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On graphs associated to topological spaces

Haval M. Mohammed Salih

Department of Mathematics, Faculty of Science, Soran University, Kawa St. Soran, Iraq

haval.mohammed@soran.edu.iq

Abstract

Let X be a set and T be a topology on X. A new type of graph on $P(X)$, namely the closure graph of T is introduced. The closure graph denoted by Γ^c, as an undircted simple graph whose vertex set is $P(X)$ and for distinct $A, B \in P(X)$, there is an edge $e = AB$ if $\overline{A} \cap \overline{B} \subseteq \overline{A \cap B}$. In this paper, the closure graph is shown as a connected graph with diameter bounded by two. Also, the girth of the closure graph Γ^c of T is three if X contains more than one point. Also, several graph properties are studied.

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1. Introduction

Recently, a lot of graph structure defined on groups and rings which can be found in [1, 2, 9, 4]. A lot of work has been done based on these new definitions. It becomes a new branch in abstract algebra and graph theory. The distance between two distinct vertices A and B in G, denoted by $d(A, B)$, is the length of the shortest path connecting them, if such a path exists; otherwise, we put $d(A, B) = \infty$. The diameter of a graph G is defined by $diam(G) = Sup\{d(A, B) : A \text{ and } B \text{ are distinct vertices of } G\}$. The girth of G is the length of the shortest cycle in G, denoted by $g(G)$ ($gr(G) := \infty$ if G has no cycles). A graph G is connected if there is a path between any two distinct vertices and it is complete if it is connected with $diam(G) = 1$ and it will be denoted by K_n. Two graphs G and H are isomorphic, denoted by $G \cong H$, if there is a bijection $f : V(G) \to V(H)$ such that xy is an edge in G if and only if $f(x)f(y)$ is an edge in H [8, 9].

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Throughout this paper, we will assume that (X, T) is a topological space (for short X) and let $A \subseteq X$. The interior of A is defined by $A^o = \cup\{G \subseteq A \mid G\ is\ an\ open\ set\ in\ X\}$ and the closure of A is defined by $\overline{A} = \cap\{F$ is closed set $in\ X|A \subseteq F\}$. Also, the boundary of A is defined by $b(A) = \bar{A} \bigcap \bar{A^c}$.

Throughout this paper we assume that X is a finite set unless otherwise stated. We define the graph structure on a topological space. A new type of graph on $P(X)$, namely the closure graph of T is defined. The closure graph is denoted by Γ^c where the vertex set is $P(X)$. In this graph, for $A \neq B\ in\ P(X),\ e = AB$ is an edge if $\overline{A} \cap \overline{B} \subseteq \overline{A \cap B}$. Our main goal of this work is to study some properties of a new type of graph by using closure properties.

The following results are well known in topology and graph theory.

Lemma 1.1. *[5] Let A and B be subsets of X. Then*

1. $A^o = \bar{A^c}^c$.
2. *If A is an open set, then $A \cap \overline{B} \subseteq \overline{A \cap B}$.*
3. *$b(A) = \emptyset$ if and only if A is open and closed.*

Corollary 1.1. *(Dirac, 1952) [8]. If G is a simple graph with at least three vertices, and if the degree of each vertex greater than or equal to half number of vertices, then G is Hamiltonian.*

Theorem 1.1. *[8]. A graph G is planar if and only if it does not contain a subgraph homeomorphic to $K_{3,3}$ or K_5.*

Theorem 1.2. *[5] If A and B are subsets of X, then the following are equivalent:*

1. $\bar{A} \cap \bar{B} = \overline{A \cap B}$
2. $b(A) \cap b(B) \subseteq b(A \bigcap B)$.

2. Some Properties of The Closure Graph

In this section, a new kind of graph is introduced which is called the closure graph of the topology T on a finite set X and denoted by Γ^c. Furthermore, some results related to the closure graph are given.

Definition 2.1. *Assume that X is a set and T be a topology on X. A closure graph of T denoted by Γ^c whose vertex set is $P(X)$ and for distinct A and B in $P(X)$, there is an edge $e = AB$ if $\overline{A} \cap \overline{B} \subseteq \overline{A \cap B}$.*

Example 2.1. *Let $X = \{a, b, c\}$ and $T = \{\phi, \{a\}, \{b, c\}, X\}$. The closure graph of T is*

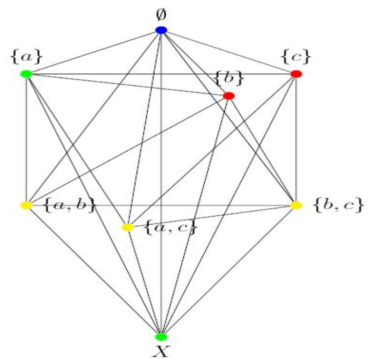

Figure 1. The closure graph of T

Proposition 2.1. *Let A and B be subsets of X and Γ^c be the closure graph of T. Then*

1. *If $A \subseteq B$, then A and B are adjacent.*
2. *Two closed sets are adjacent.*
3. *If A and B are finite sets of a Huasdorff space, then they are adjacent.*
4. *If A and B are separated sets and one of them is closed, then they are adjacent.*

Proof. 1. If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$. Now $\overline{A} \cap \overline{B} = \overline{A} = \overline{A \cap B}$. Therefore A is adjacent to B.

2. The proof is clear.
3. Every finite set in X is compact. Since X is a Huasdorff space, then A and B in X are closed sets. The rest follows from 4.
4. Let A be closed set and $\overline{A} \cap \overline{B} = A \cap \overline{B} = \emptyset \subseteq \overline{A \bigcap B}$. So A and B are adjacent.

□

Remark 2.1. The converse of 1, 2, 3 and 4 in Proposition 2.1 are not true according to Example 2.1. It is clear that $\{a, c\}$ and $\{b, c\}$ are adjacent, but they are different. Also, $\{a\}$ and $\{c\}$ are adjacent. However, $\{c\}$ is not closed. Furthermore X is not Hausdorff space. Since $\{b, c\}$ is closed set and it is adjacent to $\{b\}$. However, they are not separated.

Proposition 2.2. *Let A be a subset of X and Γ^c be the closure graph of T. Then $b(A) = \emptyset$ iff A is adjacent to all other vertices in Γ^c.*

Proof. Let $A \subseteq X$. Since $b(A) = \emptyset$, then A is closed and open set. Then $\overline{A} \cap \overline{B} = A \cap \overline{B} \subseteq \overline{A \cap B}$, the last inequality follows from Lemma 1.1. Hence A is adjacent to all other vertices. Conversely, let H is adjacent to all other vertices, that is $\overline{H} \cap \overline{B} \subseteq \overline{H \cap B}$ for all $B \subseteq X$. Now, $\emptyset = \overline{\emptyset} =$

$\overline{H \cap H^c} = \overline{H} \cap \overline{H^c}$. So $\overline{H}$ and $\overline{H^c}$ are disjoint. Also, we have $X = \overline{X} = \overline{H \cup H^c} = \overline{H} \cup \overline{H^c}$. Therefore,

$$\overline{H^c} = \overline{H}^c \tag{1}$$

Now, $H^c \subseteq \overline{H^c} = \overline{H}^c$, the equality follows from (1). This implies that $\overline{H} \subseteq H$. Thus H is closed set. By using Equation (1), we obtain $\overline{H^c}^c = \overline{H}$, so $H^o = \overline{H} = H$. Thus, H is open and $b(H) = \emptyset$. □

Corollary 2.1. *Let* Γ^c *be the closure graph of* X *and* $b(A) = \emptyset$ *where* $A \subseteq X$*, then* $\deg(A) = 2^{|X|-1}$.

Proof. The proof follows from Proposition 2.2. □

Remark 2.2.
1. If the closure graph of T is K_2, then (X, T) is indiscrete space.
2. If X contains one point, then the closure graph of T is K_2.

The converse of Remark 2.2, part 1, is not true according to the following example.

Example 2.2. *Assume that* $X = \{a, b\}$ *and* $T = \{\emptyset, X\}$*. The closure graph of* T *is the following:*

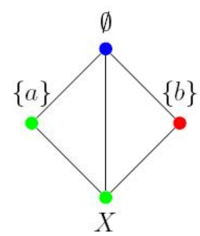

Figure 2. The closure graph of T

Proposition 2.3. *The closure graph of* T *is complete iff* (X, T) *is discrete space.*

Proof. If the closure graph of T is complete, then for each subset A of X is adjacent with other vertex in Γ^c. By Proposition 2.2, we obtain $b(A) = \emptyset$. So A is open and closed set. Since A is arbitrary set and hence X is discrete space.
Conversely, it is straightforward. □

As a consequence of Proposition 2.3, we get the following.

Corollary 2.2. *If the closure graph is complete, then it is disconnected space.*

Proof. The proof follows from Proposition 2.3. □

Theorem 2.1. *Let* A *and* B *be subsets of* X*. Then, the following statements are equivalent:*

1. *The closure graph of T is complete.*
2. $\overline{A} \cap \overline{B} = \overline{A \cap B}$ *for all subsets A and B of X.*
3. $A^o \cup B^o = (A \cup B)^o$ *for all subsets A and B of X.*
4. (X, T) *is discrete.*
5. $b(A) \cap b(B) \subseteq b(A \cap B)$ *for all subsets A and B of X.*

Proof. The proof follows form Proposition 2.3 and Theorem 1.2. □

Theorem 2.2. *If X contains more than one point, then the closure graph Γ^c is a connected with $diam(\Gamma^c) \leq 2, rad(\Gamma^c) = 1$ and $g(\Gamma^c) = 3$.*

Proof. Suppose that A and B are two distinct vertices of Γ^c. If A is adjacent to B, thus $d(A, B) = 1$. If A is not adjacent to B, then the vertices A and B are adjacent with X; hence $d(A, B) = 2$. This means that Γ^c is connected with diameter at most two and radius one. It well known that there exist a vertex A such that $X \neq A$ and $A \neq \emptyset$. These vertices produce a cycle of order three in Γ^c , hence the girth of Γ^c is three. The proof is complete. □

Proposition 2.4. *The closure graph of T is Hamiltonian.*

Proof. If X contains n elements, then there are 2^n subsets of X. Let A be a subset of X, by property 2 of Proposition 2.3, A is adjacent with all super sets of A and ϕ. There are $\frac{2^n}{2}$ super sets of A. So $deg(A) \geq \frac{2^n}{2}$. The rest follows from Corollary 1.2. □

Lemma 2.1. *If there is at least five clopen sets in X, then the closure graph of T is non-planar.*

Proof. The proof is clear. □

Proposition 2.5. *If $f : X \to Y$ is a homeomorphism, then their closure graphs are isomorphic.*

Proof. The proof is clear. □

The converse of Proposition 2.5 is not true for the following example.

Example 2.3. *Assume that $Y = \{a, b\}$, $T = \{\emptyset, Y\}$ and $\partial = \{\emptyset, \{a\}, Y\}$. The closure graphs of T and ∂ are isomorphic.*

Remark 2.3. If (Y, T_Y) is a subspace of X, then it does not imply that the closure graph of T_Y is a subgraph of the closure graph of T.

Example 2.4. *Let $X = \{a, b, c, d\}$ and $T = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X\}$. Let $Y = \{b, c, d\}$ and $T_Y = \{\emptyset, \{b, c\}, Y\}$. The closure graph of T_Y is not a subgraph of the closure graph of T.*

Someone can consider the infinite case as follows:

Example 2.5. *Let us consider the co-finite topology on infinite set X. We need to consider the following:*

If A and B are finite subsets of X, then they are adjacent.

If A is an infinite set and B is a finite, then they are adjacent.

If A and B are infinite subsets of X, then either $A \cap B$ is finite or infinite. That is they are not adjacent or they are adjacent. It gives infinite closure graph.

3. Conclusion

In this study, we introduced a new graph called the closure graph. The graph is found for a topology on a finite set. Also, we showed that there is a relationship between properties of this graph and topological spaces. Furthermore, some properties of this graph were determined such as planarity, connectedness and so on.

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