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When is the maximal graph of an atomic domain with at least two maximal ideals connected?

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Abstract

In this paper, with any atomic domain R which admits at least two maximal ideals, we associate an undirected graph denoted by $\mathbb{MGI}(R)$ whose vertex set is $\mathcal{I}(R) = \{R\pi \mid \pi \in Irr(R) \backslash J(R)\}$ (where $Irr(R)$ is the set of all irreducible elements of R and $J(R)$ is the Jacobson radical of R) and distinct $R\pi, R\pi' \in \mathcal{I}(R)$ are adjacent if and only if $R\pi + R\pi' \subseteq M$ for some maximal ideal M of R. We call $\mathbb{MGI}(R)$ as the maximal graph of R. We denote the set of all maximal ideals of R by $Max(R)$. In this paper, some necessary (respectively, sufficient) conditions on $Max(R)$ are provided such that $\mathbb{MGI}(R)$ is connected. Also, in this paper, in some cases, a necessary and sufficient condition is determined so that $\mathbb{MGI}(R)$ is connected.

Keywords: Irreducible element, Atomic domain, prime element, diameter, radius
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1. Introduction

The rings considered in this paper are commutative with identity. We use R to denote a ring. The graphs considered here are undirected and simple. The vertex set (respectively, the edge set) of a graph G is denoted by $V(G)$ (respectively, $E(G)$). The set of all prime ideals (respectively, the set of all maximal ideals) of R is denoted by $Spec(R)$ (respectively, $Max(R)$) and the notation $J(R)$ is used to denote the Jacobson radical of R. The set of all units (respectively, the set of all

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non-units) in R is denoted in this paper by $U(R)$ (respectively, $NU(R)$). We now give a brief motivation for the problem considered in this paper. In [14], Sharma and Bhatwadekar associated a graph G with vertices as elements of R and distinct vertices a and b are adjacent if and only if $Ra + Rb = R$. It was proved in [14] that G is finitely colourable if and only if R is a finite ring. In [10], Maimani et al. called the graph studied in [14] as the *comaximal graph* of R and they denoted it by $\Gamma(R)$. The authors of [10] also studied the subgraph of $\Gamma(R)$ induced by $NU(R)$ (respectively, $NU(R)\backslash J(R)$). Several researchers have studied the comaximal graph of a commutative ring (for example, [4], [11], [13], [15]).

For a simple graph $G = (V, E)$, recall that the *complement* of G denoted by G^c is defined by taking $V(G^c) = V(G)$ and different $u, v \in V(G)$ are adjacent in G^c if and only if they are not adjacent in G [2, Definition 1.2.13].

In [5], Gaur and Sharma associated a graph G with vertices as elements of R and distinct $a, b \in R$ are adjacent if and only if $Ra + Rb \subseteq M$ for some maximal ideal M of R and they called the graph studied by them as the *maximal graph* of R. Several interesting results on the coloring of G were proved in [5]. Observe that the maximal graph of R is the complement of the comaximal graph of R. Several authors have studied the maximal graph of a commutative ring (for example, [9], [16]).

Recall that an integral domain R is said to be *atomic* if any non-zero non-unit of R can be expressed as a product of a finite number of irreducible elements of R [7, page 321]. We denote the set of all irreducible elements of R by $Irr(R)$. It is well-known that any domain which satisfies the ascending chain condition (a.c.c.) on principal ideals is atomic. Hence, any Noetherian domain is atomic. As usual, we denote the cardinality of a set A by $|A|$.

For an atomic domain R which admits at least two maximal ideals, in this paper, the collection $\{R\pi \mid \pi \in Irr(R)\backslash J(R)\}$ is denoted by $\mathcal{I}(R)$. With R, we associate an undirected graph denoted by $\mathbb{MGI}(R)$ such that $V(\mathbb{MGI}(R)) = \mathcal{I}(R)$ and distinct $R\pi, R\pi' \in \mathcal{I}(R)$ are adjacent if and only if $R\pi + R\pi' \subseteq M$ for some $M \in Max(R)$. In Section 2, some necessary conditions on $Max(R)$ are provided such that $\mathbb{MGI}(R)$ is connected. In Section 3, some sufficient conditions on $Max(R)$ are given so that $\mathbb{MGI}(R)$ is connected. In some special cases, a necessary and sufficient condition is determined such that $\mathbb{MGI}(R)$ is connected. In Section 4, some atomic domains R are provided such that $\mathbb{MGI}(R)$ is connected. Several examples are provided to illustrate the results proved in this paper. In Section 5, some problems are mentioned for which the author is not aware of their solutions. In this paper, $(\mathbb{MGI}(R))^c$ is denoted by $\mathbb{CGI}(R)$. It is useful to note that $V(\mathbb{CGI}(R)) = \mathcal{I}(R)$ and distinct $R\pi, R\pi' \in \mathcal{I}(R)$ are adjacent in $\mathbb{CGI}(R)$ if and only if $R\pi + R\pi' \nsubseteq M$ for any maximal ideal M of R and hence, $R\pi + R\pi' = R$ by [1, Corollary 1.4]. We mention this graph here as we need some properties of this graph in our discussion.

We say that a ring R is *quasi-local* if R has only one maximal ideal. A Noetherian quasi-local ring is called a *local ring*. For a ring R, we denote $R\backslash\{0\}$ by R^*, the polynomial ring in one variable X over R by $R[X]$, and for $n \geq 2$, the polynomial ring in n variables $X_1, X_2, \ldots, X_n$ over R by $R[X_1, X_2, \ldots, X_n]$. If R is a subring of a ring T, then it is assumed that R contains the identity element of T. If a set A is a proper subset of a set B, then we denote it by $A \subset B$. The Krull dimension of a ring R is referred to as the dimension of R and is denoted by $dimR$. For definitions and results from Commutative Ring Theory that are used in this paper, the reader can refer any of the following books [1, 6, 8, 12].

The reader can refer any of the standard text book in Graph Theory (for example, [2, 3]) for the standard definitions and results from graph theory that are used in this paper. If $G = (V, E)$ is a connected graph, then the diameter (respectively, the radius) of G is denoted by $diam(G)$ (respectively, $r(G)$).

2. Some necessary conditions on $Max(R)$ so that $\mathbb{MGI}(R)$ is connected

Throughout this paper, unless otherwise specified, we use R to denote an atomic domain which admits at least two maximal ideals. This section aims to determine some necessary conditions on $Max(R)$ so that $\mathbb{MGI}(R)$ is connected. We begin with the following lemma.

Lemma 2.1. *If a prime ideal P of R is such that $P \not\subseteq J(R)$, then there exists $R\pi \in \mathcal{I}(R)$ such that $\pi \in P$.*

Proof. Choose $a \in P \backslash J(R)$. Since R is atomic and $a \in NU(R)\backslash\{0\}$, there exist $\pi_1, \ldots, \pi_n \in Irr(R)$ such that $a = \prod_{i=1}^{n} \pi_i$. Observe that $\pi_i \notin J(R)$ for each i, $1 \leq i \leq n$ and as P is a prime ideal of R, $\pi_j \in P$ for some j with $1 \leq j \leq n$. With $\pi = \pi_j$, we get that $\pi \in P$ and $R\pi \in \mathcal{I}(R)$ □

Lemma 2.2. *If $M_1, M_2 \ldots, M_n$ ($n \in \mathbb{N}\backslash\{1\}$) are pairwise distinct maximal ideals of R, then for each i ($1 \leq i \leq n$), there exists $R\pi_i \in \mathcal{I}(R)$ with $\pi_i \in M_i \backslash (\bigcup_{j \in \{1,2,\ldots,n\}\backslash\{i\}} M_j)$.*

Proof. As $M_s \not\subseteq M_t$ for all distinct s, t with $1 \leq s, t \leq n$, for each i ($1 \leq i \leq n$), we can find $x_i \in M_i \backslash (\bigcup_{j \in \{1,2,\ldots,n\}\backslash\{i\}} M_j)$ by [1, Proposition 1.11(i)]. By the choice of x_i, no irreducible factor of x_i in R can belong to M_j for any j with $1 \leq j \leq n, j \neq i$ and since $M_i \in Spec(R)$, we can find an irreducible factor π_i of x_i in R with $\pi_i \in M_i$. Thus for each i, $1 \leq i \leq n$, there exists $R\pi_i \in \mathcal{I}(R)$ with $\pi_i \in M_i \backslash (\bigcup_{j \in \{1,2,\ldots,n\}\backslash\{i\}} M_j)$. □

In the following lemma, we provide a necessary condition on $|Max(R)|$ such that $\mathbb{MGI}(R)$ is connected.

Lemma 2.3. *If $\mathbb{MGI}(R)$ is connected, then the number of maximal ideals of R is greater than 2.*

Proof. Assume that $\mathbb{MGI}(R)$ is connected. By assumption on R, R has at least two maximal ideals. Suppose that $|Max(R)| = 2$. Let $Max(R) = \{M_1, M_2\}$. Define the sets V_1, V_2 as follows: $V_1 = \{R\pi \in \mathcal{I}(R) \mid \pi \in M_1\}$ and $V_2 = \{R\pi' \in \mathcal{I}(R) \mid \pi' \in M_2\}$. By Lemma 2.1, $V_i \neq \emptyset$ for each i, $1 \leq i \leq 2$ and it is clear that $V_1 \cap V_2 = \emptyset$. If $R\pi \in V_1$ and $R\pi' \in V_2$, then $\pi \in M_1$ but not in M_2 and $\pi' \in M_2$ but not in M_1. Hence, $R\pi + R\pi' \not\subseteq M_1 \cup M_2$. As $Max(R) = \{M_1, M_2\}$, we get that $R\pi + R\pi'$ is not a subset of any maximal ideal of R. Thus, there is no edge of $\mathbb{MGI}(R)$ which joins a vertex in V_1 to a vertex in V_2 and so, by [3, Theorem 2-1], we obtain that $\mathbb{MGI}(R)$ is not connected. Hence, if $\mathbb{MGI}(R)$ is connected, then $|Max(R)| \geq 3$. □

In Lemma 2.5, we provide another necessary condition on $Max(R)$ so that $\mathbb{MGI}(R)$ is connected. We use the following lemma in its proof.

Lemma 2.4. *For a simple graph $G = (V, E)$ with $|V| \geq 2$, if for some $v \in V$, v and w are adjacent in G for each $w \in V$, $w \neq v$, then G^c is not connected.*

Proof. Since V has more than one element, we can find $u \in V$ with $u \neq v$. If there is a path in G^c between v and u, then v and w are adjacent in G^c for some $w \in V\backslash\{v\}$. This is impossible, since v and w are adjacent in G. Therefore, G^c is not connected. □

We say that elements a, b of a ring R are *associates* in R if $a = ub$ for some $u \in U(R)$. If R is an integral domain and $a \in R\backslash\{0\}$, then $b \in R$ is an associate of a in R if and only if $Ra = Rb$.

Lemma 2.5. *If $\mathbb{MGI}(R)$ is connected, then for any $M \in Max(R)$, M is not principal.*

Proof. Assume that $\mathbb{MGI}(R)$ is connected. If there is a principal maximal ideal M of R, then $M = R\pi$ for some $\pi \in Irr(R)\backslash J(R)$, since R is atomic and $M \in Spec(R)$ but $M \not\subseteq J(R)$. Note that $R\pi \in \mathcal{I}(R)$. As $|Max(R)| \geq 2$, it follows from Lemma 2.2 that $|\mathcal{I}(R)| \geq 2$. Hence, we can find $R\pi' \in \mathcal{I}(R)$ with $R\pi' \neq R\pi$. Thus π and π' are not associates in R and so, $R\pi' \not\subseteq R\pi = M$. Hence, $R\pi + R\pi' = R$. Thus $R\pi$ and $R\pi'$ are adjacent in $\mathbb{CGI}(R) = (\mathbb{MGI}(R))^c$ for all $R\pi' \in \mathcal{I}(R)\backslash\{R\pi\}$. This implies by Lemma 2.4 that $\mathbb{MGI}(R)$ is not connected. This is a contradiction, since $\mathbb{MGI}(R)$ is connected by assumption. Hence, no maximal ideal of R is principal. □

If R admits a maximal ideal M such that M is principal, then it is noted in the proof of Lemma 2.5 that $M = R\pi$ for some $\pi \in Irr(R)\backslash J(R)$ and $R\pi$ is not adjacent to any $R\pi' \in \mathcal{I}(R)\backslash\{R\pi\}$ in $\mathbb{MGI}(R)$. It is natural to characterize R such that any two vertices of $\mathbb{MGI}(R)$ are not adjacent in $\mathbb{MGI}(R)$. This happens if and only if $\mathbb{MGI}(R)$ has no edges. With the help of the following lemmas, in Theorem 2.9, we characterize R such that $\mathbb{MGI}(R)$ has no edges.

Lemma 2.6. *If $\mathbb{MGI}(R)$ has no edges, then each maximal ideal of R is principal.*

Proof. Let $M \in Max(R)$. Since R is atomic and $M \not\subseteq J(R)$, we can find $R\pi \in \mathcal{I}(R)$ with $R\pi \subseteq M$ by Lemma 2.1. For any $a \in M\backslash J(R)$, it is possible to find $R\pi_1 \in \mathcal{I}(R)$ with π_1 is a factor of a in R and $\pi_1 \in M$. Note that $R\pi + R\pi_1 \subseteq M$ and so, $R\pi_1 = R\pi$, since $\mathbb{MGI}(R)$ has no edges by assumption. Hence, $a \in R\pi$. This shows that $M \subseteq R\pi \cup J(R)$. As $M \not\subseteq J(R)$, we get that $M \subseteq R\pi$ and so, $M = R\pi$ is principal. □

Lemma 2.7. *If $\mathbb{MGI}(R)$ has no edges, then each non-zero prime ideal of R is principal and maximal.*

Proof. Let $P \in Spec(R)\backslash\{(0)\}$. Since any proper ideal of a ring is contained in a maximal ideal, there exists $M \in Max(R)$ with $P \subseteq M$. Note that $M = R\pi$ for some $R\pi \in \mathcal{I}(R)$ by the proof of Lemma 2.6. For $a \in P\backslash\{0\}$, we can find an irreducible factor π' of a in R with $\pi' \in P$, since R is atomic and $P \in Spec(R)$. As $\pi' \in R\pi$, we get that $\pi' = u\pi$ for some $u \in U(R)$. Therefore, $\pi = u^{-1}\pi' \in P$ and so, $P = R\pi = M$. This shows that any non-zero $P \in Spec(R)$ is principal and maximal. □

Lemma 2.8. *If $\mathbb{MGI}(R)$ has an edge, then R has a maximal ideal which is not principal.*

Proof. By assumption, it is possible to find $R\pi, R\pi' \in \mathcal{I}(R)$ with $R\pi$ and $R\pi'$ are adjacent in $\mathbb{MGI}(R)$. Hence, $R\pi + R\pi' \subseteq M$ for some maximal ideal M of R. If $M = Rm$ for some $m \in M$, then $\pi = u_1 m$ and $\pi' = u_2 m$ for some $u_1, u_2 \in U(R)$. This implies that π and π' are associates of each other. This is a contradiction, since $R\pi \neq R\pi'$ and so, M is not principal. □

Theorem 2.9. *The following statements are equivalent:*

(1) *Every ideal of R is principal (that is, R is a principal ideal domain (PID)).*

(2) *Every maximal ideal of R is principal.*

(3) $\mathbb{MGI}(R)$ *has no edges.*

Proof. $(1) \Rightarrow (2)$ This is clear.

$(2) \Rightarrow (3)$ This follows by Lemma 2.8.

$(3) \Rightarrow (1)$ By Lemma 2.7, we get that every prime ideal of R is principal. Hence, every ideal of R is principal by [8, Exercise 10, page 8] (that is, R is a PID). □

We end this section with the following proposition in which we provide one more necessary condition such that $\mathbb{MGI}(R)$ is connected.

Proposition 2.10. *If $\mathbb{MGI}(R)$ is connected, then given $M \in Max(R)$, there exist $R\pi \in \mathcal{I}(R)$ and $N \in Max(R)\backslash\{M\}$ such that $\pi \in M \cap N$.*

Proof. Assume that $\mathbb{MGI}(R)$ is connected. Let $M \in Max(R)$. As $M \not\subseteq J(R)$ and $M \in Spec(R)$, there exists $R\pi \in \mathcal{I}(R)$ with $\pi \in M$ by Lemma 2.1. If $M \subseteq \bigcup_{N \in Max(R)\backslash\{M\}} N$, then there exists $N \in Max(R)\backslash\{M\}$ with $\pi \in N$ and so, $\pi \in M \cap N$. Hence, we assume that $M \not\subseteq \bigcup_{N \in Max(R)\backslash\{M\}} N$. Choose an element $m \in M$ with $m \notin \bigcup_{N \in Max(R)\backslash\{M\}} N$. By the choice of m, it follows that no irreducible divisor of m in R can belong to N for any maximal ideal N of R with $N \neq M$. Since $M \in Spec(R)$, we can choose an irreducible divisor π_1 of m in R with $R\pi_1 \in \mathcal{I}(R)$ and $\pi_1 \in M$. Let $N \in Max(R)\backslash\{M\}$. Note that $R\pi_1 + N = R$. Now, $r\pi_1 + s = 1$ for some $r \in R$ and $s \in N$. Since R is atomic and $N \in Spec(R)$, there exists $\pi_2 \in Irr(R)$ with π_2 is a divisor of s in R and $\pi_2 \in N$. Observe that $R\pi_1 + R\pi_2 = R$ and $R\pi_2 \in \mathcal{I}(R)$. Thus $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{MGI}(R)$. By assumption, $\mathbb{MGI}(R)$ is connected. Let $d(R\pi_1, R\pi_2) = k$ in $\mathbb{MGI}(R)$. Then $k \geq 2$ and there are elements $R\pi_{11}, \ldots, R\pi_{1k-1} \in \mathcal{I}(R)$ with $R\pi_1 - R\pi_{11} - \ldots - R\pi_{1k-1} - R\pi_2$ is a path of length k between $R\pi_1$ and $R\pi_2$ in $\mathbb{MGI}(R)$. As $R\pi_1 + R\pi_{11}$ is a subset of a maximal ideal of R, we obtain that $\pi_{11} \in M$ by the choice of π_1. If $k = 2$, then as $\pi_2 \notin M$, $R\pi_{11} + R\pi_2 \subseteq N'$ for some $N' \in Max(R)$ with $N' \neq M$. Hence, $\pi_{11} \in M \cap N'$. If $k \geq 3$, then $\pi_{12} \notin M$, as $d(\pi_1, \pi_2) = k$ by assumption. From $R\pi_{11} + R\pi_{12}$ is a subset of a maximal ideal of R, we can find $N'' \in Max(R)$ with $N'' \neq M$ and $R\pi_{11} + R\pi_{12} \subseteq N''$. Hence, $\pi_{11} \in M \cap N''$.

This shows that if $\mathbb{MGI}(R)$ is connected, then given $M \in Max(R)$, there exist $R\pi \in \mathcal{I}(R)$ and $N \in Max(R)\backslash\{M\}$ such that $\pi \in M \cap N$. □

3. Some sufficient conditions on $Max(R)$ so that $\mathbb{MGI}(R)$ is connected and some related results

As in Section 2, unless otherwise specified, we use R to denote an atomic domain with at least two maximal ideals. In this section, we provide some sufficient conditions on $Max(R)$ such that $\mathbb{MGI}(R)$ is connected.

For the sake of convenience, we introduce the following definition. We say that $Max(R)$ *satisfies* (SC_1) if given any two distinct $M, N \in Max(R)$, then there exists $R\pi \in \mathcal{I}(R)$ with $\pi \in M \cap N$.

Lemma 3.1. *If $Max(R)$ satisfies (SC_1), then $\mathbb{MGI}(R)$ is connected and its diameter is at most two.*

Proof. Let $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ be distinct. We can assume that $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{MGI}(R)$. Note that there exist $M, N \in Max(R)$ such that $\pi_1 \in M$ and $\pi_2 \in N$. As $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{MGI}(R)$ by assumption, it follows that $M \neq N$. As $Max(R)$ satisfies (SC_1) by assumption, there exists $R\pi \in \mathcal{I}(R)$ with $\pi \in M \cap N$. Hence, $R\pi_1 + R\pi \subseteq M$ and $R\pi + R\pi_2 \subseteq N$ and so, $R\pi_1 - R\pi - R\pi_2$ is a path of length 2 between $R\pi_1$ and $R\pi_2$ in $\mathbb{MGI}(R)$. This shows that $\mathbb{MGI}(R)$ is connected and $diam(\mathbb{MGI}(R)) \leq 2$. □

If $Max(R)$ satisfies (SC_1), then in Corollary 3.6, we prove $diam(\mathbb{MGI}(R)) = 2 = r(\mathbb{MGI}(R))$ with the help of the following results.

Lemma 3.2. *For a simple graph $G = (V, E)$ with $|V| \geq 2$, if both G and G^c are connected, then $r(G^c) \geq 2$ and $r(G) \geq 2$.*

Proof. Let $v \in V$. Since $|V| \geq 2$ and G is connected by hypothesis, we can find $u \in V$ such that v and u are adjacent in G and so, $d_{G^c}(v, u) \geq 2$. Hence, we get that $e_{G^c}(v) \geq 2$. Therefore, $r(G^c) \geq 2$. Similarly, it can be shown that $r(G) \geq 2$. □

Note that $(\mathbb{MGI}(R))^c = \mathbb{CGI}(R)$. In Proposition 3.4, we prove $\mathbb{CGI}(R)$ is connected and $diam(\mathbb{CGI}(R)) \leq 3$ so that we can apply Lemma 3.2 when $\mathbb{MGI}(R)$ is connected. We use the following lemma in its proof.

Lemma 3.3. *If $\pi_1\pi_2 \notin J(R)$ for some distinct $R\pi_1, R\pi_2 \in \mathcal{I}(R)$, then there is a path of length at most two between $R\pi_1$ and $R\pi_2$ in $\mathbb{CGI}(R)$.*

Proof. Note that $\pi_1\pi_2 \notin M$ for some $M \in Max(R)$, since $\pi_1\pi_2 \notin J(R)$ by assumption. Hence, $R\pi_1\pi_2 + M = R$. Now, $r\pi_1\pi_2 + m = 1$ for some $r \in R$ and $m \in M$. Since $m \notin J(R)$ and R is atomic by hypothesis, we can find an irreducible factor π of m in R with $R\pi \in \mathcal{I}(R)$ and $\pi \in M$. Observe that $R\pi_1 + R\pi = R = R\pi_2 + R\pi$. Hence, $R\pi_1 - R\pi - R\pi_2$ is a path between $R\pi_1$ and $R\pi_2$ in $\mathbb{CGI}(R)$. □

Proposition 3.4. $\mathbb{CGI}(R)$ *is connected and* $diam(\mathbb{CGI}(R)) \leq 3$.

Proof. Let $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ be distinct. Using arguments found in the proof of [10, Theorem 3.1], we show that there exists a path of length at most three between $R\pi_1$ and $R\pi_2$ in $\mathbb{CGI}(R)$. We can assume that $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{CGI}(R)$.

If $\pi_1\pi_2 \notin J(R)$, then there exists a path of length two between $R\pi_1$ and $R\pi_2$ in $\mathbb{CGI}(R)$ by Lemma 3.3.

Assume that $\pi_1\pi_2 \in J(R)$. Since $\pi_1, \pi_2 \notin J(R)$, there exist $M_1, M_2 \in Max(R)$ such that $\pi_1 \notin M_1$ and $\pi_2 \notin M_2$. As $\pi_1\pi_2 \in J(R)$, we get that $\pi_1 \in M_2$ and $\pi_2 \in M_1$. Now, $r\pi_1 + m_1 = 1$ for some $r \in R$ and $m_1 \in M_1$, since $R\pi_1 + M_1 = R$. Note that $m_1 \notin M_2$ and $R\pi_1 + Rm_1 = R$. For any irreducible divisor π of m_1 in R, $\pi \notin M_2$, and $R\pi_1 + R\pi = R$. Note that there exists at least one irreducible divisor π of m_1 in R such that $\pi \in M_1$. Observe that $R\pi_1 - R\pi$ is an edge of $\mathbb{CGI}(R)$. Since $\pi\pi_2 \notin M_2$, $\pi\pi_2 \notin J(R)$. As $R\pi_2 + R\pi \subseteq M_1$, $R\pi$ and $R\pi_2$ are not adjacent in

$\mathbb{CGI}(R)$. Note that there exists $R\pi' \in \mathcal{I}(R)$ such that $R\pi - R\pi' - R\pi_2$ is a path P of length two between $R\pi$ and $R\pi_2$ in $\mathbb{CGI}(R)$ by Lemma 3.3. The union of the edge $R\pi_1 - R\pi$ and the path P between $R\pi$ and $R\pi_2$ gives a path of length three between $R\pi_1$ and $R\pi_2$ in $\mathbb{CGI}(R)$.

This proves that $\mathbb{CGI}(R)$ is connected and $diam(\mathbb{CGI}(R)) \leq 3$. □

The following corollaries are needed in our future discussion.

Corollary 3.5. *$Diam(\mathbb{CGI}(R)) = 3$ if and only if there exist $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ with $R\pi_1 \neq R\pi_2$ such that $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{CGI}(R)$ and $\pi_1\pi_2 \in J(R)$.*

Proof. Assume that $diam(\mathbb{CGI}(R)) = 3$. Then it follows from Proposition 3.4 that there exist distinct $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ such that $d(R\pi_1, R\pi_2) = 3$ in $\mathbb{CGI}(R)$. Hence, $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{CGI}(R)$ and by Lemma 3.3, it follows that $\pi_1\pi_2 \in J(R)$.

Conversely, assume that there exist distinct $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ such that $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{CGI}(R)$ and $\pi_1\pi_2 \in J(R)$. It is enough to show that there is no path of length two between $R\pi_1$ and $R\pi_2$ in $\mathbb{CGI}(R)$, since $diam(\mathbb{CGI}(R)) \leq 3$. If $R\pi_3 \in \mathcal{I}(R)$ is such that $R\pi_1$ and $R\pi_3$ are adjacent in $\mathbb{CGI}(R)$, then $R\pi_1 + R\pi_3 = R$ and so, $R\pi_2 = R\pi_1\pi_2 + R\pi_3\pi_2$. If $M \in Max(R)$ is such that $\pi_3 \in M$, then $\pi_2\pi_3, \pi_1\pi_2 \in M$ and therefore, $\pi_2 \in M$. Hence, $R\pi_3 + R\pi_2 \subseteq M$ and so, $R\pi_3$ and $R\pi_2$ are not adjacent in $\mathbb{CGI}(R)$. This proves that $d(R\pi_1, R\pi_2) \geq 3$ in $\mathbb{CGI}(R)$ and so, $d(R\pi_1, R\pi_2) = 3$ in $\mathbb{CGI}(R)$. Hence, $diam(\mathbb{CGI}(R)) = 3$. □

Corollary 3.6. *If $Max(R)$ satisfies (SC_1), then $\mathbb{MGI}(R)$ is connected and $diam(\mathbb{MGI}(R)) = 2 = r(\mathbb{MGI}(R))$.*

Proof. Assume that $Max(R)$ satisfies (SC_1). Then $\mathbb{MGI}(R)$ is connected and $diam(\mathbb{MGI}(R)) \leq 2$ by Lemma 3.1. Note that $(\mathbb{MGI}(R))^c = \mathbb{CGI}(R)$ is connected by Proposition 3.4. By applying Lemma 3.2 with $G = \mathbb{MGI}(R)$, we obtain that $r(\mathbb{MGI}(R)) \geq 2$. If G is any connected graph, then it is well-known that $diam(G) \geq r(G)$. Hence, $diam(\mathbb{MGI}(R)) = 2 = r(\mathbb{MGI}(R))$. □

In Corollary 3.8, we provide another sufficient condition so that $\mathbb{MGI}(R)$ is connected. We use the following lemma in its proof.

Lemma 3.7. *If a simple graph G is not connected, then G^c is connected and $diam(G^c) \leq 2$.*

Proof. This follows from the proof of [2, Theorem 1.5.7]. □

Corollary 3.8. *If $diam(\mathbb{CGI}(R)) = 3$, then $\mathbb{MGI}(R)$ is connected.*

Proof. If $\mathbb{MGI}(R)$ is not connected, then as $(\mathbb{MGI}(R))^c = \mathbb{CGI}(R)$, we get that $diam(\mathbb{CGI}(R)) \leq 2$ by Lemma 3.7. This contradicts the assumption $diam(\mathbb{CGI}(R)) = 3$. Therefore, $\mathbb{MGI}(R)$ is connected. □

If $|Max(R)| = 3$, then we prove in the following theorem that $\mathbb{MGI}(R)$ is connected if and only if $diam(\mathbb{CGI}(R)) = 3$.

Theorem 3.9. *If R is an atomic domain with $|Max(R)| = 3$, then the following statements are equivalent:*

(1) $\mathbb{MGI}(R)$ *is connected.*

(2) *If $M \in Max(R)$, then there exist $R\pi \in \mathcal{I}(R)$ and $N \in Max(R)\backslash\{M\}$ with $\pi \in M \cap N$.*

(3) $diam(\mathbb{CGI}(R)) = 3$.

Proof. Assume that $Max(R) = \{M_1, M_2, M_3\}$.

$(1) \Rightarrow (2)$ This follows from Proposition 2.10. (For this part of the proof, we do not need the assumption that $|Max(R)| = 3$.)

$(2) \Rightarrow (3)$ By (2), there exist $R\pi \in \mathcal{I}(R)$ and $M_k \in Max(R)\backslash\{M_1\}$ with $\pi \in M_1 \cap M_k$. Observe that k is either 2 or 3. For $j \in \{1, 2, 3\}\backslash\{1, k\}$, again by (2), there exist $R\pi' \in \mathcal{I}(R)$, $M_{j'} \in Max(R)$ for some $j' \in \{1, 2, 3\}\backslash\{j\}$ with $\pi' \in M_j \cap M_{j'}$. Observe that $R\pi \neq R\pi'$, since $\pi \notin M_j$ and $\pi' \in M_j$. As $j' \in \{1, k\}$, we get that $R\pi + R\pi' \neq R$. Thus $R\pi$ and $R\pi'$ are not adjacent in $\mathbb{CGI}(R)$. Note that $\pi\pi' \in J(R)$. Hence, $diam(\mathbb{CGI}(R)) = 3$ by Corollary 3.5.

$(3) \Rightarrow (1)$ This follows from Corollary 3.8. □

In the following example, we provide a unique factorization domain (UFD) R such that $Max(R)$ satisfies (SC_1) and $diam(\mathbb{CGI}(R)) = 3$.

Example 3.10. *Let $n \geq 3$. Let $p_1, p_2, p_3, \ldots, p_n$ be distinct prime numbers with $p_1 = 2$. With $T = \mathbb{Z}[X]$, let $M_1 = T2 + T(X-1)$, $M_2 = Tp_2 + TX$, $M_3 = Tp_3 + TX, \ldots, M_n = Tp_n + TX$ and $S = T\backslash(\bigcup_{i=1}^{n} M_i)$. Then $R = S^{-1}T$ is a UFD, $Max(R)$ satisfies (SC_1), $\mathbb{MGI}(R)$ is connected with $diam(\mathbb{MGI}(R)) = 2 = r(\mathbb{MGI}(R))$, and $diam(\mathbb{CGI}(R)) = 3$.*

Proof. It is well-known that T is a UFD. Note that $M_1, M_2, M_3, \ldots, M_n$ are maximal ideals of T and are pairwise distinct. As T is a UFD and $S = T\backslash(\bigcup_{i=1}^{n} M_i)$ is a multiplicatively closed subset (m.c. subset) of T, $R = S^{-1}T$ is a UFD follows by applying [1, Proposition 3.11(iv)] and [8, Theorem 5]. Hence, R is atomic. By [1, Proposition 1.11(i)], it follows that $\{M_i \mid i \in \{1, 2, 3, \ldots, n\}\}$ is the set of prime ideals of T maximal with respect to not meeting S and so, we obtain by [1, Proposition 3.11(iv)] that $Max(R) = \{S^{-1}M_i \mid i \in \{1, 2, 3, \ldots, n\}\}$. Thus $|Max(R)| = n$. Observe that $TX \in Spec(T)$ and $TX \cap S = \emptyset$ and so, $S^{-1}TX \in Spec(R)$. Hence, X is a prime element of R. Note that $X \notin J(R)$, since $X \notin S^{-1}M_1$. Thus $RX \in \mathcal{I}(R)$. Note that $X \in \bigcap_{k=2}^{n} S^{-1}M_k$. For $2 \leq k \leq n$, note that $p_k = 2t_k + 1$ for some $t_k \in \mathbb{N}$ and $X - p_k \in M_1 \cap M_k$ and $X - p_k \notin M_j$ for any j with $2 \leq j \leq n$ and $j \neq k$. As $T(X - p_k) \in Spec(T)$ and it does not meet S, we get that $X - p_k$ is a prime element of R. Since $X - p_k \notin J(R)$, we arrive at $R(X - p_k) \in \mathcal{I}(R)$. It is clear that $X - p_k \in S^{-1}M_1 \cap S^{-1}M_k$. This shows that $Max(R)$ satisfies (SC_1) and hence, $\mathbb{MGI}(R)$ is connected with $diam(\mathbb{MGI}(R)) = r(\mathbb{MGI}(R)) = 2$ by Corollary 3.6.

For $2 \leq k \leq n$, note that $X \in S^{-1}M_k$ and $S^{-1}M_k = Rp_k + RX = R(X - p_k) + RX$. Also, $X - p_k \in S^{-1}M_1$, and so, $(X - p_k)X \in \bigcap_{i=1}^{n} S^{-1}M_i = J(R)$. Hence, by Corollary 3.5, we obtain that $diam(\mathbb{CGI}(R)) = 3$. □

In the following example, we provide a UFD R with $|Max(R)| = 3$ but $Max(R)$ does not satisfy (SC_1).

Example 3.11. *Consider $T = \mathbb{Z}[X]$ and the prime ideals of T given by $P_1 = 2T, P_2 = T3 + TX$, and $P_3 = T5 + TX$. If $S = T\backslash(\bigcup_{i=1}^{3} P_i)$, then $R = S^{-1}T$ is a UFD, $|Max(R)| = 3$, and $Max(R)$ does not satisfy (SC_1).*

Proof. Note that $P_1 \in Spec(T)$ and $P_2, P_3 \in Max(T)$. Observe that P_i and P_j are not comparable under inclusion for all distinct i, j with $1 \leq i, j \leq 3$. As $S = T\backslash(\bigcup_{i=1}^{3} P_i)$ is a m.c. subset of T, T is a UFD, we get that $R = S^{-1}T$ is a UFD by applying [1, Proposition 3.11(iv)] and [8, Theorem 5]. Hence, R is atomic. It can be shown using [1, Proposition 1.11(i)] that the set of prime ideals of T maximal with respect to not meeting S equals $\{P_i \mid 1 \leq i \leq 3\}$. Hence, by [1, Proposition 3.11(iv)], we obtain that $Max(S^{-1}T) = \{S^{-1}P_i \mid 1 \leq i \leq 3\}$ and so, $|Max(R)| = 3$. As the maximal ideal $S^{-1}P_1 = S^{-1}2T$ of R is principal, $\mathbb{MGI}(R)$ is not connected by Lemma 2.5. With $M_1 = S^{-1}P_1$ and $M_2 = S^{-1}P_2$, note that $M_1, M_2 \in Max(R)$ and are distinct. If $\pi \in Irr(R) \cap M_1$, then $\pi = 2u$ for some $u \in U(R)$. As $M_1 \neq M_2$, it follows that $2 \notin M_2$ and so, $\pi \notin M_2$. This shows that $Max(R)$ does not satisfy (SC_1). Since $\mathbb{MGI}(R)$ is not connected, one also use Lemma 3.1 to conclude that $Max(R)$ does not satisfy (SC_1). □

In the following example, we provide a UFD R such that $Max(R)$ is not finite and $Max(R)$ does not satisfy (SC_1).

Example 3.12. *If $T = \mathbb{Z}_{2\mathbb{Z}}$, then $R = T[X]$ is a UFD, $Max(R)$ is not finite, and $Max(R)$ does not satisfy (SC_1).*

Proof. Note that $T = \mathbb{Z}_{2\mathbb{Z}}$ is a local PID. Hence, $R = T[X]$ is a UFD. Observe that $J(R) = (0)$ by [1, Exercise 4, p.11] and so, $Max(R)$ is not finite. Since $\frac{R}{R(1+2X)} \cong \mathbb{Q}$ as rings, we get that $R(1 + 2X) \in Max(R)$. Since $R(1 + 2X)$ is principal, $\mathbb{MGI}(R)$ is not connected by Lemma 2.5. Hence, $Max(R)$ does not satisfy (SC_1) by Lemma 3.1. □

For a ring T and $f(X) \in T[X]\backslash\{0\}$, the degree of $f(X)$ is denoted by $deg(f(X))$.

In Example 3.14, we provide a Noetherian domain R such that $Max(R)$ is not finite and $Max(R)$ satisfies (SC_1). We use the following lemma in its verification.

Lemma 3.13. *The subring $R = K[X^2, X^3]$ of the ring $T = K[X]$ (where K is a field) is a Noetherian domain, $dim R = 1$, $M = X^2K[X] \in Max(R)$, $Max(R)$ is not finite, and for each $\alpha \in K$, $(X - \alpha)X^2$ is an irreducible element of R.*

Proof. It is well-known that $T = K[X]$ is a PID. Observe that R is a domain, since R is a subring of T. By [1, Corollary 7.7], $R = K[X^2, X^3]$ is Noetherian and observe that $R = K + X^2K[X]$. Thus $M = X^2K[X]$ is an ideal of R and $M = X^2K[X] \in Max(R)$, since $\frac{R}{X^2K[X]} \cong K$ as rings and K is a field. Observe that $T = K[X] = R + RX$ is a finitely generated R-module. For any $f(X) \in T$, $R[f(X)] \subseteq T$ and T is a finitely generated R-module. Hence, $f(X)$ is integral over R by $(iii) \Rightarrow (i)$ of [1, Proposition 5.1]. This shows that T is an integral extension of R. By [6, 11.8], we get that $dim R = 1$, since $dim T = 1$. Note that $U(T) = U(R) = K^*$. If $r \in J(R)$, then $1 - r \in U(R)$ by [1, Proposition 1.9]. Hence, $1 - r = \alpha$ for some $\alpha \in K^*$ and so, $r = 1 - \alpha \in J(R) \cap K = (0)$. This shows that $J(R) = (0)$. This implies that $Max(R)$ is not finite.

For any $\alpha \in K$, we claim that $(X-\alpha)X^2 \in Irr(R)$. Let $r_1, r_2 \in R$ be such that $(X-\alpha)X^2 = r_1r_2$. Since $(X-\alpha)X^2 \in M$ and $M \in Max(R)$, either r_1 or r_2 belongs to M. If $r_1 \in M$, then $r_1 = X^2f(X)$ for some $f(X) \in T$. Hence, $(X-\alpha)X^2 = X^2f(X)r_2$ and so, $X-\alpha = f(X)r_2$. This implies that $1 = deg(f(X)) + deg(r_2)$. From $X \notin R$, we get that $deg(r_2) = 0$ and so, $r_2 \in K^* = U(R)$. Similarly, if $r_2 \in M$, then it can be shown that $r_1 \in U(R)$. This shows that $(X-\alpha)X^2 \in Irr(R)$. □

Example 3.14. *If $R = \mathbb{C}[X^2, X^3]$, then $Max(R)$ is not finite and $Max(R)$ satisfies (SC_1).*

Proof. From the proof of Lemma 3.13, we get that R is a Noetherian domain, $T = \mathbb{C}[X]$ is an integral extension of R, $dimR = 1$, $M = X^2\mathbb{C}[X] \in Max(R)$, and $J(R) = (0)$. Hence, $Max(R)$ is not finite. We now verify that $Max(R)$ satisfies (SC_1). Consider $M_1, M_2 \in Max(R)$ with $M_1 \neq M_2$. Since T is an integral extension of R, there exist $N_1, N_2 \in Spec(T)$ with $N_i \cap R = M_i$ for $1 \leq i \leq 2$ by [1, Theorem 5.10]. Note that $N_1 \neq N_2$ and $N_1, N_2 \in Max(T)$, since $dimT = 1$. As $\mathbb{C}$ is an algebraically closed field, we obtain that $N_1 = T(X-\alpha_1)$ and $N_2 = T(X-\alpha_2)$ for some $\alpha_1, \alpha_2 \in \mathbb{C}$ by [8, Theorem 32]. Observe that $\alpha_1 \neq \alpha_2$. Either $\alpha_i = 0$ for some i between 1 and 2 or both α_1 and α_2 are not equal to 0.

Assume that exactly one between α_1 and α_2 equals 0. We can assume $\alpha_1 = 0$ but $\alpha_2 \neq 0$. Then $N_1 = TX$ and $M_1 = X^2\mathbb{C}[X] = M$. Observe that $(X-\alpha_2)X^2 \in Irr(R)$ by Lemma 3.13 and $R\pi \in \mathcal{I}(R)$ with $\pi = (X-\alpha_2)X^2$. It is clear that $\pi \in \bigcap_{i=1}^{2} M_i$.

Assume that $\alpha_i \neq 0$ for each i, $1 \leq i \leq 2$. Either $\alpha_1 + \alpha_2 = 0$ or $\alpha_1 + \alpha_2 \neq 0$. If $\alpha_1 + \alpha_2 = 0$, then $\prod_{i=1}^{2}(X-\alpha_i) = X^2 - \alpha_1^2 \in R$. We claim that $X^2 - \alpha_1^2 \in Irr(R)$. Assume that $X^2 - \alpha_1^2 = r_1r_2$ for some $r_1, r_2 \in R$. Hence, $2 = deg(X^2 - \alpha_1^2) = deg(r_1) + deg(r_2)$. Observe that $deg(r_i) = 0$ for some i with $1 \leq i \leq 2$, since R does not contain any $g(X) \in T$ with $deg(g(X)) = 1$. Hence, one between r_1 and r_2 is a unit in R and so, $X^2 - \alpha_1^2 \in Irr(R)$. With $\pi = X^2 - \alpha_1^2$, we get that $R\pi \in \mathcal{I}(R)$ and $\pi \in (\bigcap_{i=1}^{2} N_i) \cap R = \bigcap_{i=1}^{2} M_i$.

If $\alpha_1 + \alpha_2 \neq 0$, then we claim that $(\prod_{i=1}^{2}(X-\alpha_i))X^2 \in Irr(R)$. Observe that $(\prod_{i=1}^{2}(X-\alpha_i))X^2 \in M \subset R$. Assume that $(\prod_{i=1}^{2}(X-\alpha_i))X^2 = r_1r_2$ for some $r_1, r_2 \in R$. Either r_1 or r_2 belongs to M. If $r_1 \in M$, then $r_1 = X^2f(X)$ for some $f(X) \in T$. Hence, we get that $\prod_{i=1}^{2}(X-\alpha_i) = f(X)r_2$. Note that $2 = deg(\prod_{i=1}^{2}(X-\alpha_i)) = deg(f(X)) + deg(r_2)$. This implies that $deg(r_2) = 0$ or 2, since R does not contain any $g(X) \in T$ with $deg(g(X)) = 1$. As $0 \neq -(\alpha_1 + \alpha_2) =$ the coefficient of X in $\prod_{i=1}^{2}(X-\alpha_i)$ = the coefficient of X in $f(X)$ multiplied by the constant term of r_2, we get that $f(X)$ must be of positive degree. Hence, $deg(r_2) = 0$ and so, r_2 is a unit in R. Similarly, if $r_2 \in M$, then it can be shown that r_1 is a unit in R. This shows that $(\prod_{i=1}^{2}(X-\alpha_i))X^2 \in Irr(R)$. With $\pi = (\prod_{i=1}^{2}(X-\alpha_i))X^2$, we obtain that $R\pi \in \mathcal{I}(R)$ and $\pi == ((X-\alpha_1)X)((X-\alpha_2)X) \in (N_1 \cap N_2) \cap R = \bigcap_{i=1}^{2} M_i$.

From the above discussion, we obtain that $Max(R)$ satisfies (SC_1). Hence by Corollary 3.6, we get that $\mathbb{MGI}(R)$ is connected and $diam(\mathbb{MGI}(R)) = 2 = r(\mathbb{MGI}(R))$. □

If $\mathbb{C}$ is replaced by $\mathbb{R}$ in the previous example, then we verify in the following example that the conclusion of the previous example fails to hold.

Example 3.15. *If $K = \mathbb{R}$ and T, R are as in Lemma 3.13, then $Max(R)$ does not satisfy (SC_1).*

Proof. In the notation of the statement of Lemma 3.13, $T = \mathbb{R}[X], R = \mathbb{R}[X^2, X^3] = \mathbb{R} + X^2\mathbb{R}[X]$. Note that $T(X^2+1) \in Max(T)$, since X^2+1 is irreducible over $\mathbb{R}$. As $T(X^2+1)\cap R = R(X^2+1)$, we get that $R(X^2+1) \in Max(R)$ and is principal. Hence, $\mathbb{MGI}(R)$ is not connected by Lemma 2.5. Therefore, $Max(R)$ does not satisfy (SC_1) by Lemma 3.1. □

Note that $|\mathcal{I}(R)| \geq 2$ by Lemma 2.2. If $\mathbb{MGI}(R)$ is connected, then as $\mathbb{CGI}(R)$ is connected by Proposition 3.4, it follows that $\mathbb{CGI}(R)$ has at least one edge and so, $diam(\mathbb{MGI}(R)) \geq 2$. In fact, we obtain from Lemma 3.2 that $r(\mathbb{MGI}(R)) \geq 2$. It is natural to ask whether there is a necessary and sufficient condition so that $\mathbb{MGI}(R)$ is connected with $diam(\mathbb{MGI}(R)) = 2$. The following proposition provides an answer to this question in the case $3 \leq |Max(R)| < \infty$.

Proposition 3.16. *If $3 \leq |Max(R)| < \infty$, then the following statements are equivalent:*
(1) $\mathbb{MGI}(R)$ *is connected and* $diam(\mathbb{MGI}(R)) = 2$.
(2) $Max(R)$ *satisfies* (SC_1).
Moreover, if (2) *holds, then* $r(\mathbb{MGI}(R)) = 2$.

Proof. Assume that R has exactly n maximal ideals with $n \in \mathbb{N}$ ($n \geq 3$) and $Max(R) = \{M_1, M_2, M_3, \ldots, M_n\}$.

(1) ⇒ (2) Assume that $\mathbb{MGI}(R)$ is connected and $diam(\mathbb{MGI}(R)) = 2$. Consider $M_i, M_j \in Max(R)$ with $M_i \neq M_j$. By Lemma 2.2, we know that there exist $\pi_i, \pi_j \in Irr(R)$ such that M_i (respectively, M_j) is the only maximal ideal of R which contains π_i (respectively, π_j). Therefore, $R\pi_i, R\pi_j \in \mathcal{I}(R)$ and $R\pi_i + R\pi_j$ is not contained in any maximal ideal of R. Hence, $R\pi_i + R\pi_j = R$. This implies that $R\pi_i$ and $R\pi_j$ are not adjacent in $\mathbb{MGI}(R)$. As $diam(\mathbb{MGI}(R)) = 2$ by assumption, there exists $R\pi \in \mathcal{I}(R)$ with $R\pi$ is adjacent to both $R\pi_i$ and $R\pi_j$ in $\mathbb{MGI}(R)$. From the choice of π_i and π_j, it is now clear that $\pi \in M_i \cap M_j$.

(2) ⇒ (1) Assume that $Max(R)$ satisfies (SC_1). Then by Corollary 3.6, we obtain that $\mathbb{MGI}(R)$ is connected and $diam(\mathbb{MGI}(R)) = 2 = r(\mathbb{MGI}(R))$. Note that the proof of (2) ⇒ (1) does not need the assumption $|Max(R)| < \infty$.

Assume that (2) holds. Then it is already noted in the proof of (2) ⇒ (1) of this proposition that $r(\mathbb{MGI}(R)) = 2$. □

4. Some atomic domains R such that $\mathbb{MGI}(R)$ is connected

As in Section 2, unless otherwise specified, we use R to denote an atomic domain with at least two maximal ideals. In this section, we provide some atomic domains R such that $\mathbb{MGI}(R)$ is connected.

Some unique factorization domains R are provided in Section 3 such that $\mathbb{MGI}(R)$ is not connected. Let $n \in \mathbb{N}\backslash\{1\}$ and let $R = K[X_1, X_2, \ldots, X_n]$, the polynomial ring in n variables $X_1, X_2, \ldots, X_n$ over a field K. It is well-known that R is a UFD. Note that $J(R) = (0)$ by [1, Exercise 4, p.11] and so, $Max(R)$ is not finite. It is known that each $M \in Max(R)$ is generated by n elements and cannot be generated by less than n elements (see [12, Theorem 3, p.281 and Theorem 22, p.217]). Thus no maximal ideal of R is principal. We are interested to know whether or not $\mathbb{MGI}(R)$ is connected.

For the sake of convenience, we first introduce the following definition. We say that $Max(R)$ *satisfies* (SC_2), if given any two distinct $M_1, M_2 \in Max(R)$, there exist $M_3 \in Max(R)$ and $R\pi, R\pi' \in \mathcal{I}(R)$ with $\pi \in M_1 \cap M_3$ and $\pi' \in M_2 \cap M_3$.

We verify in the following lemma that (SC_1) implies (SC_2).

Lemma 4.1. *If $Max(R)$ satisfies (SC_1), then $Max(R)$ satisfies (SC_2).*

Proof. Let $M_1, M_2 \in Max(R)$ be distinct. Since $Max(R)$ satisfies (SC_1) by assumption, there exists $R\pi \in \mathcal{I}(R)$ such that $\pi \in M_1 \cap M_2$. With $M_3 = M_1$ and $\pi' = \pi$, we obtain that $R\pi = R\pi' \in \mathcal{I}(R)$, $\pi \in M_1 \cap M_3$, and $\pi' \in M_2 \cap M_3$. Therefore, $Max(R)$ satisfies (SC_2). □

The following proposition is used in Example 4.6 to verify that $\mathbb{MGI}(K[X_1, X_2, \ldots, X_n])$ is connected.

Proposition 4.2. *If $Max(R)$ satisfies (SC_2), then $\mathbb{MGI}(R)$ is connected with $diam(\mathbb{MGI}(R)) \leq 3$ and $r(\mathbb{MGI}(R)) \geq 2$.*

Proof. Assume that $Max(R)$ satisfies (SC_2). Let $R\pi_1, R\pi_2 \in \mathcal{I}(R)$ be distinct. Assume that $R\pi_1$ and $R\pi_2$ are not adjacent in $\mathbb{MGI}(R)$. Hence, $R\pi_1 + R\pi_2 = R$. Since π_1, π_2 are not units in R, we can find $M_1, M_2 \in Max(R)$ with $\pi_1 \in M_1$ and $\pi_2 \in M_2$. As $R\pi_1 + R\pi_2 = R$, it follows that $M_1 \neq M_2$. Since $Max(R)$ satisfies (SC_2) by assumption, there exist $M_3 \in Max(R)$ and $R\pi, R\pi' \in \mathcal{I}(R)$ with $\pi \in M_1 \cap M_3$ and $\pi' \in M_2 \cap M_3$. Note that $R\pi_1 + R\pi \subseteq M_1$, $R\pi_2 + R\pi' \subseteq M_2$, and $R\pi + R\pi' \subseteq M_3$. If $R\pi_1$ and $R\pi'$ are adjacent in $\mathbb{MGI}(R)$, then $R\pi_1 - R\pi' - R\pi_2$ is a path between $R\pi_1$ and $R\pi_2$ in $\mathbb{MGI}(R)$. Similarly, if $R\pi_2$ and $R\pi$ are adjacent in $\mathbb{MGI}(R)$, then $R\pi_1 - R\pi - R\pi_2$ is a path between $R\pi_1$ and $R\pi_2$ in $\mathbb{MGI}(R)$. If $R\pi_1$ and $R\pi'$ are not adjacent in $\mathbb{MGI}(R)$ and $R\pi_2$ and $R\pi$ are not adjacent in $\mathbb{MGI}(R)$, then $R\pi \neq R\pi'$ and $R\pi_1 - R\pi - R\pi' - R\pi_2$ is a path between $R\pi_1$ and $R\pi_2$ in $\mathbb{MGI}(R)$. This shows that $\mathbb{MGI}(R)$ is connected and $diam(\mathbb{MGI}(R)) \leq 3$. Now, by Proposition 3.4 and Lemma 3.2, we get that $r(\mathbb{MGI}(R)) \geq 2$. □

The following example illustrates Proposition 4.2.

Example 4.3. *If K is an algebraically closed field and $R = K[X_1, X_2, \ldots, X_n]$ $(n \geq 2)$, then $\mathbb{MGI}(R)$ is connected with $diam(\mathbb{MGI}(R)) \leq 3$ and $r(\mathbb{MGI}(R)) \geq 2$.*

Proof. It is well-known that R is a UFD and $J(R) = (0)$. Note that $U(R) = K^*$ by [1, Exercise 2(i), p.11]. Let $M_1, M_2 \in Max(R)$ be distinct. Since K is an algebraically closed field, we obtain that there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in K$ (respectively, $\beta_1, \beta_2, \ldots, \beta_n \in K$) with $M_1 = \sum_{i=1}^{n} R(X_i - \alpha_i)$ (respectively, $M_2 = \sum_{i=1}^{n} R(X_i - \beta_i)$) by [8, Theorem 32]. As $M_1 \neq M_2$, we get that $\alpha_k \neq \beta_k$ for some k with $1 \leq k \leq n$. Either $\alpha_i = \beta_i$ for some i with $1 \leq i \leq n$ or $\alpha_i \neq \beta_i$ for each i with $1 \leq i \leq n$.

Assume that $\alpha_i = \beta_i$ for some i with $1 \leq i \leq n$. As $R(X_i - \alpha_i) \in Spec(R)$, with $\pi = X_i - \alpha_i$, we get that $R\pi \in \mathcal{I}(R)$. If $M_3 = R(X_i - \alpha_i) + \sum_{j \in \{1,2,\ldots,n\} \setminus \{i\}} RX_j$, then $M_3 \in Max(R)$ and with $\pi = \pi' = X_i - \alpha_i$, $\pi \in M_1 \cap M_3$ and $\pi' \in M_2 \cap M_3$.

Assume that $\alpha_i \neq \beta_i$ for each i with $1 \leq i \leq n$. Hence, $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$. Observe that $R(X_1 - \beta_1), R(X_2 - \alpha_2) \in Spec(R)$. Hence, with $\pi = X_2 - \alpha_2$ and $\pi' = X_1 - \beta_1$,

$R\pi, R\pi' \in \mathcal{I}(R)$. With $M_3 = R\pi + R\pi'$ if $n = 2$ and $M_3 = R\pi + R\pi' + \sum_{j=3}^{n} RX_j$ if $n \geq 3$, it follows that $M_3 \in Max(R)$, $\pi \in M_1 \cap M_3$, and $\pi' \in M_2 \cap M_3$.

Thus it is shown that given any two distinct $M_1, M_2 \in Max(R)$, there exist $M_3 \in Max(R)$ and $R\pi, R\pi' \in \mathcal{I}(R)$ with $\pi \in M_1 \cap M_3$ and $\pi' \in M_2 \cap M_3$. Hence, $Max(R)$ satisfies (SC_2) and so, $\mathbb{MGI}(R)$ is connected with $diam(\mathbb{MGI}(R)) \leq 3$ and $r(\mathbb{MGI}(R)) \geq 2$ by Proposition 4.2. $\square$

In Example 4.6, for any field K (K is not necessarily algebraically closed), we verify that $R = K[X_1, X_2, \ldots, X_n]$ ($n \geq 2$) is such that $Max(R)$ satisfies (SC_2) and hence, $\mathbb{MGI}(R)$ is connected with $diam(\mathbb{MGI}(R)) \leq 3$ and $r(\mathbb{MGI}(R)) \geq 2$. First, we collect some facts that are needed in our discussion to follow.

If an integral domain T is an integral extension of its subring S, then for any non-zero ideal I of T, $I \cap S \neq (0)$. Choose $t \in I$, $t \neq 0$. We can find $f(X) \in S[X]$, a monic polynomial of least degree satisfied by t over S. Then the constant term of $f(X)$ is not equal to zero and it belongs to $I \cap S$. Hence, $I \cap S \neq (0)$. If T satisfies a.c.c. on principal ideals, then so does S.

The following proposition is needed in the verification of Example 4.6.

Proposition 4.4. *Let S be a subring of a UFD T such that T is an integral extension of S. If $J(T) = (0)$ and $Max(T)$ satisfies (SC_2), then $Max(S)$ also satisfies (SC_2).*

Proof. Assume that T is a UFD, $J(T) = (0)$, and $Max(T)$ satisfies (SC_2). As $J(T) = (0)$, $Max(T)$ is not finite. By hypothesis, S is a subring of T and T is an integral extension of S. Observe that $J(S) = (0)$, since $J(T) \cap S = J(S)$ by [1, Exercise 5(ii), p.67]. Hence, $Max(S)$ is not finite. Since T satisfies a.c.c. on principal ideals, we get that S satisfies a.c.c. on principal ideals and so, S is atomic. Let $M_1, M_2 \in Max(S)$ be distinct. By [1, Theorem 5.10], there exist $N_1, N_2 \in Spec(T)$ with $N_i \cap S = M_i$ for each i, $1 \leq i \leq 2$. Observe that $N_1, N_2 \in Max(T)$ by [1, Corollary 5.8]. Note that $N_1 \neq N_2$, since $M_1 \neq M_2$. Since $Max(T)$ satisfies (SC_2) by assumption, there exist $N_3 \in Max(T)$ and $T\xi, T\xi' \in \mathcal{I}(T)$ with $\xi \in N_1 \cap N_3$ and $\xi' \in N_2 \cap N_3$. As T is a UFD, we obtain that $T\xi, T\xi' \in Spec(T)$. Hence, $P_1 = T\xi \cap S, P_2 = T\xi' \cap S \in Spec(S)$ and $P_i \neq (0)$ for each i, $1 \leq i \leq 2$. Let $M_3 = N_3 \cap S$. By [1, Corollary 5.8], we get that $M_3 \in Max(S)$. Observe that $P_1 = T\xi \cap S \subseteq (N_1 \cap N_3) \cap S = M_1 \cap M_3$ and $P_2 = T\xi' \cap S \subseteq (N_2 \cap N_3) \cap S = M_2 \cap M_3$. Since S is atomic, $J(S) = (0)$, and $P_1, P_2 \in Spec(S)$, there are $S\pi, S\pi' \in \mathcal{I}(S)$ with $\pi \in P_1$ and $\pi' \in P_2$. Thus given any two distinct $M_1, M_2 \in Max(S)$, there exist $M_3 \in Max(S)$ and $S\pi, S\pi' \in \mathcal{I}(S)$ with $\pi \in M_1 \cap M_3$ and $\pi' \in M_2 \cap M_3$. Therefore, $Max(S)$ satisfies (SC_2). $\square$

The following example illustrates that the conclusion of Proposition 4.4 can fail to hold if the hypothesis T is a UFD in its statement is omitted.

Example 4.5. *$Max(\mathbb{C}[X^2, X^3])$ satisfies (SC_2) but $Max(\mathbb{R}[X^2, X^3])$ does not satisfy (SC_2).*

Proof. As $Max(\mathbb{C}[X^2, X^3])$ satisfies (SC_1) by Example 3.14, it satisfies (SC_2) by Lemma 4.1. It is noted in the proof of Example 3.14 that $J(\mathbb{C}[X^2, X^3]) = (0)$. Since $\mathbb{C}$ is a finite extension of $\mathbb{R}$, it follows that $\mathbb{C}[X^2, X^3]$ is a finitely generated $\mathbb{R}[X^2, X^3]$-module. Hence, $\mathbb{C}[X^2, X^3]$ is an integral extension of $\mathbb{R}[X^2, X^3]$ by $(iii) \Rightarrow (i)$ of [1, Proposition 5.1]. It is already verified in Example 3.15 that $\mathbb{MGI}(\mathbb{R}[X^2, X^3])$ is not connected. Therefore, $Max(\mathbb{R}[X^2, X^3])$ does not satisfy (SC_2) by Proposition 4.2. $\square$

Example 4.6. *If $R = K[X_1, X_2, \ldots, X_n]$ ($n \geq 2$), where K is a field which is not necessarily algebraically closed, then $Max(R)$ satisfies (SC_2) and hence, $\mathbb{MGI}(R)$ is connected with $diam(\mathbb{MGI}(R)) \leq 3$ and $r(\mathbb{MGI}(R)) \geq 2$.*

Proof. Let $f(X_1, X_2, \ldots, X_n) \in T = \overline{K}[X_1, X_2, \ldots, X_n]$, where $\overline{K}$ is an algebraic closure of K. Then $f(X_1, X_2, \ldots, X_n)$ is a finite sum of elements of the type $\alpha X_1^{k_1} X_2^{k_2} \cdots X_n^{k_n}$, where $\alpha \in \overline{K}$ and $k_1 \geq 0, k_2 \geq 0, \ldots, k_n \geq 0$. Since $\overline{K}$ is an integral extension of K and $K \subset R$, we get that any $\alpha \in \overline{K}$ is integral over R. Therefore, by [1, Corollary 5.3], we obtain that $f(X_1, X_2, \ldots, X_n)$ is integral over R. Hence, T is an integral extension of R. Now, T is a UFD, $J(T) = (0)$, and it is shown in the proof of Example 4.3 that $Max(T)$ satisfies (SC_2). Hence, $Max(R)$ also satisfies (SC_2) by Proposition 4.4. Therefore, $\mathbb{MGI}(R)$ is connected with $diam(\mathbb{MGI}(R)) \leq 3$ and $r(\mathbb{MGI}(R)) \geq 2$ by Proposition 4.2. □

Note that $T = \mathbb{C}[X_1, X_2, \ldots, X_n]$ ($n \geq 2$) is a UFD, $J(T) = (0)$, and $Max(T)$ satisfies (SC_2) by the proof of Example 4.3. Consider any subring S of T with $R = \mathbb{R}[X_1, X_2, \ldots, X_n] \subseteq S$. Since T is an integral extension of R (indeed, T is a finitely generated R-module), we get that T is an integral extension of S. Hence, by Proposition 4.4, $Max(S)$ also satisfies (SC_2) and so, $\mathbb{MGI}(S)$ is connected with $diam(\mathbb{MGI}(S)) \leq 3$ and $r(\mathbb{MGI}(S)) \geq 2$ by Proposition 4.2. As T is a finitely generated R-module and R is Noetherian, it follows that S is a finitely generated R-module and so, S is Noetherian. If $S = \mathbb{R} + (X_1, X_2, \ldots, X_n)T$, then $Max(S)$ satisfies (SC_2). Note that T and S have the same quotient field, i is integral over S but $i \notin S$. Therefore, S is not integrally closed and so, S is not a UFD.

Assume that I is a non-zero proper ideal of $R = K[X_1, X_2, \ldots, X_n]$ ($n \geq 2$), where K is a field. Consider the subring $S = K + I$ of R. As $U(R) = U(S) = K^*$ and R is Noetherian, we get that S satisfies a.c.c. on principal ideals and so, S is atomic. It follows as in the proof of Lemma 3.13 that $J(S) = (0)$ and so, $Max(S)$ is not finite. We verify in Example 4.8 that $Max(S)$ satisfies (SC_2). We use the following lemma in its proof.

Lemma 4.7. *Let S be a subring of a ring T. If a non-zero proper ideal I of T belongs to $Max(S)$, then given any $M \in Max(S)$, there exists $N \in Max(T)$ such that $N \cap S = M$.*

Proof. Though this lemma is well-known, yet for the sake of completeness, we include a proof of this lemma. Consider any $M \in Max(S)$. Either $M = I$ or $M \neq I$. Assume that $M = I$. Since I is a proper ideal of T, we can find $N \in Max(T)$ with $I \subseteq N$ and so, $M = I \subseteq N \cap S$. Hence, $N \cap S = M$. Assume that $M \neq I$. Since $I \in Max(S)$ by hypothesis, we get that $I \not\subseteq M$. Choose $a \in I \backslash M$. Observe that $MTa \subseteq MI$, since I is an ideal of T. This implies that $MTa \subseteq M$. As $a \notin M$, we obtain that $MT \neq T$. Hence, there exists $N \in Max(T)$ with $MT \subseteq N$ and so, $M \subseteq MT \cap S \subseteq N \cap S$. Therefore, $M = N \cap S$, since $M \in Max(S)$. □

Example 4.8. *Let $R = K[X_1, X_2, \ldots, X_n]$ ($n \geq 2$), where K is a field. Consider a non-zero proper ideal I of R and the subring S of R given by $S = K + I$. Then $Max(S)$ satisfies (SC_2) and hence, $\mathbb{MGI}(S)$ is connected with $diam(\mathbb{MGI}(S)) \leq 3$ and $r(\mathbb{MGI}(S)) \geq 2$. Moreover, there are ideals I of R with $S = K + I$ is not Noetherian.*

Proof. Note that S satisfies a.c.c. on principal ideals and $J(S) = (0)$. Consider any two different $M_1, M_2 \in Max(S)$. It is clear that I is an ideal of both R and S. Since K is a field and $\frac{S}{I} \cong K$ as rings, $I \in Max(S)$. By Lemma 4.7, there exist $N_1, N_2 \in Max(R)$ with $N_i \cap S = M_i$ for each i, $1 \leq i \leq 2$. Note that $N_1 \neq N_2$. As $Max(R)$ satisfies (SC_2) by Example 4.6, there exist $N_3 \in Max(R)$ and $R\pi, R\pi' \in \mathcal{I}(R)$ with $\pi \in N_1 \cap N_3$ and $\pi' \in N_2 \cap N_3$. Now, $N_3 \cap S$ is a proper ideal of S and hence, there exists $M_3 \in Max(S)$ with $N_3 \cap S \subseteq M_3$. If J is any non-zero ideal of R, then JI is a non-zero ideal of both R and S and $JI \subseteq J \cap S$ and so, $J \cap S \neq (0)$. Hence, $R\pi \cap S, R\pi' \cap S$ are non-zero proper ideals of S. Since, R is a UFD, $R\pi, R\pi' \in Spec(R)$ and so, $R\pi \cap S, R\pi' \cap S \in Spec(S)$. As $J(S) = (0)$ and S is atomic, there are $S\xi, S\xi' \in \mathcal{I}(S)$ with $\xi \in R\pi \cap S$ and $\xi' \in R\pi' \cap S$. Note that $S\xi \subseteq R\pi \cap S \subseteq (N_1 \cap N_3) \cap S \subseteq M_1 \cap M_3$ and $S\xi' \subseteq R\pi' \cap S \subseteq (N_2 \cap N_3) \cap S \subseteq M_2 \cap M_3$. Thus given any two different $M_1, M_2 \in Max(S)$, there exist $M_3 \in Max(S)$ and $S\xi, S\xi' \in \mathcal{I}(S)$ with $\xi \in M_1 \cap M_3$ and $\xi' \in M_2 \cap M_3$. Hence, $Max(S)$ satisfies (SC_2) and so, we obtain that $\mathbb{MGI}(S)$ is connected with $diam(\mathbb{MGI}(S)) \leq 3$ and $r(\mathbb{MGI}(S)) \geq 2$.

If I is a non-zero ideal of R with $I \subseteq P$ for some $P \in Spec(R) \backslash Max(R)$, then we claim that S is not Noetherian. Suppose that S is Noetherian. If $a \in I \backslash \{0\}$, then $aR \subseteq I \subset S$. If S is Noetherian, then as $R \subseteq \frac{1}{a}S$, we get that R is a finitely generated S-module. Hence, R is an integral extension of S by $(iii) \Rightarrow (i)$ of [1, Proposition 5.1]. This implies that $\frac{R}{I}$ is an integral extension of $\frac{S}{I}$. Since, $\frac{S}{I} \cong K$ as rings, we get that $dim\frac{S}{I} = 0$ and so, $dim\frac{R}{I} = 0$ by [6, 11.8]. Hence, we obtain that $P \in Max(R)$ and this contradicts the choice of P. This proves that S is not Noetherian. With $I = P = X_1R$, note that $P \in Spec(R) \backslash Max(R)$ and so, $S = K + P$ is not Noetherian. □

5. Conclusion

For an atomic domain R with $|Max(R)| \geq 2$, in this paper, we introduce the graph $\mathbb{MGI}(R)$ and try to find out when $\mathbb{MGI}(R)$ is connected. In such a study, some necessary (respectively, sufficient) conditions are determined such that $\mathbb{MGI}(R)$ is connected and we arrive at some problems for which I am not aware of their solutions. In this section, we mention some of them. Does $(1) \Rightarrow (2)$ of Proposition 3.16 hold even if $Max(R)$ is not finite? Regarding the sufficient conditions (SC_1) and (SC_2), in view of Lemma 4.1, it would be interesting to know whether or not there exists an atomic domain R for which $Max(R)$ satisfies (SC_2) but does not satisfy (SC_1). The exact value of $diam(\mathbb{MGI}(K[X_1, X_2, \ldots, X_n]))$ is not known.

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