_n^{i_2}); i_1, i_2 \in \{1,2,...,m\}, j \in \{1,2,...,n\}, i_1 \neq i_2$

  If v_{i_1} and v_{i_2} are adjacent, then there is a path $P : v_{i_1} - v_{i_2} - v_{i_2,j}$ of length 2. The coloring of $v_{i_1}v_{i_2}$ refers to Subcase 4.1 and $v_{i_2}v_{i_2,j}$ refers to Case 4.2. If v_{i_1} and v_{i_2} are not adjacent, then there is a path $P' : v_{i_1} - ... - v_{i_2} - v_{i_2,j}$ of length more than 2. Hence, we can ignore distinct vertices that are not adjacent.

Thus, it is proved that the $lsrc_2(C_m \odot C_n) = \lceil n/3 \rceil + 2$ for $m > 3$ and $n \geq 6$. The 2-local strong rainbow coloring of $C_4 \odot C_n$ is illustrated in Figure 4.

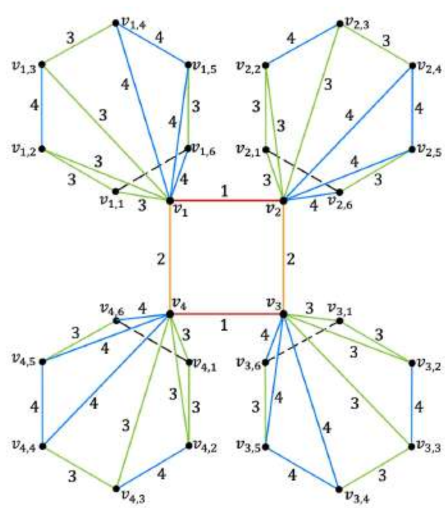

Figure 4. 2-local strong rainbow coloring of $C_4 \odot C_n$

□

Theorem 3.2. *For $m \geq 3$ and $n \geq 3$, the d-local strong rainbow connection number of $C_m \odot C_n$ for $d = 3$ is*

$$lsrc_3(C_m \odot C_n) = \begin{cases} 3, & m = 3, n \in \{3,4,5\}, \\ (\lceil n/3 \rceil \cdot 3) + 1, & m = 3, n \geq 6, \\ \lceil \frac{m}{2} \rceil + 2, & m \in \{4,5\}, n \in \{3,4,5\}, \\ 2 + (\lceil \frac{n}{3} \rceil \cdot \lceil \frac{m}{2} \rceil), & m \in \{4,5\}, n \geq 6, \\ \lceil \frac{m}{\lfloor m/3 \rfloor} \rceil + 2, & m \geq 6 (m \neq 3x), n \in \{3,4,5\}, \\ 6, & m = 3x, n \in \{3,4,5\}, \\ \lceil \frac{m}{\lfloor m/3 \rfloor} \rceil + (\lceil \frac{n}{3} \rceil \cdot \lceil \frac{m}{\lfloor m/2 \rfloor} \rceil), & m \geq 6 \text{ (m even) or } 3|m, n \geq 6, \\ (\lceil n/3 \rceil \cdot 3) + 3, & m \geq 6, 3 \nmid m \text{ (m odd)}, n \geq 6, \end{cases}$$

where x is an odd number larger than 1.

Proof. We will prove the $lsrc_3$ of $C_m \odot C_n$ in eight cases. Take any pair of vertices of $C_m \odot C_n$ with distance at most 3. For $d = 3$, we have to pay attention to geodesic whose length is 2 and 3. For geodesic of length 2, the explanation may refer to Theorem 3.1.

Case 1: For $m = 3$ and $n \in \{3,4,5\}$

Defined the coloring $c_1 : E(C_3 \odot C_n) \rightarrow \{1,2,3\}$ as follows:

For $i \in \{1,2,3\}$

- $c_1(v_1v_2) = 1, c_1(v_2v_3) = 2, c_1(v_3v_1) = 3,$
- $c_1(v_iv_{i,j}) = \begin{cases} 2, & i = 1, \\ 3, & i = 2, \\ 1, & i = 3, \end{cases}$ where $j \in \{1,2,...,n\}$,
- for $n = 3$: $c_1(v_{i,1}v_{i,2}) = c_1(v_{i,2}v_{i,3}) = c_1(v_{i,3}v_{i,1}) = 1,$
- for $n = 4$: $c_1(v_{i,j}v_{i,j+1}) = \begin{cases} 1, & j \in \{1,3\}, \\ 2, & j \in \{2,4\}, \end{cases}$
- for $n = 5$: $c_1(v_{i,j}v_{i,j+1}) = \begin{cases} 1, & j \in \{1,4\}, \\ 2, & j \in \{2,5\}, \\ 3, & j = 3. \end{cases}$

Take a look at Figure 5 below. In this case, we can see that the greatest distance between any two vertices is 3. So, 3 colors are enough to color the edges of the graph. For the edges of the inner cycle C_3, we can put color $1,2,3$, consecutively. Furthermore, it can be seen that the graph $C_3 \odot C_n$ has three outer cycle subgraphs that are connected to each other and $d(v_{i_1,j}, v_{i_2,k}) = 3$, where $i_1, i_2 \in \{1,2,3\}, j,k \in \{1,2,...,n\}, i_1 \neq i_2$. Therefore, each outer cycle subgraph need

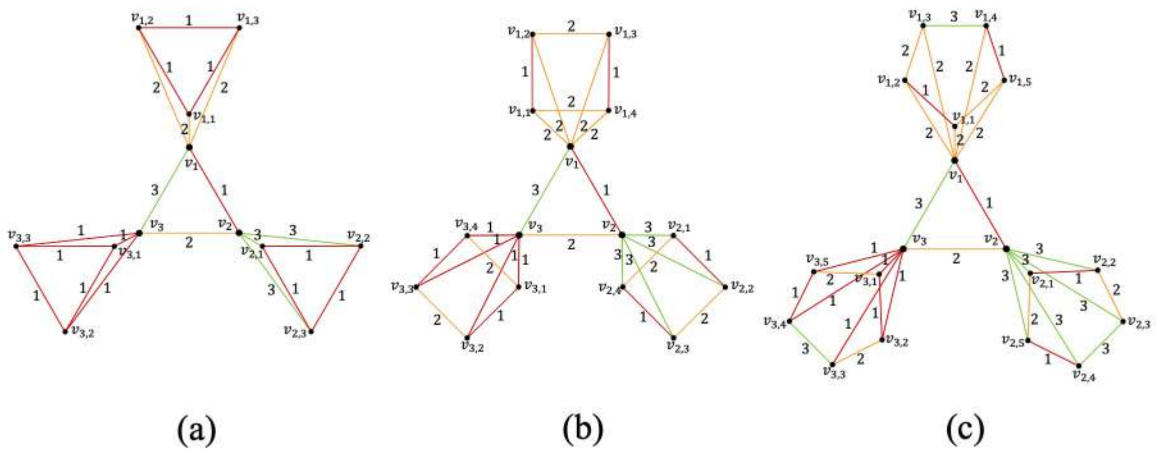

Figure 5. 3-local strong rainbow coloring of (a) $C_3 \odot C_3$, (b) $C_3 \odot C_4$, and (c) $C_3 \odot C_5$

to be colored differently. From Figure 5, it can be seen that the $lsrc_3$ of $C_3 \odot C_3$, $C_3 \odot C_4$, and $C_3 \odot C_5$ is 3. Thus, it is proved that the $lsrc_3(C_3 \odot C_n) = 3$ for $n \in \{3, 4, 5\}$.

Case 2: For $m = 3$ and $n \geq 6$

Defined the coloring $c_2 : E(C_3 \odot C_n) \to \{1, 2, ..., (\lceil n/3 \rceil \cdot 3) + 1\}$ as follows:

- $c_1(v_1v_2) = c_1(v_2v_3) = c_1(v_3v_1) = 1,$
- $c_2(v_iv_{i,j}) = \begin{cases} p+2, & i = 1; j \in \{3p+1, 3p+2, 3p+3\}, 0 \leq p \leq \lceil \frac{n}{3} \rceil - 1, \\ q+1+p_1, & i = 2; j \in \{3q+1, 3q+2, 3q+3\}, 0 \leq q \leq \lceil \frac{n}{3} \rceil - 1, \\ r+1+q_1, & i = 3; j \in \{3r+1, 3r+2, 3r+3\}, 0 \leq r \leq \lceil \frac{n}{3} \rceil - 1, \end{cases}$

  where $p_1 = max(p+2)$ and $q_1 = max(q+1+p_1)$,
- $c_2(v_{i,j}v_{i,j+1}) = \begin{cases} 2, & i = 1; j \text{ odd}, \\ 3, & i = 1; j \text{ even}, \\ 4, & i = 2; j \text{ odd}, \\ 5, & i = 2; j \text{ even}, \\ 6, & i = 3; j \text{ odd}, \\ 7, & i = 3; j \text{ even}., \end{cases}$ where $j \in \{1, 2, ..., n\}$.

Consider the outer cycle subgraphs as wheel subgraphs with v_i as its center vertex, where $v_i \in \{v_1, v_2, v_3\}$.

- **Subcase 2.1** If $v_{i_1} \in V(C_3), v_{i_2} \in V(C_3); i_1, i_2 \in \{1, 2, 3\}, i_1 \neq i_2$

  We knew that $C_3 = K_3$ and based on Theorem 2.1, so the edges of C_3 only need one color.

- **Subcase 2.2** If $v_{i,j} \in V(C_n^i), v_{i,k} \in V(C_n^i); i \in \{1,2,3\}, j,k \in \{1,2,...,n\}, j \neq k$

  Since the geodesic length of a wheel graph is 2, the coloring may refer to Subcase 2.2 on Theorem 3.1.

- **Subcase 2.3** If $v_{i_1,j} \in V(C_n^{i_1}), v_{i_2,k} \in V(C_n^{i_2}); i_1, i_2 \in \{1,2,3\}, j,k \in \{1,2,...,n\}, i_1 \neq i_2$

  Based on Theorem 2.3, we knew that $src(W_n) = \lceil n/3 \rceil$. Because the inner cycle is a graph C_3, then there are three wheel subgraphs that are connected to each other. Hence, $\lceil n/3 \rceil \cdot 3$ colors are needed.

- **Subcase 2.4** If $v_{i_1} \in V(C_3), v_{i_2,j} \in V(C_n^{i_2}); i_1, i_2 \in \{1,2,3\}, j \in \{1,2,...,n\}, i_1 \neq i_2$

  Note that for any v_{i_1} and v_{i_2} in C_3, the geodesic length of $v_{i_1} - v_{i_2} - v_{i_2,j}$ path is 2. The coloring of $v_{i_1}v_{i_2}$ refers to Subcase 2.1 and $v_{i_2}v_{i_2,j}$ refers to Subcase 2.3.

Thus, it is proved that the $lsrc_3(C_3 \odot C_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + 1$ for $n \geq 6$. The 3-local strong rainbow coloring of $C_3 \odot C_n$, $n \geq 6$ is illustrated in Figure 6.

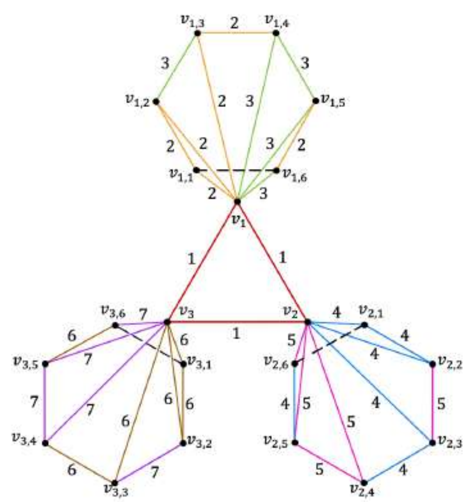

Figure 6. 3-local strong rainbow coloring of $C_3 \odot C_n$

Case 3: For $m \in \{4,5\}$ and $n \in \{3,4,5\}$

Defined the coloring $c_3 : E(C_m \odot C_n) \rightarrow \{1,2,...,\lceil \frac{m}{2} \rceil + 2\}$ as follows:

- The coloring of v_iv_{i+1} is the same as $c_3(v_iv_{i+1})$ on Theorem 3.1,

- for $m = 4$: $c_3(v_iv_{i,j}) = \begin{cases} 3, & i \in \{1,3\}, \\ 4, & i \in \{2,4\}, \end{cases}$ where $j \in \{1,2,...,n\}$,

- for $m = 5$: $c_3(v_iv_{i,j}) = \begin{cases} 3, & i = 1, \\ 4, & i \in \{2,4\}, \\ 5, & i \in \{3,5\}, \end{cases}$

where $j \in \{1, 2, ..., n\}$,

- the coloring of $v_{i,j}v_{i,j+1}$ is the same as $c_3(v_{i,j}v_{i,j+1})$ on Theorem 3.1.

Figure 7 shows the 3-local strong rainbow coloring of $C_5 \odot C_4$. It can be seen that the $lsrc_3(C_5 \odot C_4) = 5$.

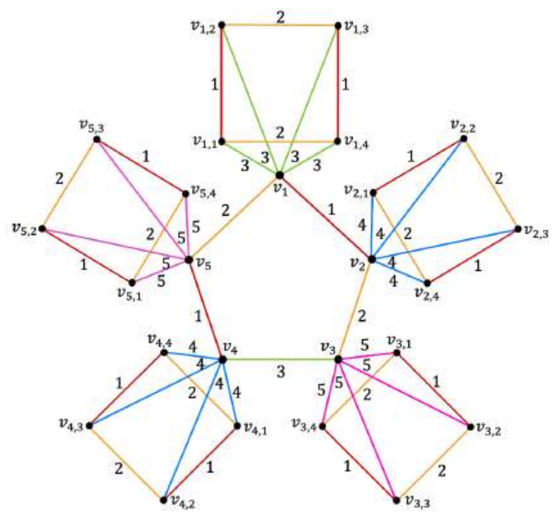

Figure 7. 3-local strong rainbow coloring of $C_5 \odot C_4$

- **Subcase 3.1** If $v_{i_1} \in V(C_m), v_{i_2} \in V(C_m); i_1, i_2 \in \{1, 2, ..., m\}, i_1 \neq i_2$

  Note that the $v_{i_1} - v_{i_2}$ geodesic has a length of 2. Then, the coloring may refer to Subcase 3.1 on Theorem 3.1.

- **Subcase 3.2** If $v_{i_1,j} \in V(C_n^{i_1}), v_{i_2,k} \in V(C_n^{i_2}); i_1, i_2 \in \{1, 2, ..., m\}, j, k \in \{1, 2, ..., n\}, i_1 \neq i_2$

  If v_{i_1} and v_{i_2} are adjacent, the distance between $v_{i_2,j}$ and $v_{i_2,k}$ is 3. Hence, if v_{i_1} and v_{i_2} are adjacent then the connecting edges $v_{i_1}v_{i_1,j}$ and $v_{i_2}v_{i_2,k}$ cannot be given the same color. For $m = 4$, we can put color $3, 4, 3, 4$ consecutively on the edges $v_iv_{i,j}$. For $m = 5$, we can put color 3 on $v_1v_{1,j}$ and $4, 5, 4, 5$ consecutively on the remaining $v_iv_{i,j}$.

- **Subcase 3.3** If $v_{i_1} \in V(C_m), v_{i_2,j} \in V(C_n^{i_2}); i_1, i_2 \in \{1, 2, ..., m\}, j \in \{1, 2, ..., n\}, i_1 \neq i_2$

  If v_{i_1} and v_{i_2} are adjacent, then there is a path $P : v_{i_1} - v_{i_2} - v_{i_2,j}$ of length 2 and if v_{i_1} and v_{i_2} are not adjacent, then there is a path $P' : v_{i_1} - v_i - v_{i_2} - v_{i_2,j}$, which is a $v_{i_1} - v_{i_2,j}$ geodesic for every j, of length 3. The coloring of $v_{i_1}v_{i_2}$ and $v_{i_1}v_iv_{i_2}$ refers to Subcase 3.1 while the coloring of $v_{i_2}v_{i_2,j}$ refers to Subcase 3.2.

- **Subcase 3.4** If $v_{i,j} \in V(C_n^i), v_{i,k} \in V(C_n^i); i \in \{1, 2, ..., m\}, j, k \in \{1, 2, ..., n\}, j \neq k$

  Note that the $v_{i,j} - v_{i,k}$ geodesic has a length of 2. Then, the coloring may refer to Subcase 1.3 on Theorem 3.1.

Thus, it is proved that the $lsrc_3(C_m \odot C_n) = \lceil \frac{m}{2} \rceil + 2$ for $m \in \{4,5\}$ and $n \in \{3,4,5\}$.

Case 4: For $m \in \{4,5\}$ and $n \geq 6$

Defined the coloring $c_4 : E(C_m \odot C_n) \rightarrow \{1, 2, ..., 2 + (\lceil \frac{n}{3} \rceil \cdot \lceil \frac{m}{2} \rceil)\}$ as follows:

- The coloring of v_iv_{i+1} is the same as $c_3(v_iv_{i+1})$ on Theorem 3.1,
- for $m = 4$:
$$c_4(v_iv_{i,j}) = \begin{cases} p+3, & i \in \{1,3\}; j \in \{3p+1, 3p+2, 3p+3\}, 0 \leq p \leq \lceil \frac{n}{3} \rceil - 1, \\ q+1+p_1, & i \in \{2,4\}; j \in \{3q+1, 3q+2, 3q+3\}, 0 \leq q \leq \lceil \frac{n}{3} \rceil - 1, \end{cases}$$
where $p_1 = max(p+3)$,
- for $m = 5$:
$$c_4(v_iv_{i,j}) = \begin{cases} p+3, & i = 1; j \in \{3p+1, 3p+2, 3p+3\}, 0 \leq p \leq \lceil \frac{n}{3} \rceil - 1, \\ q+1+p_1, & i \in \{2,4\}; j \in \{3q+1, 3q+2, 3q+3\}, 0 \leq q \leq \lceil \frac{n}{3} \rceil - 1, \\ r+1+q_1, & i \in \{3,5\}; j \in \{3r+1, 3r+2, 3r+3\}, 0 \leq r \leq \lceil \frac{n}{3} \rceil - 1, \end{cases}$$
where $p_1 = max(p+3)$ and $q_1 = max(q+1+p_1)$,
- for $m = 4$: $c_4(v_{i,j}v_{i,j+1}) = \begin{cases} 3, & i \in \{1,3\}; j \text{ odd}, \\ 4, & i \in \{1,3\}; j \text{ even}, \\ 5, & i \in \{2,4\}; j \text{ odd}, \\ 6, & i \in \{2,4\}; j \text{ even}, \end{cases}$ where $j \in \{1, 2, ..., n\}$
- for $m = 5$: $c_4(v_{i,j}v_{i,j+1}) = \begin{cases} 3, & i = 1; j \text{ odd}, \\ 4, & i = 1; j \text{ even}, \\ 5, & i \in \{2,4\}; j \text{ odd}, \\ 6, & i \in \{2,4\}; j \text{ even}, \\ 7, & i \in \{3,5\}; j \text{ odd}, \\ 8, & i \in \{3,5\}; j \text{ even}, \end{cases}$ where $j \in \{1, 2, ..., n\}$.

Consider the outer cycle subgraphs as wheel subgraphs with v_i as its center vertex, where $v_i \in \{v_1, v_2, ..., v_m\}$.

- **Subcase 4.1** If $v_{i_1} \in V(C_m), v_{i_2} \in V(C_m); i_1, i_2 \in \{1, 2, ..., m\}, i_1 \neq i_2$

  The coloring refers to Subcase 3.1.
- **Subcase 4.2** If $v_{i,j} \in V(C_n^i), v_{i,k} \in V(C_n^i); i \in \{1, 2, ..., m\}, j, k \in \{1, 2, ..., n\}, j \neq k$

  Since the geodesic length of a wheel graph is 2, the coloring may refer to Subcase 2.2 on Theorem 3.1.

- **Subcase 4.3** If $v_{i_1,j} \in V(C_n^{i_1}), v_{i_2,k} \in V(C_n^{i_2}); i_1, i_2 \in \{1, 2, ..., m\}, j, k \in \{1, 2, ..., n\}, i_1 \neq i_2$

  For $m = 4$, because there are an even number of outer cycle subgraphs, then $\lceil n/3 \rceil \cdot 2$ additional colors are needed. For $m = 5$, because there are an odd number of outer cycle subgraphs, then there will be one subgraph that cannot be given those new colors. Then, we can put color 3 on the edges $v_1v_{1,1}, v_1v_{1,2}$, and $v_1v_{1,3}$, while the remaining $v_1v_{1,j}$ are colored with $\lceil n/3 \rceil - 1$ additional colors.

- **Subcase 4.4** If $v_{i_1} \in V(C_m), v_{i_2,j} \in V(C_n^{i_2}); i_1, i_2 \in \{1, 2, ..., m\}, j \in \{1, 2, ..., n\}, i_1 \neq i_2$

  The coloring may refer to Subcase 3.3 but in this case, the coloring of $v_{i_1}v_{i_2}$ and $v_{i_1}v_iv_{i_2}$ refers to Subcase 4.1 while the coloring of $v_{i_2}v_{i_2,j}$ refers to Subcase 4.3.

Thus, it is proved that the $lsrc_3(C_m \odot C_n) = 2 + (\lceil \frac{n}{3} \rceil \cdot \lceil \frac{m}{2} \rceil)$ for $m \in \{4, 5\}$ and $n \geq 6$. The 3-local strong rainbow coloring of $C_5 \odot C_n$ is illustrated in Figure 8.

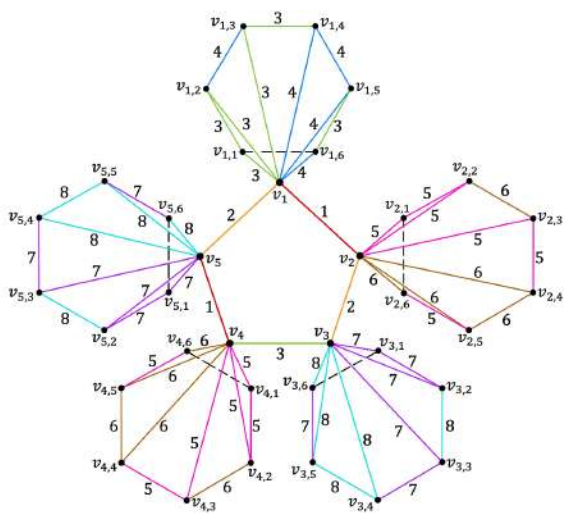

Figure 8. 3-local strong rainbow coloring of $C_5 \odot C_n$

Case 5: For $m \geq 6$ ($m \neq 3x$) and $n \in \{3, 4, 5\}$

Defined the coloring $c_5 : E(C_m \odot C_n) \to \{1, 2, ..., \lceil \frac{m}{\lfloor m/3 \rfloor} \rceil + 2\}$ as follows:

- For $3|m$: $c_5(v_iv_{i+1}) = \begin{cases} 1, & i \equiv 1 \pmod 3, \\ 2, & i \equiv 2 \pmod 3, \\ 3, & i \equiv 0 \pmod 3, \end{cases}$

- for $m \equiv 1 \pmod 3$: $c_5(v_iv_{i+1}) = \begin{cases} 1, & i = 1 \text{ and } i \equiv 2 \pmod 3, \\ 2, & i = 2 \text{ and } i \equiv 0 \pmod 3, \\ 3, & i = 3 \text{ and } i \equiv 1 \pmod 3, \\ 4, & i = 4, \end{cases}$

- for $m \equiv 2 \pmod 3$: $c_5(v_iv_{i+1}) = \begin{cases} 1, & i \in \{1,5\} \text{ and } i \equiv 0 \pmod 3, \\ 2, & i \in \{2,6\} \text{ and } i \equiv 1 \pmod 3, \\ 3, & i \in \{3,7\} \text{ and } i \equiv 2 \pmod 3, \\ 4, & i \in \{4,8\}, \end{cases}$

- for $6|m$: $c_5(v_iv_{i,j}) = \begin{cases} 4, & i \text{ odd}, \\ 5, & i \text{ even}, \end{cases}$ where $j \in \{1, 2, ..., n\}$,

- for $m = 8$: $c_5(v_iv_{i,j}) = \begin{cases} 5, & i \text{ odd}, \\ 6, & i \text{ even}, \end{cases}$ where $j \in \{1, 2, ..., n\}$,

- for other m: $c_5(v_iv_{i,j}) = \begin{cases} 4, & i = 1, \\ 5, & i \text{ even}, \\ 6, & i \text{ odd } (i \neq 1), \end{cases}$ where $j \in \{1, 2, ..., n\}$,

- the coloring of $v_{i,j}v_{i,j+1}$ is the same as $c_3(v_{i,j}v_{i,j+1})$ on Theorem 3.1.

Let x be an odd number greater than 1. Thus, what it meant by $3x$ is $9, 15, 21$ and so on. Figure 9 shows an example of 3-local strong rainbow coloring of $C_6 \odot C_4$ and it can be seen from the figure that the $lsrc_3(C_6 \odot C_4) = 5$.

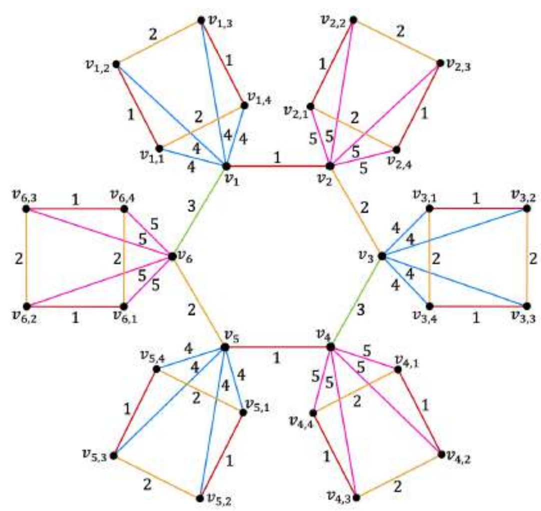

Figure 9. 3-local strong rainbow coloring of $C_6 \odot C_4$

- **Subcase 5.1** If $v_{i_1} \in V(C_m), v_{i_2} \in V(C_m); i_1, i_2 \in \{1, 2, ..., m\}, i_1 \neq i_2$

  Because the graph is a cycle graph, the coloring refers to the 3-local strong rainbow coloring of cycle graph C_m. Based on Theorem 2.5, for $m \geq 6$, $lsrc_3(C_m) = 3$ if $3|m$ and $lsrc_3(C_m) = 4$ for other m.

- **Subcase 5.2** If $v_{i_1,j} \in V(C_n^{i_1}), v_{i_2,k} \in V(C_n^{i_2}); i_1, i_2 \in \{1, 2, ..., m\}, j, k \in \{1, 2, ..., n\}, i_1 \neq i_2$

  We will only look at a pair of vertices that are adjacent since the distance between those vertices is 3. If $6|m$, we can put color $4, 5, 4, ..., 5$ consecutively on the connecting edges $v_{i_1}v_{i_1,j}$ and $v_{i_2}v_{i_2,k}$. If $6 \nmid m$, we can put color 4 on $v_1v_{1,j}$ and the remaining $v_iv_{i,j}$ are colored with 2 additional colors. Specifically for $m = 8$, the connecting edges $v_{i_1}v_{i_1,j}$ and $v_{i_2}v_{i_2,k}$ can be colored with $5, 6, 5, ..., 6$ consecutively.

- **Subcase 5.3** If $v_{i_1} \in V(C_m), v_{i_2,j} \in V(C_n^{i_2}); i_1, i_2 \in \{1, 2, ..., m\}, j \in \{1, 2, ..., n\}, i_1 \neq i_2$

  If v_{i_1} and v_{i_2} are adjacent, then there is a path $P : v_{i_1} - v_{i_2} - v_{i_2,j}$ of length 2. If v_{i_1} and v_{i_2} are not adjacent, then there is a pair of vertices with distance $3 \leq d(v_{i_1}, v_{i_2,j}) \leq \lfloor m/2 \rfloor + 1$. Note that we will only pay attention to a geodesic of length 3. Notice that the path P' : $v_{i_1} - v_i - v_{i_2} - v_{i_2,j}$ is a $v_{i_1} - v_{i_2,j}$ geodesic of length 3. The coloring of $v_{i_1}v_{i_2}$ and $v_{i_1}v_iv_{i_2}$ refers to Subcase 5.1 while $v_{i_2}v_{i_2,j}$ refers to Subcase 5.2.

- **Subase 5.4** If $v_{i,j} \in V(C_n^i), v_{i,k} \in V(C_n^i); i \in \{1, 2, ..., m\}, j, k \in \{1, 2, ..., n\}, j \neq k$

  The coloring refers to Subcase 3.3 on Theorem 3.1.

Thus, it is proved that the $lsrc_3(C_m \odot C_n) = \lceil \frac{m}{\lfloor m/3 \rfloor} \rceil + 2$ for $m \geq 6$ ($m \neq 3x$) and $n \in \{3, 4, 5\}$.

Case 6: For $m = 3x$ and $n \in \{3, 4, 5\}$

Defined the coloring $c_6 : E(C_m \odot C_n) \to \{1, 2, ..., 6\}$ as follows:

- $c_6(v_iv_{i+1}) = \begin{cases} 1, & i \equiv 1 \pmod 3, \\ 2, & i \equiv 2 \pmod 3, \\ 3, & i \equiv 0 \pmod 3, \end{cases}$

- $c_6(v_iv_{i,j}) = \begin{cases} 4, & \text{if } i \text{ is odd}, \\ 5, & \text{if } i \text{ is even}, \\ 6, & i = m, \end{cases}$

  where $j \in \{1, 2, ..., n\}$,

- the coloring of $v_{i,j}v_{i,j+1}$ is the same as $c_3(v_{i,j}v_{i,j+1})$ on Theorem 3.1.

From the coloring c_6 above, we got six colors. Now, we divided the proof into subcases.

- **Subcase 6.1** If $v_{i_1} \in V(C_m), v_{i_2} \in V(C_m); i_1, i_2 \in \{1, 2, ..., m\}, i_1 \neq i_2$

  The coloring refers to Subcase 5.1.

- **Subcase 6.2** If $v_{i_1,j} \in V(C_n^{i_1}), v_{i_2,k} \in V(C_n^{i_2}); i_1, i_2 \in \{1, 2, ..., m\}, j, k \in \{1, 2, ..., n\}, i_1 \neq i_2$

  Since $m = 3x$, there are an odd number of outer cycle subgraphs. Therefore, three additional colors are needed.

- **Subcase 6.3** If $v_{i_1} \in V(C_m), v_{i_2,j} \in V(C_n^{i_2}); i_1, i_2 \in \{1, 2, ..., m\}, j \in \{1, 2, ..., n\}, i_1 \neq i_2$

  The coloring may refer to Subcase 5.3 but in this case, the coloring of $v_{i_2}v_{i_2,j}$ refers to Subcase 6.2.

- **Subcase 6.4** If $v_{i,j} \in V(C_n^i), v_{i,k} \in V(C_n^i); i \in \{1, 2, ..., m\}, j, k \in \{1, 2, ..., n\}, j \neq k$

  The coloring refers to Subcase 5.4.

Thus, it is proved that the $lsrc_3(C_m \odot C_n) = 6$ for $m = 3x$ and $n \in \{3, 4, 5\}$. The 3-local strong rainbow coloring of $C_m \odot C_3$ for $m = 3x$ is illustrated in Figure 10 and it can be seen that the $lsrc_3(C_m \odot C_3) = 6$.

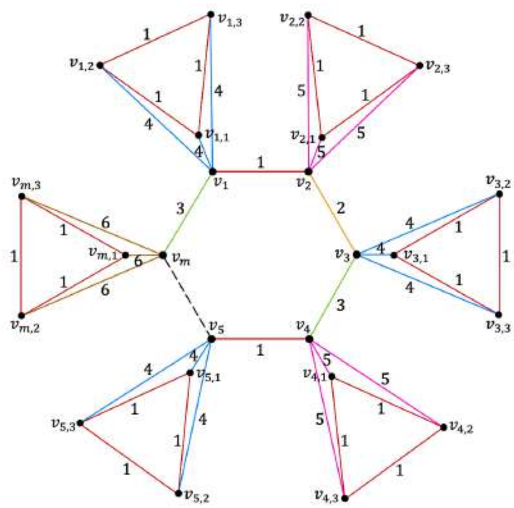

Figure 10. 3-local strong rainbow coloring of $C_m \odot C_3$

Case 7: For $m \geq 6$ (m even) or $3|m$ and $n \geq 6$

Defined the coloring $c_7 : E(C_m \odot C_n) \rightarrow \{1, 2, ..., \lceil \frac{m}{\lfloor m/3 \rfloor} \rceil + (\lceil \frac{n}{3} \rceil \cdot \lceil \frac{m}{\lfloor m/2 \rfloor} \rceil)\}$ as follows:

For $i \in \{1, 2, ..., m\}$,

- the coloring of v_iv_{i+1} is the same as $c_5(v_iv_{i+1})$,
- for $6|m$:

$$c_7(v_iv_{i,j}) = \begin{cases} p + 4, & i \text{ odd}; j \in \{3p+1, 3p+2, 3p+3\}, 0 \leq p \leq \lceil \frac{n}{3} \rceil - 1, \\ q + \lceil \frac{n}{3} \rceil + 4, & i \text{ even}; j \in \{3q+1, 3q+2, 3q+3\}, 0 \leq q \leq \lceil \frac{n}{3} \rceil - 1, \end{cases}$$

- for $m = 8$:

$$c_7(v_iv_{i,j}) = \begin{cases} p + 5, & i \text{ odd}; j \in \{3p+1, 3p+2, 3p+3\}, 0 \leq p \leq \lceil \frac{n}{3} \rceil - 1, \\ q + \lceil \frac{n}{3} \rceil + 5, & i \text{ even}; j \in \{3q+1, 3q+2, 3q+3\}, 0 \leq q \leq \lceil \frac{n}{3} \rceil - 1, \end{cases}$$

- for $3|m$ (m odd):

$$c_7(v_iv_{i,j}) = \begin{cases} p+4, & i \text{ odd}; j \in \{3p+1, 3p+2, 3p+3\}, 0 \le p \le \lceil \frac{n}{3} \rceil - 1, \\ q+1+p_1, & i \text{ even}; j \in \{3q+1, 3q+2, 3q+3\}, 0 \le q \le \lceil \frac{n}{3} \rceil - 1, \\ r+1+q_1, & i \text{ odd } (i \neq 1); j \in \{3r+1, 3r+2, 3r+3\}, 0 \le r \le \lceil \frac{n}{3} \rceil - 1, \end{cases}$$

where $p_1 = max(p+4)$ and $q_1 = max(q+1+p_1)$,

- for other m:

$$c_7(v_iv_{i,j}) = \begin{cases} p+4, & i = 1; j \in \{3p+1, 3p+2, 3p+3\}, 0 \le p \le \lceil \frac{n}{3} \rceil - 1, \\ q+1+p_1, & i \text{ even}; j \in \{3q+1, 3q+2, 3q+3\}, 0 \le q \le \lceil \frac{n}{3} \rceil - 1, \\ r+5, & i \text{ odd}; j \in \{3r+1, 3r+2, 3r+3\}, 0 \le r \le \lceil \frac{n}{3} \rceil - 2, \\ 1+q_1, & i \text{ odd}; j \in \{3s+1, 3s+2, 3s+3\}, s = \lceil \frac{n}{3} \rceil - 1, \end{cases}$$

where $p_1 = max(p+4)$ and $q_1 = max(q+1+p_1)$,

- for $6|m$: $c_7(v_{i,j}v_{i,j+1}) = \begin{cases} 4, & i \text{ odd}; j \text{ odd}, \\ 5, & i \text{ odd}; j \text{ even}, \\ 6, & i \text{ even}; j \text{ odd}, \\ 7, & i \text{ even}; j \text{ even}, \end{cases}$ where $j \in \{1, 2, ..., n\}$

- for $m = 8$: $c_7(v_{i,j}v_{i,j+1}) = \begin{cases} 5, & i \text{ odd}; j \text{ odd}, \\ 6, & i \text{ odd}; j \text{ even}, \\ 7, & i \text{ even}; j \text{ odd}, \\ 8, & i \text{ even}; j \text{ even}, \end{cases}$ where $j \in \{1, 2, ..., n\}$

- for $m = 3x$: $c_7(v_{i,j}v_{i,j+1}) = \begin{cases} 4, & i \text{ odd}; j \text{ odd}, \\ 5, & i \text{ odd}; j \text{ even}, \\ 6, & i \text{ even}; j \text{ odd}, \\ 7, & i \text{ even}; j \text{ even}, \\ 8, & i = m; j \text{ odd}, \\ 9, & i = m; j \text{ even}, \end{cases}$ where $j \in \{1, 2, ..., n\}$

- for other m: $c_7(v_{i,j}v_{i,j+1}) = \begin{cases} 4, & i = 1; j \text{ odd}, \\ 5, & i = 1; j \text{ even}, \\ 6, & i \text{ even}; j \text{ odd}, \\ 7, & i \text{ even}; j \text{ even}, \\ 5, & i \text{ odd}; j \text{ odd}, \\ 8, & i \text{ odd}, r = 0; j \text{ even}, \\ 6, & i \text{ odd}, r > 0; j \text{ even}, \end{cases}$ where $0 \le r \le \lceil \frac{n}{3} \rceil - 2$ and $j \in \{1, 2, ..., n\}$.

Consider the outer cycle subgraphs as wheel subgraphs with v_i as its center vertex, where $v_i \in \{v_1, v_2, ..., v_m\}$. Figure 11 shows the illustration of 3-local strong rainbow coloring of $C_m \odot C_n$ for $m \geq 6$, where m is even, or $3|m$ and $n \geq 6$.

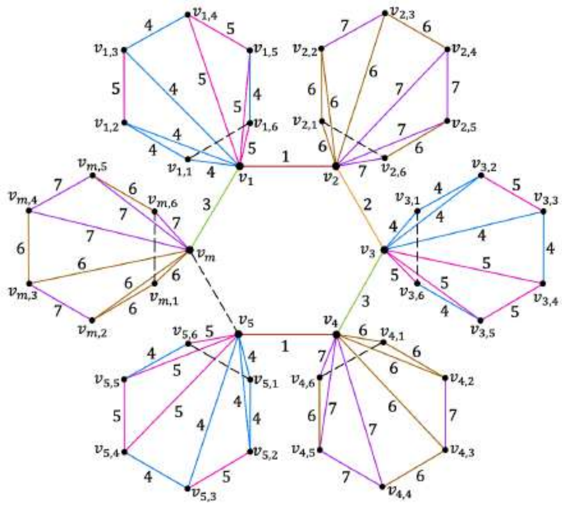

Figure 11. 3-local strong rainbow coloring of $C_m \odot C_n$

- **Subcase 7.1** If $v_{i_1} \in V(C_m), v_{i_2} \in V(C_m); i_1, i_2 \in \{1, 2, ..., m\}, i_1 \neq i_2$

  The coloring refers to Subcase 5.1.

- **Subcase 7.2** If $v_{i,j} \in V(C_n^i), v_{i,k} \in V(C_n^i); i \in \{1, 2, ..., m\}, j, k \in \{1, 2, ..., n\}, j \neq k$

  The coloring refers to Subcase 4.2.

- **Subcase 7.3** If $v_{i_1,j} \in V(C_n^{i_1}), v_{i_2,k} \in V(C_n^{i_2}); i_1, i_2 \in \{1, 2, ..., m\}, j, k \in \{1, 2, ..., n\}, i_1 \neq i_2$

  The coloring may refer to Subcase 4.3. However, in this case, for m even, $\lceil n/3 \rceil \cdot 2$ additional colors are needed to color $v_{i_1,j}v_{i_1}$ and $v_{i_2}v_{i_2,k}$ while for $3|m$, $\lceil n/3 \rceil \cdot 3$ additional colors are needed to color $v_{i_1,j}v_{i_1}$ and $v_{i_2}v_{i_2,k}$.

- **Subcase 7.4** If $v_{i_1} \in V(C_m), v_{i_2,j} \in V(C_n^{i_2}); i_1, i_2 \in \{1, 2, ..., m\}, j \in \{1, 2, ..., n\}, i_1 \neq i_2$

  The coloring may refer to Subcase 5.3 but in this case, the coloring of $v_{i_2}v_{i_2,j}$ refers to Case 7.3.

Thus, it is proved that the $lsrc_3(C_m \odot C_n) = \lceil \frac{m}{\lfloor m/3 \rfloor} \rceil + (\lceil \frac{n}{3} \rceil \cdot \lceil \frac{m}{\lfloor m/2 \rfloor} \rceil)$ for $m \geq 6$ (m is even) or $3|m$ and $n \geq 6$.

Case 8: For $m \geq 6, 3 \nmid m$ (m is odd) and $n \geq 6$

Defined the coloring $c_8 : E(C_m \odot C_n) \rightarrow \{1, 2, ..., (\lceil n/3 \rceil \cdot 3) + 3\}$ as follows:

- The coloring of v_iv_{i+1} is the same as $c_5(v_iv_{i+1})$,

- $c_8(v_iv_{i,j}) = \begin{cases} p+4, & i=1; j \in \{3p+1, 3p+2, 3p+3\}, 0 \le p \le \lceil \frac{n}{3} \rceil - 1, \\ q+1+p_1, & i \text{ even}; j \in \{3q+1, 3q+2, 3q+3\}, 0 \le q \le \lceil \frac{n}{3} \rceil - 1, \\ r+1+q_1, & i \text{ odd } (i \neq 1); j \in \{3r+1, 3r+2, 3r+3\}, 0 \le r \le \lceil \frac{n}{3} \rceil - 1, \end{cases}$

  where $p_1 = max(p+4)$ and $q_1 = max(q+1+p_1)$,

- $c_8(v_{1,j}v_{1,j+1}) = \begin{cases} 4, & j \text{ is odd}, \\ 5, & j \text{ is even}, \end{cases}$ where $j \in \{1, 2, ..., n\}$,

- $c_8(v_{i,j}v_{i,j+1}) = \begin{cases} 6, & j \text{ is odd}, \\ 7, & j \text{ is even}, \end{cases}$ where $i \in \{2, 4, 6, ..., m-1\}$ and $j \in \{1, 2, ..., n\}$,

- $c_8(v_{i,j}v_{i,j+1}) = \begin{cases} 8, & j \text{ is odd}, \\ 9, & j \text{ is even}, \end{cases}$ where $i \in \{3, 5, ..., m\}$ and $j \in \{1, 2, ..., n\}$

Consider the outer cycle subgraphs as wheel subgraphs with v_i as its center vertex, where $v_i \in \{v_1, v_2, ..., v_m\}$. Figure 12 shows the 3-local strong rainbow coloring illustration of $C_7 \odot C_n$, $n \ge 6$.

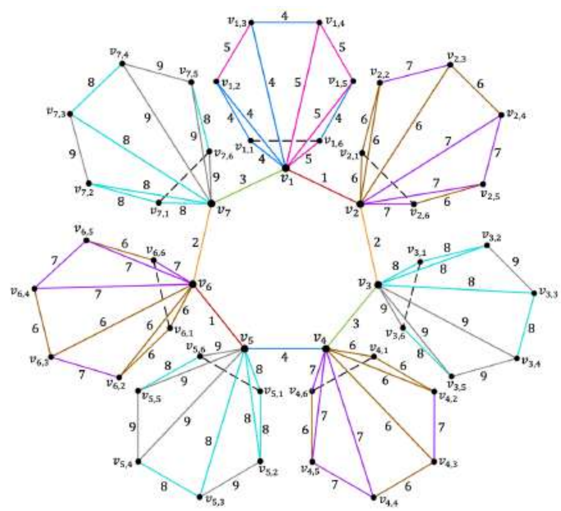

Figure 12. 3-local strong rainbow coloring of $C_7 \odot C_n$

- **Subcase 8.1** If $v_{i_1} \in V(C_m), v_{i_2} \in V(C_m); i_1, i_2 \in \{1, 2, ..., m\}, i_1 \neq i_2$

  The coloring refers to Subcase 5.1.

- **Subcase 8.2** If $v_{i,j} \in V(C_n^i), v_{i,k} \in V(C_n^i); i \in \{1, 2, ..., m\}, j, k \in \{1, 2, ..., n\}, j \neq k$

  The coloring refers to Subcase 4.2.

- **Subcase 8.3** If $v_{i_1,j} \in V(C_n^{i_1}), v_{i_2,k} \in V(C_n^{i_2}); i_1, i_2 \in \{1, 2, ..., m\}, j, k \in \{1, 2, ..., n\}, i_1 \neq i_2$

  According to Subcase 7.3, $\lceil n/3 \rceil \cdot 2$ additional colors are needed to color $v_{i_1,j}v_{i_1}$ and $v_{i_2}v_{i_2,k}$. However, one subgraph cannot be given those new colors because there are an odd number of outer cycle subgraphs. Thus, we can put color 4 on $v_1v_{1,1}, v_1v_{1,2},$ and $v_1v_{1,3}$, then the remaining $v_1v_{1,j}$ are colored with $\lceil n/3 \rceil - 1$ additional colors.

- **Subcase 8.4** If $v_{i_1} \in V(C_m), v_{i_2,j} \in V(C_n^{i_2}); i_1, i_2 \in \{1, 2, ..., m\}, j \in \{1, 2, ..., n\}, i_1 \neq i_2$

  The coloring may refer to Subcase 5.3 but in this case, the coloring of $v_{i_2}v_{i_2,j}$ refers to Subcase 8.3.

Thus, it is proved that the $lsrc_3(C_m \odot C_n) = (\lceil n/3 \rceil \cdot 3) + 3$ for $m \geq 6, 3 \nmid m$ (m is odd) and $n \geq 6$.

□

4. Conclusion

From the discussion in Section 3, it can be concluded that for any cycle graph C_m and C_n, where $m \geq 3$ and $n \geq 3$:

1. The 2-local strong rainbow connection number of $C_m \odot C_n$ is

$$lsrc_2(C_m \odot C_n) = \begin{cases} \lceil n/2 \rceil, & \text{for } m = 3, n \in \{3,4,5\}, \\ \lceil n/3 \rceil + 1, & \text{for } m = 3, n \geq 6, \\ 3, & \text{for } m > 3, n \in \{3,4,5\}, \\ \lceil n/3 \rceil + 2, & \text{for } m > 3, n \geq 6. \end{cases}$$

2. The 3-local strong rainbow connection number of $C_m \odot C_n$ is

$$lsrc_3(C_m \odot C_n) = \begin{cases} 3, & m = 3, n \in \{3,4,5\}, \\ (\lceil n/3 \rceil \cdot 3) + 1, & m = 3, n \geq 6, \\ \lceil \frac{m}{2} \rceil + 2, & m \in \{4,5\}, n \in \{3,4,5\}, \\ 2 + (\lceil \frac{n}{3} \rceil \cdot \lceil \frac{m}{2} \rceil), & m \in \{4,5\}, n \geq 6, \\ \lceil \frac{m}{\lfloor m/3 \rfloor} \rceil + 2, & m \geq 6 (m \neq 3x), n \in \{3,4,5\}, \\ 6, & m = 3x, n \in \{3,4,5\}, \\ \lceil \frac{m}{\lfloor m/3 \rfloor} \rceil + (\lceil \frac{n}{3} \rceil \cdot \lceil \frac{m}{\lfloor m/2 \rfloor} \rceil), & m \geq 6 \ (m \text{ even) or } 3|m, n \geq 6, \\ (\lceil n/3 \rceil \cdot 3) + 3, & m \geq 6, 3 \nmid m \ (m \text{ odd}), n \geq 6, \end{cases}$$

   where x is an odd number larger than 1.

For further research, we can study local rainbow coloring for other product graphs.

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References

[1] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs and digraphs*, Fifth Edition, Chapman and Hall/CRC (2010).

[2] A. J. Bondy and U. S. R. Murty, *Graph theory with applications*, Macmillan, London (1976).

[3] R. Frucht and F. Harary, On the corona of two graphs, *Aeq. Math.*, **4** (1970), 322-325.

[4] G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang, Rainbow connection in graphs, *Math. Bohem.*, **133** (2008), 85–98.

[5] A. Maulani, S. Pradini, D. Setyorini, and K. A. Sugeng, Rainbow connection number of $C_m \odot P_n$ and $C_m \odot C_n$, *Indonesian Journal of Combinatorics*, **3** (2020), 95–108.

[6] F. Septyanto and K. A. Sugeng, Distance-local rainbow connection number, *Discussiones Math. Graph Theory*, **42** (2022), 1027–1039.