Indonesian Journal of Combinatorics **7** (2) (2023), 63–72

INDONESIAN JOURNAL OF COMBINATORICS

On the metric dimension of Buckminster-fullerene-net graph

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Abstract

The metric dimension of an arbitrary connected graph G, denoted by $\dim(G)$, is the minimum cardinality of the resolving set W of G. An ordered set $W = \{w_1, w_2, \cdots, w_k\}$ is a resolving set of G if for all two different vertices in G, their metric representations are different with respect to W. The metric representation of a vertex v with respect to W is defined as k-tuple $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$, where $d(v, w_j)$ is the distance between v and w_j for $1 \leq j \leq k$. The Buckminsterfullerene graph is a 3-reguler graph on 60 vertices containing some cycles C_5 and C_6. Let B_{60}^t denotes the t^{th} B_{60} for $1 \leq t \leq m$ and $m \geq 2$. Let v_t be a terminal vertex for each B_{60}^t. The Buckminsterfullerene-net, denoted by $H := Amal\{B_{60}^t, v|1 \leq t \leq m; m \geq 2\}$ is a graph constructed from the identification of all terminal vertices v_t, for $1 \leq t \leq m$ and $m \geq 2$, into a new vertex, denoted by v. This paper will determine the metric dimension of the Buckminsterfullerene-net graph H.

Keywords: Buckminsterfullerene, amalgamation, Buckminsterfullerene-net, metric dimension
Mathematics Subject Classification : 05C12, 05C15

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Received: 1 December 2022, Revised: 1 November 2023, Accepted: 11 December 2023.

1. Introduction

Let $G = (V, E)$ be a simple, finite and undirected graph, where $V = V(G)$ and $E = E(G)$ are the vertex-set and the edge-set of G, respectively. The distance between two arbitrary vertices u, v in G, denoted by $d(u, v)$, is the length of the shortest path between them. Let W be an ordered subset of V. The metric representation of some vertex $v \in V(G)$ with respect to W is defined as k-vector $r(v \mid W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$. If $r(u \mid W) \neq r(v \mid W)$ for every two vertices u, v in G, then W is called the resolving set of G. The minimum cardinality of W is defined as the metric dimension of G, denoted by $\dim(G)$ [5]. Other graph terminologies and notations are taken from [6].

The fundamental results regarding the metric dimension of a graph are given by Chartrand *et al.* [5]. They stated the characterizations of some connected graph G with $\dim(G) = 1$, $\dim(G) = n - 1$ or $\dim(G) = n - 2$, and determined the metric dimension of cycle C_n and an arbitrary tree T. Another significant results are the metric dimension of regular bipartite graphs [3], fan F_n [4], unicyclic graphs [8], n-partite complete graphs [10], the lexicographic product of graphs [11], wheels, generalized wheel [14] and Jahangir graph [15]. Next, Yulianti *et al.* [16] determined the metric dimension of thorn-subdivided graph $TD(G)$ of an arbitrary connected graph G on n vertices. Recent result is the metric dimension of the triangle-net graph by Yulianti *et al.* in [17].

Akhter *et al.* [1] stated that the Fullerene molecule discovered by Kroto *et al.* [7] can be represented as a Fullerene graph. In the same paper, they considered the metric dimension of $(3, 6)$-Fullerene and $(4, 6)$-Fullerene, where $(k, 6)$-Fullerene is a planar 3-connected graph containing cycles on k and 6 vertices. The Buckminsterfullerene graph, denoted by B_{60}, is one of the $(5, 6)$-Fullerene on 60 vertices. The definition of B_{60} was taken from Andova *et al.* [2].

Putri *et al.* [9] stated that the metric dimension of the Buckminsterfullerene graph B_{60} is three. Using this result, we constructed the Buckminsterfullerene-net graph as follows. Denote B_{60}^t as the t^{th} Buckminsterfullerene graph B_{60} for $1 \leq t \leq m$ and $m \geq 2$; and define v_t as the terminal vertex for every B_{60}^t. The Buckminsterfullerene-net, denoted by $H := Amal\{B_{60}^t, v | 1 \leq t \leq m; m \geq 2\}$ is a graph constructed from the identification of all terminal vertices v_t, for $1 \leq t \leq m$ and $m \geq 2$, into a new vertex, denoted by v. In this paper we determine the metric dimension of the Buckminsterfullerene-net graph H.

2. The Buckminsterfullerene-net Graph

Putri *et al.* [9] gave the vertex set and the edge set of B_{60} as follows.

$$V(B_{60}) = \{v_i, z_i \mid 1 \leq i \leq 5\} \cup \{w_j, y_j \mid 1 \leq j \leq 15\} \cup \{x_k \mid 1 \leq k \leq 20\}, \quad (1)$$

$$\begin{aligned} E(B_{60}) = \; & \{v_l v_{l+1}, z_l z_{l+1} \mid 1 \leq l \leq 4\} \cup \{w_m w_{m+1}, y_m y_{m+1} \mid 1 \leq m \leq 14\} \\ & \cup \{x_k x_{k+1} \mid 1 \leq k \leq 19\} \cup \{v_1 v_5, z_1 z_5, w_1 w_{15}, y_1 y_{15}, x_1 x_{20}\} \\ & \cup \{v_i w_{3i-2}, w_{3i-1} x_{4i-2}, x_{4i-1} y_{3i-1} \mid 1 \leq i \leq 5\} \\ & \cup \{w_{3l} x_{4l+1}, x_{4l} y_{3l+1}, y_{3l} z_{l+1} \mid 1 \leq l \leq 4\} \\ & \cup \{w_{15} x_1, x_{20} y_1, y_{15}, z_1\}. \end{aligned} \quad (2)$$

The Buckminsterfullerene graph B_{60} is given in Figure 1.

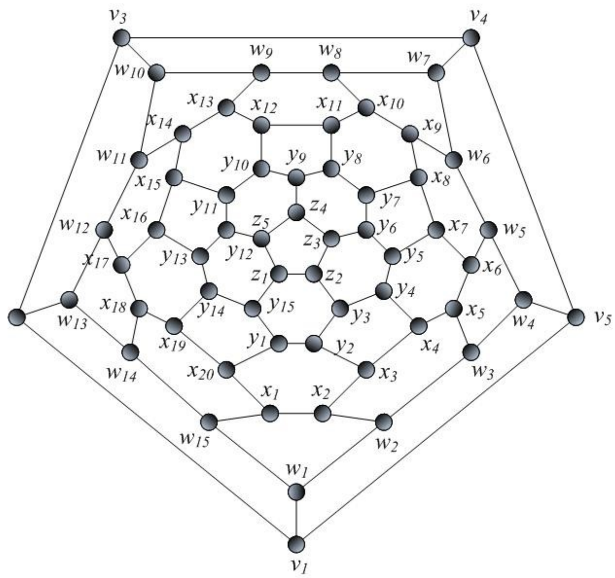

Figure 1. [9] The Buckminsterfullerene graph B_{60}

Let t be a positive integer, $1 \leq t \leq m$, and $m \geq 2$. Denote $B_{60}^{(t)}$ as the t^{th} Buckminsterfullerene. The vertex set and the edge set of $B_{60}^{(t)}$ are defined similarly as in (1) and (2).

$$\begin{aligned}
V(B_{60}^{(t)}) &= \{v_{t,i}, z_{t,i} \mid 1 \leq i \leq 5\} \cup \{w_{t,j}, y_{t,j} \mid 1 \leq j \leq 15\} \cup \{x_{t,k} \mid 1 \leq k \leq 20\},\\
E(B_{60}^{(t)}) &= \{v_{t,l}v_{t,l+1}, z_{t,l}z_{t,l+1} \mid 1 \leq l \leq 4\} \cup \{w_{t,m}w_{t,m+1}, y_{t,m}y_{t,m+1} \mid 1 \leq m \leq 14\},\\
&\quad \cup \{x_{t,n}x_{t,n+1} \mid 1 \leq n \leq 19\} \cup \{v_{t,1}v_{t,5}, z_{t,1}z_{t,5}, w_{t,1}w_{t,15}, y_{t,1}y_{t,15}, x_{t,1}x_{t,20}\},\\
&\quad \cup \{v_{t,i}w_{t,3i-2} \mid 1 \leq i \leq 5\} \cup \{w_{t,3i-1}x_{t,4i-2} \mid 1 \leq i \leq 5\} \cup \{w_{t,3l}x_{t,4l+1} \mid 1 \leq l \leq 4\},\\
&\quad \cup \{x_{t,4i-1}y_{t,3i-1} \mid 1 \leq i \leq 5\} \cup \{x_{t,4l}y_{t,3l+1} \mid 1 \leq l \leq 4\}\\
&\quad \cup \{y_{t,3l}z_{t,l+1} \mid 1 \leq l \leq 4\} \cup \{w_{t,15}x_{t,1}, x_{t,20}y_{t,1}, y_{t,15}z_{t,1}\}.
\end{aligned}$$

We construct the Buckminsterfullerene-net $H = Amal\{B_{60}^{t}, v|1 \leq t \leq m, m \geq 2\}$ by identifying the vertices $v_{t,1}$ for $1 \leq t \leq m$, into a new vertex, namely v. The vertex set and edge set of H are as follows.

$$V(H) = \bigcup_{t=1}^{m} V(B_{60}^{(t)}) \cup \{v\} \setminus \{v_{t,1} \mid 1 \leq t \leq m\}, \tag{3}$$

$$\begin{aligned}
E(H) &= \bigcup_{t=1}^{m} E(B_{60}^{(t)}) \cup \{vv_{t,2}, vv_{t,5}, vw_{t,1} \mid 1 \leq t \leq m\}\\
&\quad \setminus \{v_{t,1}v_{t,2}, v_{t,1}v_{t,5}, v_{t,1}w_{t,1} \mid 1 \leq t \leq m\}.
\end{aligned} \tag{4}$$

3. The Metric Dimension of *H*

Simanjuntak *et al.* [13] gave the lower and upper bounds for the metric dimension of amalgamation of arbitrary connected graphs, as stated in Theorem 3.1.

Table 1. The representation of B_{60}

v	$r(v \mid W)$		v	$r(v \mid W)$		v	$r(v \mid W)$		v	$r(v \mid W)$
v_1	$(0,3,5)$		x_6	$(4,2,6)$		w_1	$(1,2,6)$		y_1	$(5,5,7)$
v_2	$(1,4,4)$		x_7	$(5,3,6)$		w_2	$(2,1,7)$		y_2	$(5,4,8)$
v_3	$(2,4,3)$		x_8	$(6,4,5)$		w_3	$(3,0,7)$		y_3	$(6,4,7)$
v_4	$(2,3,4)$		x_9	$(5,4,4)$		w_4	$(2,1,6)$		y_4	$(6,3,7)$
v_5	$(1,2,5)$		x_{10}	$(5,5,3)$		w_5	$(3,2,5)$		y_5	$(6,4,6)$
z_1	$(7,6,6)$		x_{11}	$(6,6,2)$		w_6	$(4,3,4)$		y_6	$(7,5,5)$
z_2	$(7,5,6)$		x_{12}	$(6,7,1)$		w_7	$(3,4,3)$		y_7	$(7,5,4)$
z_3	$(8,6,5)$		x_{13}	$(5,7,0)$		w_8	$(4,5,2)$		y_8	$(7,6,3)$
z_4	$(9,7,4)$		x_{14}	$(5,7,1)$		w_9	$(4,6,1)$		y_9	$(8,7,3)$
z_5	$(8,7,5)$		x_{15}	$(6,8,2)$		w_{10}	$(3,5,2)$		y_{10}	$(7,8,2)$
x_1	$(3,3,7)$		x_{16}	$(5,7,3)$		w_{11}	$(4,6,2)$		y_{11}	$(7,9,3)$
x_2	$(3,2,8)$		x_{17}	$(4,6,4)$		w_{12}	$(3,6,3)$		y_{12}	$(7,8,4)$
x_3	$(4,3,9)$		x_{18}	$(4,5,5)$		w_{13}	$(2,5,4)$		y_{13}	$(6,7,4)$
x_4	$(5,2,8)$		x_{19}	$(5,5,6)$		w_{14}	$(3,4,5)$		y_{14}	$(6,6,5)$
x_5	$(4,1,7)$		x_{20}	$(4,4,7)$		w_{15}	$(2,3,6)$		y_{15}	$(6,6,6)$

Theorem 3.1. *[13] For $m \in \mathbb{N}$, $m \geq 2$, let $\{G_1, G_2, \cdots, G_m\}$ be the collection of nontrivial arbitrary connected graphs, and each G_t has a terminal vertex $v_{t,1}$, for $1 \leq t \leq m$. Denote v as the new vertex coming from identifying all of the terminal vertices. If $G := Amal\{G_1, G_2, \cdots, G_m, v\}$ then:*

$$\sum_{t=1}^{m} \dim(G_t) - m \leq \dim(G) \leq \sum_{i=t}^{m} \dim(G_t) + m - 1. \tag{5}$$

The definition of a near-distance basis of a graph is given in Definition 3.1, while in Lemma 3.1 we use the concept of a near-distance basis on the Buckminsterfullerene graph B_{60}.

Definition 3.1. *Let W be a basis of B_{60} and $v \in W$. A basis W is called a near-distance basis of v if for every $u \in N(v)$, there exists $w \in W$ such that $d(u, w) \leq d(v, w)$.*

Lemma 3.1. *The graph B_{60} has a near-distance basis of v_1.*

Proof. Putri *et al.* [9] have shown that $\dim(B_{60}) = 3$. We will provide a basis of B_{60} containing v_1 and near-distance to a vertex v_1. Define $W = \{v_1, w_3, x_{13}\}$. The metric representation of every vertex of B_{60} can be seen in the Table 1. Since the metric representation of all vertices are different, then W is the resolving set of B_{60}. Now, let us consider the vertex v_1. Note that $N(v_1) = \{v_2, v_5, w_1\}$ and we have $d(v_2, x_{13}) < d(v_2, v_1)$ and for $u \in \{v_5, w_1\}$, $d(u, w_3) < d(u, v_1)$. Thus, the set W is a near-distance basis of v_1.

□

Next, we determine the metric dimension of Buckminsterfullerene-net $H = Amal\{B_{60}^t, v|1 \leq t \leq m, m \geq 2\}$ in Theorem 3.2.

Theorem 3.2. *Let* $v \in \{v_{t,1}, v_{t,2}, v_{t,3}, v_{t,4}, v_{t,5}\}$ *of* B_{60}^t *for* $1 \leq t \leq m$ *and* $m \geq 2$. *Let* $H = Amal\{B_{60}^t, v \mid 1 \leq t \leq m, m \geq 2\}$. *Then* $\dim(H) = 2m$.

Proof. Without loss of generality, let $c = v_{t,1}$ be the terminal vertices of H for $1 \leq t \leq m$ and $m \geq 2$. The vertex and edge sets of H are defined in (3) and (4).

For the upper bound of the metric dimension of H, define $W_1 = \{v_{t,2}, x_{t,9} \mid 1 \leq t \leq m\}$. For $1 \leq t \leq m$, the metric representations of every vertex of H with respect to W_1 are given in Table 2. Because all vertices have different metric representations, then W_1 is the resolving set of H. Therefore, $dim(H) \leq 2m$.

Next, we assume that $\dim(H) = 2m - 1$, and W^* is the resolving set of H on $2m - 1$ vertices. Consider the following cases.

(1) Let $c \notin W^*$.
At least one of the subgraphs B_{60}^t, $1 \leq t \leq m$, contains a maximum of one member of W^*. Without loss of generality, assume that B_{60}^1 is the subgraph that contains a maximum of one member of W^*. Define W_1^* as the resolving set of B_{60}^1, where $|W_1^*| \leq 1$ and $W_1^* \subseteq W^*$. Note that every (v_a, v_b)-path in H always goes through the point c, where $v_a \in V(B_{60}^1)$ and $v_b \in V(H \setminus \{B_{60}^1\})$. Define a vertex set

$$D_6 = \{w_{1,6}, w_{1,8}, w_{1,9}, w_{1,11}, x_{1,3}, x_{1,5}, x_{1,6}, x_{1,17}, x_{1,18}, w_{1,20}\},$$

where $d(u, c) = 5$, for all $u \in D_6$. Since $|D_6| = 10 > diam(B_{60}^1) = 9$, then $|W_1^*| \geq 2$. This contradicts the assumption that $|W_1^*| \leq 1$.

(2) Let $c \in W^*$.
At least one of the subgraphs B_{60}^t, $1 \leq t \leq m$, contains a maximum of two members of W^*. Without loss of generality, assume that B_{60}^2 is the subgraph that contains a maximum of two members of W_1^*. Define W_2^* as the resolving set of B_{60}^2, where $\mid W_2^*| \leq 2$ and $W_2^* \subseteq W_1^*$. Define the vertex set

$$D_5 = \{x_{2,4}, x_{2,7}, x_{2,9}, x_{2,10}, x_{2.13}, x_{2.14}, x_{2,16}, x_{2,19}, y_{2,1}, y_{2,2}\},$$

where $d(v, c) = 5$, for all $v \in D_5$. Since $c \in W_1^*$ and $|D_5| = 10 > diam(B_{60}^1) = 9$, then $|W_2^*| \geq 3$. This contradicts the assumption that $|W_2^*| \leq 2$.

From these cases, we have that $dim(H) \geq 2m$. It is easy to show that $\dim(H)$ fulfills the bounds in (5) in Theorem 3.1. □

Graph $H = Amal\{B_{60}^t, v \mid 1 \leq t \leq m, m \geq 2\}$ and its metric dimension for $m = 3$ is given in Figure 2.

4. Conclusion

In this paper, we have determined that the metric dimension of the Buckminsterfullerene-net graph $H = Amal\{B_{60}^t, v \mid 1 \leq t \leq m, m \geq 2\}$ is $2m$.

Table 2. The representation of $H = Amal\{B_{60}^t, v|1 \leq t \leq m, m \geq 2\}$

v	$r(v \mid W)$		v	$r(v \mid W)$
c	$(\underbrace{1, 5, \cdots, 1, 5}_{2m})$		$z_{t,1}$	$(\underbrace{8, 12, \cdots, 8, 12}_{2(t-1)}, 7, 6, \underbrace{8, 12, \cdots, 8, 12}_{2(m-t)}),$
$v_{t,2}$	$(\underbrace{2, 6, \cdots, 2, 6}_{2(t-1)}, 0, 5, \underbrace{2, 6, \cdots, 0, 6}_{2(m-t)})$		$z_{t,2}$	$(\underbrace{8, 12, \cdots, 8, 12}_{2(t-1)}, 8, 5, \underbrace{8, 12, \cdots, 8, 12}_{2(m-t)}),$
$v_{t,3}$	$(\underbrace{3, 7, \cdots, 3, 7}_{2(t-1)}, 1, 4, \underbrace{3, 7, \cdots, 3, 7}_{2(m-t)})$		$z_{t,3}$	$(\underbrace{9, 13, \cdots, 9, 13}_{2(t-1)}, 9, 4, \underbrace{9, 13, \cdots, 9, 13}_{2(m-t)}),$
$v_{t,4}$	$(\underbrace{3, 7, \cdots, 3, 7}_{2(t-1)}, 2, 3, \underbrace{3, 7, \cdots, 3, 7}_{2(m-t)})$		$z_{t,4}$	$(\underbrace{10, 14, \cdots, 10, 14}_{2(t-1)}, 8, 5, \underbrace{10, 14, \cdots, 10, 14}_{2(m-t)}),$
$v_{t,5}$	$(\underbrace{2, 6, \cdots, 2, 6}_{2(t-1)}, 2, 4, \underbrace{2, 6, \cdots, 2, 6}_{2(m-t)})$		$z_{t,5}$	$(\underbrace{9, 13, \cdots, 9, 13}_{2(t-1)}, 7, 6, \underbrace{9, 13, \cdots, 9, 13}_{2(m-t)})$
$x_{t,1}$	$(\underbrace{4, 8, \cdots, 4, 8}_{2(t-1)}, 4, 7, \underbrace{4, 8, \cdots, 4, 8}_{2(m-t)})$		$x_{t,11}$	$(\underbrace{7, 11, \cdots, 7, 11}_{2(t-1)}, 6, 2, \underbrace{7, 11, \cdots, 7, 11}_{2(m-t)}),$
$x_{t,2}$	$(\underbrace{4, 8, \cdots, 4, 8}_{2(t-1)}, 4, 6, \underbrace{4, 8, \cdots, 4, 8}_{2(m-t)})$		$x_{t,12}$	$(\underbrace{7, 11, \cdots, 7, 11}_{2(t-1)}, 5, 3, \underbrace{7, 11, \cdots, 7, 11}_{2(m-t)}),$
$x_{t,3}$	$(\underbrace{5, 9, \cdots, 5, 9}_{2(t-1)}, 5, 6, \underbrace{5, 9, \cdots, 5, 9}_{2(m-t)})$		$x_{t,13}$	$(\underbrace{6, 10, \cdots, 6, 10}_{2(t-1)}, 4, 4, \underbrace{6, 10, \cdots, 6, 10}_{2(m-t)}),$
$x_{t,4}$	$(\underbrace{6, 10, \cdots, 6, 10}_{2(t-1)}, 6, 5, \underbrace{6, 10, \cdots, 6, 10}_{2(m-t)})$		$x_{t,14}$	$(\underbrace{6, 10, \cdots, 6, 10}_{2(t-1)}, 4, 5, \underbrace{6, 10, \cdots, 6, 10}_{2(m-t)}),$
$x_{t,5}$	$(\underbrace{5, 9, \cdots, 5, 9}_{2(t-1)}, 5, 4, \underbrace{5, 9, \cdots, 5, 9}_{2(m-t)})$		$x_{t,15}$	$(\underbrace{7, 11, \cdots, 7, 11}_{2(t-1}, 5, 6, \underbrace{7, 11, \cdots, 7, 11}_{2(m-t}),$
$x_{t,6}$	$(\underbrace{5, 9, \cdots, 5, 9}_{2(t-1)}, 5, 3, \underbrace{5, 9, \cdots, 5, 9}_{2(m-t)})$		$x_{t,16}$	$(\underbrace{6, 10, \cdots, 6, 10}_{2(t-1}, 4, 7, \underbrace{6, 10, \cdots, 6, 10}_{2(m-t}),$
$x_{t,7}$	$(\underbrace{6, 10, \cdots, 6, 10}_{2(t-1)}, 6, 2, \underbrace{6, 10, \cdots, 6, 10}_{2(m-t)})$		$x_{t,17}$	$(\underbrace{5, 9, \cdots, 5, 9}_{2(t-1)}, 3, 7, \underbrace{5, 9, \cdots, 5, 9}_{2(m-t)}),$
$x_{t,8}$	$(\underbrace{7, 11, \cdots, 7, 11}_{2(t-1)}, 6, 1, \underbrace{7, 11, \cdots, 7, 11}_{2(m-t)})$		$x_{t,18}$	$(\underbrace{5, 9, \cdots, 5, 9}_{2(t-1}), 3, 8, \underbrace{5, 9, \cdots, 5, 9}_{2(m-t})),$
$x_{t,9}$	$(\underbrace{6, 10, \cdots, 6, 10}_{2(t-1)}, 5, 0, \underbrace{6, 10, \cdots, 6, 10}_{2(m-t)})$		$x_{t,19}$	$(\underbrace{6, 10, \cdots, 6, 10}_{2(t-1)}, 4, 9, \underbrace{6, 10, \cdots, 6, 10}_{2(m-t)}),$
$x_{t,10}$	$(\underbrace{6, 10, \cdots, 6, 10}_{2(t-1)}, 5, 1, \underbrace{6, 10, \cdots, 6, 10}_{2(m-t)})$		$x_{t,20}$	$(\underbrace{5, 9, \cdots, 5, 9}_{2(t-1)}, 5, 8, \underbrace{5, 9, \cdots, 5, 9}_{2(m-t)}),$

v	$r(v \mid W)$		v	$r(v \mid W)$
$w_{t,1}$	$(\underbrace{2,6,\cdots,2,6}_{2(t-1)},2,6,\underbrace{2,6,\cdots,2,6}_{2(m-t)})$		$y_{t,1}$	$(\underbrace{6,10,\cdots,6,10}_{2(t-1)},6,7,\underbrace{6,10,\cdots,6,10}_{2(m-t)})$,
$w_{t,2}$	$(\underbrace{3,7,\cdots,3,7}_{2(t-1)},3,5,\underbrace{3,7,\cdots,3,7}_{2(m-t)})$		$y_{t,2}$	$(\underbrace{6,10,\cdots,6,10}_{2(t-1)},6,6,\underbrace{6,10,\cdots,6,10}_{2(m-t)})$,
$w_{t,3}$	$(\underbrace{4,8,\cdots,4,8}_{2(t-1)},4,4,\underbrace{4,8,\cdots,4,8}_{2(m-t)})$		$y_{t,3}$	$(\underbrace{7,11,\cdots,7,11}_{2(t-1)},7,5,\underbrace{7,11,\cdots,7,11}_{2(m-t)})$,
$w_{t,4}$	$(\underbrace{3,7,\cdots,3,7}_{2(t-1)},3,3,\underbrace{3,7,\cdots,3,7}_{2(m-t)})$		$y_{t,4}$	$(\underbrace{7,11,\cdots,7,11}_{2(t-1)},7,4,\underbrace{7,11,\cdots,7,11}_{2(m-t)})$,
$w_{t,5}$	$(\underbrace{4,8,\cdots,4,8}_{2(t-1)},4,2,\underbrace{4,8,\cdots,4,8}_{2(m-t)})$		$y_{t,5}$	$(\underbrace{7,11,\cdots,7,11}_{2(t-1)},7,3,\underbrace{7,11,\cdots,7,11}_{2(m-t)})$,
$w_{t,6}$	$(\underbrace{5,9,\cdots,5,9}_{2(t-1)},4,1,\underbrace{5,9,\cdots,5,9}_{2(m-t)})$		$y_{t,6}$	$(\underbrace{8,12,\cdots,8,12}_{2(t-1)},8,3,\underbrace{8,12,\cdots,8,12}_{2(m-t)})$,
$w_{t,7}$	$(\underbrace{4,8,\cdots,4,8}_{2(t-1)},3,2,\underbrace{4,8,\cdots,4,8}_{2(m-t)})$		$y_{t,7}$	$(\underbrace{8,12,\cdots,8,12}_{2(t-1)},7,2,\underbrace{8,12,\cdots,8,12}_{2(m-t)})$,
$w_{t,8}$	$(\underbrace{5,9,\cdots,5,9}_{2(t-1)},4,2,\underbrace{5,9,\cdots,5,9}_{2(m-t)})$		$y_{t,8}$	$(\underbrace{8,12,\cdots,8,12}_{2(t-1)},7,3,\underbrace{8,12,\cdots,8,12}_{2(m-t)})$,
$w_{t,9}$	$(\underbrace{5,9,\cdots,5,9}_{2(t-1)},3,3,\underbrace{5,9,\cdots,5,9}_{2(m-t)})$		$y_{t,9}$	$(\underbrace{9,13,\cdots,9,13}_{2(t-1)},7,4,\underbrace{9,13,\cdots,9,13}_{2(m-t)})$,
$w_{t,10}$	$(\underbrace{4,8,\cdots,4,8}_{2(t-1)},2,4,\underbrace{4,8,\cdots,4,8}_{2(m-t)})$		$y_{t,10}$	$(\underbrace{8,12,\cdots,8,12}_{2(t-1)},6,4,\underbrace{8,12,\cdots,8,12}_{2(m-t)})$,
$w_{t,11}$	$(\underbrace{5,9,\cdots,5,9}_{2(t-1)},3,5,\underbrace{5,9,\cdots,5,9}_{2(m-t)})$		$y_{t,11}$	$(\underbrace{8,12,\cdots,8,12}_{2(t-1)},6,5,\underbrace{8,12,\cdots,8,12}_{2(m-t)})$,
$w_{t,12}$	$(\underbrace{4,8,\cdots,4,8}_{2(t-1)},2,6,\underbrace{4,8,\cdots,4,8}_{2(m-t)})$		$y_{t,12}$	$(\underbrace{8,12,\cdots,8,12}_{2(t-1)},6,6,\underbrace{8,12,\cdots,8,12}_{2(m-t)})$,
$w_{t,13}$	$(\underbrace{3,7,\cdots,3,7}_{2(t-1)},1,6,\underbrace{3,7,\cdots,3,7}_{2(m-t)})$		$y_{t,13}$	$(\underbrace{7,11,\cdots,7,11}_{2(t-1)},5,7,\underbrace{7,11,\cdots,7,11}_{2(m-t)})$,
$w_{t,14}$	$(\underbrace{4,8,\cdots,4,8}_{2(t-1)},2,7,\underbrace{4,8,\cdots,4,8}_{2(m-t)})$		$y_{t,14}$	$(\underbrace{7,11,\cdots,7,11}_{2(t-1)},5,8,\underbrace{7,11,\cdots,7,11}_{2(m-t)})$,
$w_{t,15}$	$(\underbrace{3,7,\cdots,3,7}_{2(t-1)},3,7,\underbrace{3,7,\cdots,3,7}_{2(m-t)})$		$y_{t,15}$	$(\underbrace{7,11,\cdots,7,11}_{2(t-1)},6,7,\underbrace{7,11,\cdots,7,11}_{2(m-t)})$,

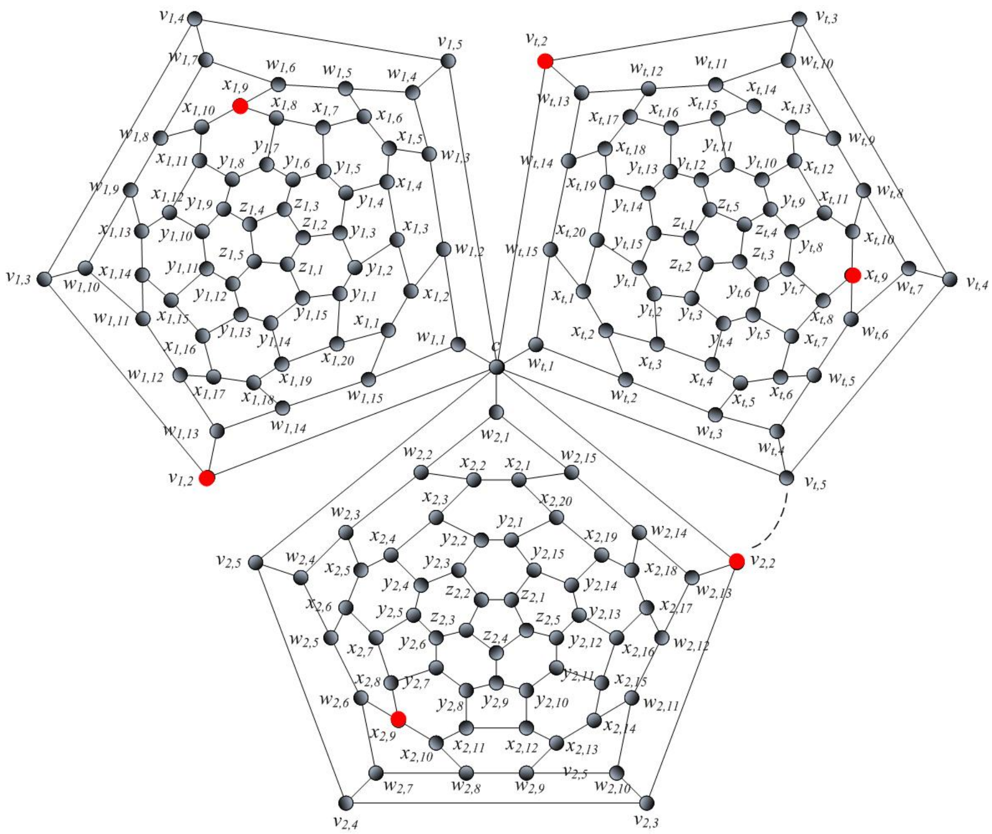

Figure 2. $Amal\{B_{60}^t, v \mid 1 \leq t \leq 3\}$ and $W_1 = \{v_{t,2}, x_{t,9} \mid 1 \leq t \leq 3\}$

5. Acknowledgments

This paper is partially supported by Hibah Riset Dasar FMIPA Universitas Andalas No. 18/UN.16.03.D/PP/FMIPA.2021.

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