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Zonal Labeling of Graphs

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Abstract

A planar graph is said to be zonal when is possible to label its vertices with the nonzero elements of $\mathbb{Z}_3$, in such a way that the sum of the labels of the vertices on the boundary of each zone is 0 in $\mathbb{Z}_3$. In this work we present some conditions that guarantee the existence of a zonal labeling for a number of families of graphs such as unicyclic and outerplanar, including the family of bipartite graphs with connectivity at least 2 whose stable sets have the same cardinality; additionally, we prove that when any edge of a zonal graph is subdivided twice, the resulting graph is zonal as well. Furthermore, we prove that the Cartesian product $G \times P_2$ is zonal, when G is a tree, a unicyclic graph, or certain variety of outerplanar graphs. Besides these results, we determine the number of different zonal labelings of the cycle C_n.

Keywords: zonal graph, zonal labeling
Mathematics Subject Classification : 05C15, 05C30

1. Introduction

This work is motivated by a question proposed by Chartrand et al. [1]; in that monograph, they introduce, to a wider audience, the concept of zonal labeling. A *vertex labeling* of a graph G is a function $f : V(G) \to S$, where S is a set of numbers; if G is a plane graph and R is a zone (or region) of G, the *label* of R is the addition of the labels of the vertices that form its boundary. Let G be a plane graph; the function $\ell : V(G) \to \{1, 2\}$ is called a *zonal labeling* of G if the label of each zone is 0 in $\mathbb{Z}_3$. A plane graph is *zonal* when it admits a zonal labeling. In Figure 1 we

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show a zonal labeling of an Archimedean graph, where the dark vertices are labeled 2 and the light vertices are labeled 1 (or vice versa).

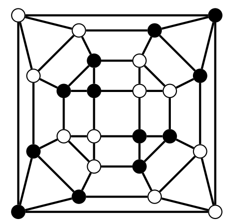

Figure 1. Zonal labeling of the rhombicuboctahedron

Let G be a graph of order n; suppose that ℓ is a zonal labeling of G, the labeling $\overline{\ell}$ of G defined by $\overline{\ell}(v) = 3 - \ell(v)$ for each $v \in V(G)$ is also a zonal labeling of G. Indeed, suppose that F is any zone of G and that its boundary, denoted by ∂F, has m vertices. Since ℓ is zonal, there exists a positive integer ξ such that $\sum_{v\in\partial F} \ell(v) = 3\xi$, then

$$\sum_{v\in\partial F} \overline{\ell}(v) = \sum_{v\in\partial F} (3 - \ell(v)) = \sum_{v\in\partial F} 3 - \sum_{v\in\partial F} \ell(v) = 3m - 3\xi = 3(m - \xi).$$

The function $\overline{\ell}$ is called the *complementary labeling* of ℓ. For the smallest cycles, we have that C_3, C_4, and C_5 only have two different zonal labelings, while C_6 has three different zonal labelings; the labelings of C_6 are self complementary. In order to clarify this point, we show the labelings of these cycles using the symbols x and y in cyclic notation, assuming that if $x = 1$, then $y = 2$ or vice versa. For C_3: (x, x, x), for C_4: (x, x, y, y) and (x, y, x, y); for C_5: (x, x, x, x, y); for C_6: (x, x, x, y, y, y), (x, x, y, x, y, y), and (x, y, x, y, x, y).

Chartrand et al. [1] proved several results about this type of labeling. In particular, they proved all the results in this introduction; we have included them for the sake of completeness and to illustrate the arguments used to prove the existence of such a labeling.

Proposition 1.1. *If T is a nontrivial tree, then T is zonal.*

The proof of this result, given in [1], is very informative, that is the reason why we sketch it here. Suppose that T is a tree of order n. A plane representation of T determines only one zone which boundary is formed by all the vertices of T. If $n \equiv 0 (\text{mod } 3)$, then a zonal labeling of T is obtained by assigning the label 1 to exactly $3k$ vertices, where $0 \le k \le \frac{n}{3}$; thus, the sum of the vertex labels is $3k + 2(n - 3k) = 2n - 3k \equiv 0 (\text{mod } 3)$. If $n \equiv 1 (\text{mod } 3)$, a zonal labeling is obtained by assigning the label 2 to exactly $3k + 2$ vertices, where $0 \le k \le \frac{n-4}{3}$; thus, the label of the unique zone is $1(n - 3k - 2) + 2(3k + 2) = n + 3k + 2$, which is zero in $\mathbb{Z}_3$. Finally, if $n \equiv 2 (\text{mod } 3)$, a zonal labeling is obtained by assigning the label 2 to exactly $3k+1$ vertices; thus, the sum of the vertex labels is $1(n - 3k - 1) + 2(3k + 1) = n + 3k + 1$, which is zero in $\mathbb{Z}_3$ as well. Since the order is the only parameter of T used to obtain the zonal labeling, we can replace

T with a forest, and the results is still valid. If T has order 1, a zonal labeling fails to exist because the unique vertex can be labeled 1 or 2, and neither of them is 0 in $\mathbb{Z}_3$.

Equally important, is the fact that all cycles are zonal graphs. The relevance of this result can be seen in the fact that, for many graphs, the boundary of each zone is a cycle.

Proposition 1.2. *If C is a cycle, then C is zonal.*

As we did before, we do not prove this result, but provide a sketch of a procedure used to obtain a zonal labeling of C_{n+1} if a zonal labeling of C_n is known. The following definitions are taken from the work of Gross [2]. Let w be a vertex of a graph G, $N_G(w)$ be the neighborhood of w, and let U and V be disjoint nonempty subsets of $V(G)$ such that $U \cup V = N_G(w)$. In the graph $G - w$, let every vertex of U be joined to a new vertex u and let every vertex of V be joined to a new vertex v, and join the vertices u and v. This operation is called *splitting graph G at vertex w*, and the resulting graph is called a *split of the graph G at the vertex w*. The vertex w is called the *split vertex*. Since the selection of U and V is arbitrary, the outcome of splitting a vertex is not necessarily unique. For $n \geq 3$, the cycle C_{n+1} is obtained by splitting any vertex of C_n because C_n is 2-regular. This fact can be used to obtain a zonal labeling of C_{3n+1} and C_{3n+2} starting with a zonal labeling of C_{3n}. Since the boundary of the two zones determined by C_{3n} has $3n$ vertices, a zonal labeling can be attained by assigning the label 2 to every vertex of the cycle. If any vertex of C_{3n} is split to obtain C_{3n+1} and the new vertices are labeled 1, then the sum of all the vertex labels that form the boundary is still $3n$ which is 0 in $\mathbb{Z}_3$. We repeat the process by splitting any vertex labeled 2 in C_{3n+1} to get a zonal labeling of C_{3n+2}.

Remark 1.1. Let ℓ be any zonal labeling of C_n; we denote by r the number of vertices labeled 1 and by $s = n - r$ the number of vertices labeled 2. Thus, $r + 2(n - r) = 2n - r$ is the label of each zone determined by C_n; but this number is 0 in $\mathbb{Z}_3$ if and only if $n + r \equiv 0 (\text{mod } 3)$. In other terms,

- If $n \equiv 0(\text{mod } 3)$, then $r, s \equiv 0(\text{mod } 3)$.
- If $n \equiv 1(\text{mod } 3)$, then $r, s \equiv 2(\text{mod } 3)$.
- If $n \equiv 2(\text{mod } 3)$, then $r, s \equiv 1(\text{mod } 3)$.

Therefore, for any value of $n > 3$, there is a zonal labeling of C_n that uses at least once the label 2.

In [1], Chartrand et al. also studied the subfamily of cubic graphs formed by the prisms $C_n \times P_2$, proving that each member of this family is zonal.

Proposition 1.3. *The prism $D_n = C_n \times P_2$ is zonal.*

After this result, Chartrand et al. [1] characterized the family of connected zonal cubic graphs.

Theorem 1.1. *A connected cubic plane graph G is zonal if and only if G is bridgeless.*

Motivated by these results, Chartrand et al. [1] posed the following question:

Question. *Which connected plane graphs are zonal?*

This question is the principal motivation for the present work; here we consider some classes of planar graphs under the perspective of zonal labelings. The results included in Section 2 generalize some of the propositions presented above; in particular, we give some conditions for a bipartite graph to be zonal, characterize the zonal unicyclic graphs, study a variation of zonal labelings for the class of triangulations. In Section 3 we present new results about zonal labelings of outerplanar graphs. In Section 4 we extend the conclusions about the Cartesian product given in [1], by proving that $G \times P_2$ is zonal when G is a tree, zonal outerplanar graph, unicyclic graph, or a 2-connected bipartite graph which stable sets have the same cardinality. We close this work in Section 5, where the concepts of zonal coloring of a cycle and bracelet are connected, in addition we determine the number of zonal colorings of C_n that use the label 1 exactly r times.

2. Extensions of Previous Results

In this section we study the existence of a zonal labeling for certain families of graphs, in particular we characterize the family of zonal unicyclic graphs, and use the concept of inner zonal labeling (introduced in [1]) to prove that every triangulation is zonal and every near-triangulation is inner zonal; we also consider the family of 2-connected bipartite graphs whose stable sets have the same cardinality, proving that they form a family of zonal graphs. Furthermore, we prove that if any edge of a zonal graph is subdivided twice, then the resulting graph is zonal.

Proposition 2.1. *If G is a 2-connected bipartite graph with both stable sets of cardinality n, then G is zonal.*

Proof. Suppose that G is a 2-connected bipartite graph of order $2n$, whose stable sets, denoted by S_1 and S_2, have the same cardinality. Let $\ell : V(G) \to \{1, 2\}$ be a labeling of G, where $\ell(v) = i$ for each $v \in S_i$. Since G is 2-connected, every vertex of G is a vertex of at least one cycle of G. The fact that G is bipartite guarantees that very cycle of G has even order; therefore, the label of any cycle of G is 0 in $\mathbb{Z}_3$ because half of its vertices are labeled 1 and the other half are labeled 2. Consequently, ℓ is a zonal labeling of G. □

An interesting family of graphs that satisfy the hypotheses of this last proposition, is formed by the polyominoes. A *polyomino* is a plane graph, obtained via edge amalgamation, of copies of the cycle C_4 (each copy of C_4 is called *cell*), in such a way that any two cells are disjoint or share an edge. A well-known example of polyomino is the grid graph $P_n \times P_m$. As an extension of the concept of polyomino we have polyhexes; in a *polyhex* each cell is an hexagon (the cycle C_6). This type of graph also satisfies the hypotheses of the last proposition. Thus, honeycombs are zonal graphs too.

Recall that the *girth* of a connected graph, other than a tree, is the length of any of its shortest cycles. In the next proposition we present a characterization of the unicyclic graphs of order n and girth g. Within the proof of this proposition, we assume that if G is a unicyclic graph, then all bridges of G are edges of the boundary of the exterior zone, i.e., the boundary of the exterior zone includes all vertices of G, or the boundary of the interior zone is a cycle. Before the proposition, we prove that all unicyclic graphs of order n and girth $n-1$ are not zonal.

Lemma 2.1. *If G is a unicyclic graph of order n and girth $n-1$, then G is not zonal.*

Proof. By contradiction. Suppose that ℓ is a zonal labeling of G. Any plane representation of G determines two zones, one of these zones has by boundary the cycle C_{n-1}. Thus, if $v_1, v_2, \ldots, v_n$ are the vertices of G and v_n is the only vertex of degree 1, then $\ell(v_1)+\ell(v_2)+\cdots+\ell(v_n) = 3\xi$, for some $\xi \in \mathbb{N}$. Since $\ell(v_n)$ is either 1 or 2, then $\ell(v_1)+\ell(v_2)+\cdots+\ell(v_{n-1})$ is not a multiple of 3. Therefore, one of the two zones has a label other than 0, but this is a contradiction. Consequently, G is not zonal. □

The subfamily of unicyclic graphs of order n and girth $n-1$ constitutes the only instance where a unicyclic graph is not zonal. We prove this statement in the coming proposition.

Proposition 2.2. *A unicyclic graph of order n and girth g is zonal if and only if $n-g \neq 1$.*

Proof. Let G be a unicyclic graph of order n and girth g. Suppose that G is zonal; by Lemma 2.1 we know that $n-g \neq 1$. Suppose now that $n-g \neq 1$. If $n-g = 0$, then G is a cycle; consequently G is zonal. If $n-g > 1$, then the interior zone determined by G has by boundary the unique cycle of G, then we label this cycle using any of its zonal labelings. On the boundary of the exterior zone, there are $n-g > 1$ vertices that have not been labeled; the graph induced by these vertices is a forest of order at least 2, this forest can be labeled following the method described in Proposition 1.1. Thus, the sum of the labels of all these vertices is 0 in $\mathbb{Z}_3$; consequently, the label of the exterior zone is 0 in $\mathbb{Z}_3$, which implies that the labeling of G is zonal. □

In Figure 2 we show an example of a zonal labeling for a unicyclic graph G of order $n = 33$ with girth $g = 8$. In order to simplify the picture, we assume that the label 2 is assigned on each dark vertex, while the label 1 is assigned on each light vertex, or vice versa. Thus, the label of the interior zone is $2 \cdot 1 + 1 \cdot 7 = 9$ and the label of the exterior zone is $2 \cdot 3 + 1 \cdot 30 = 36$. Since 9 and 36 are 0 in $\mathbb{Z}_3$, this is a zonal labeling of G.

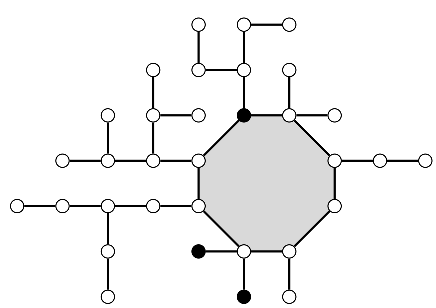

Figure 2. Zonal labeling of a unicyclic graph of order 33 and girth 8

There are two other families of planar graphs, cubic and bicubic maps, studied by Chartrand et al. [1]. A *cubic map* is a connected bridgeless cubic planar graph embedded in the plane. They proved that every cubic map has a zonal labeling. A *bicubic map* is a 2-connected planar graph embedded in the plane all of whose vertices have degree 2 or 3, where the boundary of the exterior region is a cycle C and every vertex lying interior to C has degree 3. For a bicubic map B, a labeling of the vertices of B with the nonzero elements of $\mathbb{Z}_3$ is an *inner zonal labeling* if the label of each interior zone is 0 in $\mathbb{Z}_3$. Any bicubic map that admits such a labeling is called *inner zonal*. Chartrand et al [1] asked the following question.

Question. *Which bicubic maps are inner zonal?*

Certainly, not all bicubic maps are zonal; for example, if any edge of the complete graph K_3 is subdivided once, we get a bicubic map that is not zonal. On the other side, derived from the proof of Proposition 2.4, we have an entire family of zonal bicubic maps; this family is formed by all graphs obtained by subdividing, an even number of times, any number of edges , on the boundary of the exterior zone of the prism $C_n \times P_2$.

A *plane triangulation* is a connected simple plane graph G, such that every zone of G is triangular. A plane graph G is a *near-triangulation*, if G is 2-connected and every zone of G is triangular, except possibly the unbounded zone.

Proposition 2.3. *Every triangulation is zonal. Every near-triangulation is inner zonal.*

Proof. Recall that the cycle C_3 has only two zonal labelings, where all vertices have the same label, either 1 or 2. This implies that if we want to obtain a zonal labeling of a triangulation or a near triangulation, all vertex labels are equal, either 1 or 2. The exterior face of a triangulation has boundary C_3, implying that its label is 0 in $\mathbb{Z}_3$. In the case of a near triangulation, the order of the exterior zone is not always divisible by 3; if it is, then the near-triangulation is zonal, otherwise, it is inner zonal, because the label of each interior zone is 0 in $\mathbb{Z}_3$. □

Consider the five Platonic graphs, the tetrahedron, octahedron, and icosahedron are triangulations, while the cube and dodecahedron (as well as the tetrahedron) are bridgeless cubic graphs. Therefore, all Platonic graphs are zonal. A different situation appears with the Archimedean graphs, that is, those graphs formed by the vertices and edges of the Archimedean solids. There are thirteen of these graphs, seven are bridgeless cubic, therefore they are zonal; of the remaining six, only the rhombicuboctahedron is zonal (as shown in Figure 1), four of the remaining five contain, as induced subgraph, either M_1 or M_2 (depicted in Figure 5), which are not zonal nor inner zonal (as proven in Section 3). The fact that the remaining case, the rhombicosidodecahedron, is not zonal nor inner zonal, is due to the factuality that the cycle C_5, that is the boundary of several zones, only have two zonal labelings, any of these labelings forces the labeling of the adjacent zones. In addition to these thirteen graphs, there are two graph families whose members are Archimedean graphs: prisms and antiprisms. All prisms are zonal, while antiprisms are zonal only when the order of the outerzone is a multiple of 3, otherwise they are inner zonal.

A *subdivision of an edge* $e = uv$ of a graph G is obtained by replacing e with a new vertex w and two new edges uw and wv. Two graphs are said to be *homeomorphic* if both can be obtained from the same graph by subdivisions of some edges. Suppose that G_1 and G_2 are homeomorphic and G_1 is zonal; is G_2 also zonal? In the following proposition we provide a partial answer to this question. We say that the edge e is *subdivided an even number of times* when it is replaced with the path P_{2r}, which consecutive vertices are $w_1, w_2, \ldots, w_{2r}$, and two new edges uw_1 and $w_{2r}v$. For example, a zonal labeling of C_{2n+1} or C_{2n} can be obtained by subdividing, an even number of times, any edge of C_3 or C_4, respectively. In the following result we prove that a zonal graph is obtained when any edge of a zonal graph is subdivided an even number of times.

Proposition 2.4. *If any edge of a zonal graph G is subdivided an even number of times, then the resulting graph is zonal.*

Proof. Suppose that G is a zonal graph and $e = uv$ is an edge of G. Let G' be the graph obtained from G by subdividing $2r$ times the edge e. Assuming that G is labeled using a zonal labeling, any zone that includes e on its boundary has a label that is 0 in $\mathbb{Z}_3$. When the labels 1 and 2 are evenly distributed among the $2r$ vertices introduced by the subdivision, the label of any zone, which boundary includes the vertices u and v, is also 0 in $\mathbb{Z}_3$. Therefore, G' is zonal. □

Chartrand et al. [1] proved that the prism $D_n = C_n \times P_2$ is zonal. The standard plane representation of D_n has the cycle C_n as the boundary of the exterior zone. If the any edge on this cycles is replaced by a path of odd length, where the interior vertices of this path are labeled as was done in Proposition 2.4, we get a zonal graph homeomorphic to D_n. Clearly this process can be done on any number of edges on the boundary of the exterior zone, with exactly the same result. Therefore, the graph obtained with these subdivisions is a zonal bicubic map. In Figure 3 we show two examples of a zonal labeling for the bicubic maps obtained by subdividing, an even number of times, every edge of the outer rim of the prisms D_5 and D_4.

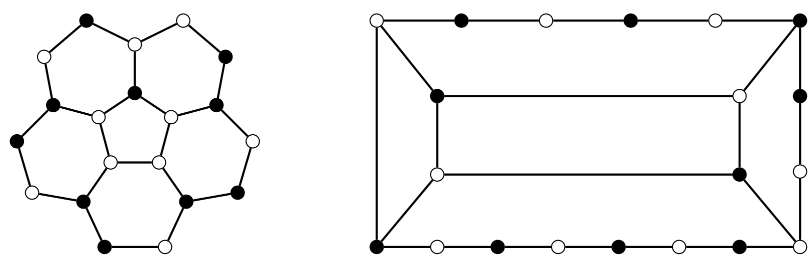

Figure 3. Zonal labelings of bicubic maps homeomorphic to D_5 and D_4, respectively

3. Zonal Labelings of Outerplanar Graphs

A robust family of plane graphs is formed by the outerplanar graphs. An *outerplanar graph* is a graph with an imbedding in the plane such that every vertex appears on the boundary of the exterior zone. Suppose that G is an outerplanar graph of order n and size m; based on propositions 1.2 and 2.3, we know that for every $n \geq 3$, the graph G is zonal when $m = n$, and in the case where $m = 2n - 3$, G is zonal if n is a multiple of 3 and inner zonal otherwise. The last result introduced in [1] includes the family of outerplanar graphs with maximum degree 3; there, Chartrand et al. proved that any member of this family is inner zonal. In the following proposition we stated this result keeping the terminology used in [1].

Theorem 3.1. *Every plane graph G with $\Delta(G) \leq 3$ where the boundary cycle of the exterior zone is a Hamiltonian cycle of G is inner zonal.*

Thus, we need to study the existence of a zonal labeling for all outerplanar graphs of order n and size $n+1 \leq m \leq 2n-4$. We assume that $v_1, v_2, \ldots, v_n$ are the consecutive vertices of the boundary of the exterior zone determined by G. We start this study with the interesting case where $m = n+1$.

Proposition 3.1. *Let G be an outerplanar graph of order n and size $m = n+1$. If G has an interior zone with boundary C_3, then G is not zonal.*

Proof. Since G has order n and size $n+1$, G has only one chord. Suppose that $v_1v_3 \in E(G)$ and that R_1, R_2 and R_3 are the zones determined by G, being R_1 the zone with boundary v_1, v_2, v_3 and R_3 the exterior zone, then the boundary of R_2 has order $n-1$. We proceed by contradiction. Suppose that ℓ is a zonal labeling of G, without loss of generality we assume that the vertices v_1, v_2 and v_3 are labeled 1. When ℓ is restricted to the boundary of R_2 we get that $\ell(v_1)+\ell(v_3)+\ell(v_4)+\ldots,+\ell(v_n) = 3\xi$, for some $\xi \in \mathbb{N}$. This implies that $\ell(v_1)+\ell(v_2)+\ldots,+\ell(v_n)$ is not a multiple of 3 because $\ell(v_2) = 1$, which is a contradiction. Consequently, G is not zonal. □

For $i \in \{1,2\}$, let G_i be a zonal graph of order n_i and $e_i = u_iv_i$ be an edge of G_i that lies on the boundary of the exterior zone determined by G_i. The *edge amalgamation* of G_1 and G_2, denoted by $G_1 \boxplus G_2$, obtained by identifying the edges e_1 and e_2, is a planar graph G of order $n = n_1+n_2-2$. We want to investigate the conditions on G_1 and G_2 that make G a zonal graph. We start assuming that each G_i is a cycle; based on Proposition 3.1, we know that if G_i is any cycle and $G_2 \cong C_3$, then G is not zonal. However, if none of these graphs is a copy of C_3, then G is zonal as we prove next. Before that, in Table 1 we exhibit the values in $\mathbb{Z}_3$ of $n = n_1+n_2-2$ and $s = s_1+s_2-1$, for the different combinations of n_1, n_2, s_1, and s_2, where s_i is the number of vertices of G_i labeled 2 by a zonal labeling.

$C_{n_1} \boxplus C_{n_2}$	$(n_2 \equiv 0, s_2 \equiv 0)$	$(n_2 \equiv 1, s_2 \equiv 2)$	$(n_2 \equiv 2, s_2 \equiv 1)$
$(n_1 \equiv 0, s_1 \equiv 0)$	$(n \equiv 1, s \equiv 2)$	$(n \equiv 2, s \equiv 1)$	$(n \equiv 0, s \equiv 0)$
$(n_1 \equiv 1, s_1 \equiv 2)$	$(n \equiv 2, s \equiv 1)$	$(n \equiv 0, s \equiv 0)$	$(n \equiv 1, s \equiv 2)$
$(n_1 \equiv 2, s_1 \equiv 1)$	$(n \equiv 0, s \equiv 0)$	$(n \equiv 1, s \equiv 2)$	$(n \equiv 2, s \equiv 1)$

Table 1. The values in $\mathbb{Z}_3$ of $n = n_1+n_2-2$ and $s = s_1+s_2-1$

Proposition 3.2. *If G_1 and G_2 are cycles other than C_3, then the edge amalgamation of G_1 and G_2 is zonal.*

Proof. For each $i \in \{1,2\}$, let G_i be the cycle C_{n_i} where $n_i \geq 4$. This last condition implies that there exists a zonal labeling of C_{n_i} that assigns $r_i > 0$ times the label 1 and $s_i > 0$ times the label 2. Therefore, there is an edge on each C_{n_i} which extremes are labeled 1 and 2. We amalgamate these cycles identifying these edges in such a way that the vertices with the same label are merged. We denote by G the graph obtained from this edge amalgamation, i.e., $G = C_{n_1} \boxplus C_{n_2}$. If s denotes the number of vertices labeled 2 in G, then $s = s_1 + s_2 - 1$. In Table 1 we found the different combinations of s_1 and s_2 and the value of the corresponding s in $\mathbb{Z}_3$. Thus, the value of s, and consequently the value of $r = r_1 + r_2 - 1$, corresponds exactly to the amount of vertices labeled 2 and 1, respectively, by a zonal labeling of a graph of order n, as it was described in Remark 1.1. Therefore, the graph G is zonal. □

Remark 3.1. If $G_1 \cong C_3$ and G_2 is any cycle, the edge amalgamation $G_1 \boxminus G_2$ is inner zonal, because there exists a zonal labeling of G_2 that assigns the label 1 to the end-vertices of an edge that can be amalgamated with an edge of G_1.

In the last proposition, we used the fact that each G_i is not a triangle to guarantee the existence of a zonal labeling of G_i that assigns the labels 1 and 2 on its vertices; in other terms, to be sure that G_i has an edge whose end-vertices are labeled 1 and 2. Triangulations are planar graphs which zonal labelings only use one of the two possible labels; in other terms, G_1 and G_2 cannot be triangulations neither. In the next proposition, we extend the result presented in Proposition 3.2.

Proposition 3.3. *For $i \in \{1, 2\}$, let G_i be a zonal graph. If there exists a zonal labeling of each G_i, that assigns the labels 1 and 2 on the vertices that form the boundary of the exterior zone, then there is an edge amalgamation of G_1 and G_2 that is zonal.*

Proof. For $i \in \{1, 2\}$, let ℓ_i be a zonal labeling of a graph G_i and let ∂_i be the boundary of the exterior zone of G_i. Suppose that n_i is the order of ∂_i and that ℓ_i assigns the label 1 to r_i vertices of ∂_i and the label 2 to the remaining s_i vertices of ∂_i, where r_i and s_i are positive integers. Suppose that $e_i = u_iv_i$ is an edge in ∂_i such that $\ell_i(u_i) = 1$ and $\ell_i(v_i) = 2$. Thus, the addition of the vertex labels on $V(\partial_i) - \{u_i, v_i\}$ is zero in $\mathbb{Z}_3$. In other terms, the label of the exterior zone of $G_1 \boxminus G_2$ is zero in $\mathbb{Z}_3$. Since the interior zones of each G_i are not affected by the amalgamation, we have obtained a zonal labeling of an edge amalgamation of G_1 and G_2. □

Recall that for an outerplanar graph of order n, we have denoted by $v_1, v_2, \ldots, v_n$ the consecutive vertices of the boundary of the exterior zone. In the next proposition we deal with a class of outerplanar graphs of order n and size $n + \chi - 2$, where the subgraph induced by $v_1, v_2, \ldots, v_\chi$ is a near-triangulation.

Proposition 3.4. *Let G be an outerplanar graph of order n with $\chi - 2$ chords such that the subgraph induced by $v_1, v_2, \ldots, v_\chi$ is a near-triangulation. The graph G is zonal if and only if $\chi \equiv 2(mod\, 3)$.*

Proof. Since the subgraph of G induced by $v_1, v_2, \ldots, v_\chi$ is a near-triangulation of order χ, then it has $\chi - 3$ chords. Consequently, the subgraph induced by the vertices $v_1, v_\chi, v_{\chi+1}, \ldots, v_n$ is a cycle; we denote this cycle by C.

Suppose that G is zonal; without loss of generality we assume that there exists a zonal labeling ℓ of G such that $\ell(v_1) = \ell(v_2) = \cdots = \ell(v_\chi) = 1$. The zone R, with boundary $v_1, v_\chi, v_{\chi+1}, \ldots, v_n, v_1$ has label 0 in $\mathbb{Z}_3$. Thus

$$\sum_{i=1}^{n} \ell(v_i) \equiv 0(\text{mod } 3) \text{ and } \ell(v_1) + \sum_{i=\chi}^{n} \ell(v_i) \equiv 0(\text{mod } 3).$$

Then, $\ell(v_2) + \ell(v_3) + \cdots + \ell(v_{\chi-1}) \equiv 0(\text{mod } 3)$. Since $\ell(v_i) = 1$ for each $2 \leq i \leq \chi - 1$, we get that $\chi - 2 = 3\xi$ for some integer ξ. Hence, $\chi \equiv 2(\text{mod } 3)$.

Suppose now that $\chi \equiv 2(\text{mod } 3)$. Since the subgraph of G induced by $v_1, v_2, \ldots, v_\chi$ is a near-triangulation, we have that in any possible zonal labeling of G, all these vertices must have the same label, either 1 or 2. Let ℓ be a zonal labeling of C that assigns the label 1 at least twice. We assume that $\ell(v_1) = \ell(v_\chi) = 1$. Thus, $2 + \sum_{i=\chi+1}^{n} \ell(v_i) = 3\sigma$ for some integer σ. Because $\chi \equiv 2(\text{mod } 3)$,

$$\chi + \sum_{i=\chi+1}^{n} \ell(v_i) = 3\sigma + \chi - 2 = 3\kappa \equiv 0(\text{mod } 3)$$

for some integer κ. Consequently, the label of each zone determined by G is 0 in $\mathbb{Z}_3$ and G is zonal. □

In the following result, we use the fact that the maximum size of an outerplanar graph of order n is $2n - 3$. The following proposition considers the class of outerplanar graphs of order n and size $2n - 4$.

Proposition 3.5. *For each $i \in \{1, 2\}$, let G_i be an outerplanar graph of order n_i and size $2n_i - 3$, and $e_i = u_iv_i$ be any edge on the boundary of the exterior zone of G_i. The outerplanar graph G, obtained from G_1 and G_2 by adding the edges u_1u_2 and v_1v_2, is zonal if and only if $n_1 \equiv n_2(\text{mod } 3)$.*

Proof. Since G_i has order n_i and size $2n_i - 3$, G_i is a near-triangulation; by Proposition 2.3 we know that G_i is inner zonal. An inner zonal labeling of G_i is obtained by assigning the label 1 (or the label 2) to every vertex of G_i.

Suppose that G is zonal. The zone with boundary u_1, u_2, v_2, v_1 has label 0 in $\mathbb{Z}_3$. This label is achieved by assigning the label 1 to exactly two of these vertices. But both G_1 and G_2 are inner zonal; this implies that all vertex labels on G_1 must be 1 and all vertex labels on G_2 must be 2, or vice versa. Either way, u_i and v_i have the same label. Thus, the label of the exterior zone of G is $n_1 + 2n_2 \equiv 0(\text{mod } 3)$. In Table 2 we show the equivalence class of $n_1 + 2n_2$ in $\mathbb{Z}_3$ for all the combinations of n_1 and n_2. Since $n_1 + 2n_2 \equiv 0(\text{mod } 3)$, we conclude that $n_1 \equiv n_2(\text{mod } 3)$.

$n_1 + 2n_2$	$n_2 \equiv 0(\text{mod } 3)$	$n_2 \equiv 1(\text{mod } 3)$	$n_2 \equiv 2(\text{mod } 3)$
$n_1 \equiv 0(\text{mod } 3)$	0	2	1
$n_1 \equiv 1(\text{mod } 3)$	1	0	2
$n_1 \equiv 2(\text{mod } 3)$	2	1	0

Table 2. The values of $n_1 + 2n_2$ in $\mathbb{Z}_3$

Suppose now that $n_1 \equiv n_2(\text{mod } 3)$. We use on G_i the inner zonal labeling that assigns the label i to each of its vertices. In this way, each internal zone of G_i has label 0 in $\mathbb{Z}_3$. The vertices u_1 and v_1 of G_1 have label 1, while the vertices u_2 and v_2 of G_2 have label 2; consequently, the zone with boundary u_1, u_2, v_2, v_1 has label 0 in $\mathbb{Z}_3$ because $n_1 \equiv n_2(\text{mod } 3)$. Furthermore, the label of the exterior zone of G is $n_1 + 2n_2 \equiv 0(\text{mod } 3)$ because $n_1 \equiv n_2(\text{mod } 3)$. Therefore, G is zonal. □

Note that the outerplanar graph G obtained in this last proposition has order n and size $2n-4$. In Figure 4 we show an example of this construction where $n_1 = 8$ and $n_2 = 5$.

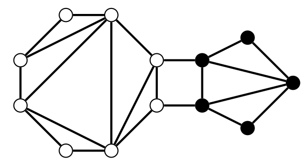

Figure 4. Zonal labeling of a graph obtained by connecting, with 2 edges, two maximal outerplanar graphs whose orders are equivalent in $\mathbb{Z}_3$

Consider the graph M_1 in Figure 5. Assume that M_1 is zonal, then every vertex on a triangular zone has the same label (or color). Since the zones R_1 and R_2 share a vertex, the labels on the boundaries of these two zones are identical. The boundary of R_3 is the cycle C_4, any zonal labeling of C_4 must use both labels twice, but this is not possible in the case of M_1 because R_3 has three vertices with the same label; thus, regardless of the label that could be used on the still unlabeled vertex of R_3, the label of this zone is not 0 in $\mathbb{Z}_3$, which is a contradiction. Therefore, M_1 is not zonal nor inner zonal. Using the catalog of outerplanar graphs of order up to 9 in [4], we found that M_2 and M_3, shown in Figure 5, are the only outerplanar graphs of order 8 that are not zonal nor inner zonal and do not have M_1 as an induced subgraph. Indeed, given that the vertices on the boundary of the highlighted zone, must have the same label, and the other vertices need to be labeled in the way shown (or its complement), the zone R cannot have label 0 in $\mathbb{Z}_3$. Therefore, both graphs are not zonal nor inner zonal.

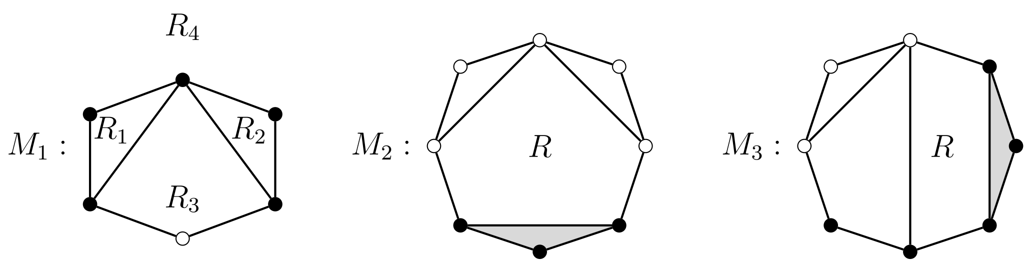

Figure 5. Neither zonal nor inner zonal outerplanar graphs of order 6 and 8

Based on these observations, the proof of the following result is straightforward and it is omitted.

Proposition 3.6. *Any outerplanar graph that contains M_1, M_2, or M_3, as an induced subgraph, is neither zonal nor inner zonal.*

In Figure 6 we show all outerplanar graphs of order 9 that are neither zonal nor inner zonal; the graphs on the ith row have M_i as an induced subgraph. A question, that raises naturally in this context, is whether or not there exists an outerplanar graph of order $n \geq 10$ and size $m \geq n+2$ that is neither zonal nor inner zonal and does not contain any of the M_i as an induced subgraph.

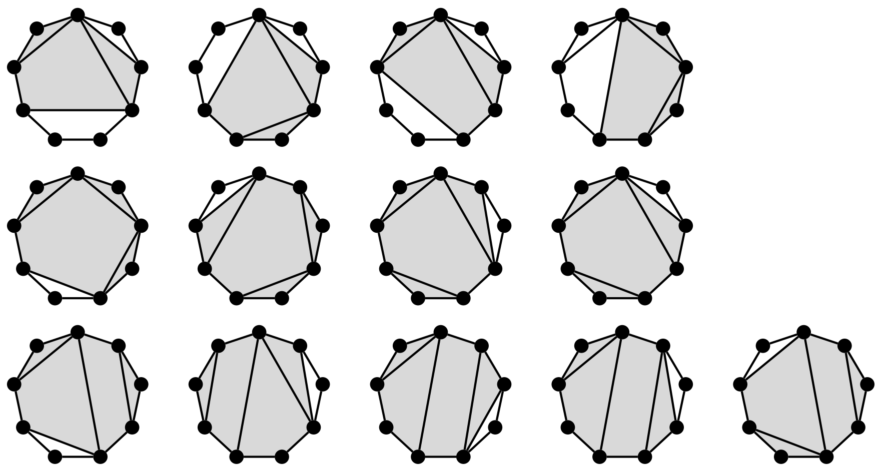

Figure 6. Not zonal nor inner zonal outerplanar graphs of order 9

Consider, for example, the outerplanar graph in Figure 7, this graph has order $n = 12$ and size $2n - 3 = 21$; the 9 chords are classified into three groups: if the thicker solid chord (in blue) is deleted, the resulting graph is zonal; when any of the regular solid chords (in green) is erased, the resulting graph is inner zonal; if any of the dashed chords (in red) is removed, the resulting graph contains the graph M_1 as an induced subgraph, then it is neither zonal nor inner zonal. In other terms, Proposition 3.5 is the essential tool to characterize zonal outerplanar graphs of order n and size $2n-4$. This characterization can be described in the following terms: Let G be an outerplanar graph of order n and size $2n - 3$, and let $e = v_1 v_j$ be a chord of G, where $i < j$. The graph $G' = G - e$ is zonal if and only if $v_{i-1}v_j$ and $v_i v_{j-1}$ are chords of G and the removal of the edges $v_{i-1}v_i$, $v_{j-1}v_j$ of G' results in two graphs G_1 and G_2 such that the order of G_1 is equivalent to the order of G_2 in $\mathbb{Z}_3$.

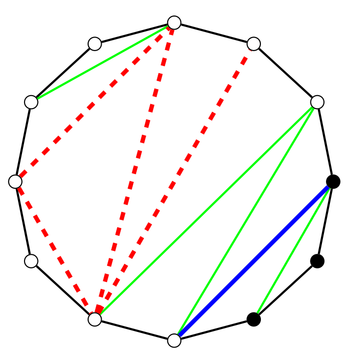

Figure 7. Different types of chords within an outerplanar graph of maximum size

Summarizing, within the family of outerplanar graphs, we encounter zonal and inner zonal graphs, and others that are neither zonal nor inner zonal. We conclude this section with the following questions.

Question. *In addition to the graphs shown in Figure 5, are there more forbidden subgraphs?*

Question. *Which outerplanar graphs are zonal?*

4. Zonal Graphs and the Cartesian Product

As mentioned in the Introduction, Chartrand et al. [1] studied the Cartesian product of cycles and the path P_2, showing that $G = C_n \times P_2$ is zonal for all $n \geq 3$. There are two copies of C_n in G, these copies are labeled with complementary zonal labelings; in this way, each vertex label is used the same number of times. Two of the zones determined by G have boundary C_n, while the remaining n zones have boundary C_4. The labeling on each C_n is zonal as well as the labelings of each copy of C_4 because two of its vertices are in the first copy of C_n and the other two are in the other copy; since the labelings used on both copies of C_n are complementary, each C_4 has two vertices labeled 1 and two vertices labeled 2, then the labeling of each C_4 in G is zonal.

In the following results we use the same technique to prove that $G \times P_2$ is zonal when G is either a tree or an outerplanar graph.

Proposition 4.1. *If T is any tree, then $T \times P_2$ is zonal.*

Proof. Let T be any tree of order $n \geq 1$ and let $\ell : V(T) \to \{1, 2\}$ be any labeling of T. The first copy of T in $T \times P_2$ is labeled using the function ℓ while the second copy of T is labeled using $\overline{\ell}$, where $\overline{\ell}(v) = 3 - \ell(v)$ for each $v \in V(T)$. In this way, the $2n$ vertices of $T \times P_2$ are labeled either 1 or 2, and each vertex label is used exactly n times. The boundary of any zone R determined by $T \times P_2$ is a cycle of order $2k$ for some $k \geq 2$; since each copy of T contributes k vertices to the boundary of any zone and the labelings on each copy of T are complementary, we have that there are k vertices labeled 1 and k vertices labeled 2; therefore, the label of R is $k + 2k = 3k$, which is 0 in $\mathbb{Z}_3$. In consequence, $T \times P_2$ is zonal. □

In Figure 8 we show a zonal labeling for $T \times P_2$, where T is a tree of order 24. Note that different representations of T can produce regions of different order; we must also observe that the labeling of each copy of T is not required to be zonal. The reason why we can use any labeling on T that assigns the labels 1 and 2 is that a plane representation of T only produces one zone.

In the next proposition we consider the Cartesian product of any zonal outerplanar graph and the path P_2.

Proposition 4.2. *If G is a zonal outerplanar graph, then $G \times P_2$ is zonal.*

Proof. Let G be an outerplanar graph and let ℓ be a zonal labeling of the first copy of G in $G \times P_2$. The second copy is labeled using its complementary labeling $\overline{\ell}$. Thus, the label on all the interior zones determined by each copy of G is 0 in $\mathbb{Z}_3$. The same occurs with any zone which boundary is formed by two adjacent vertices of the first copy with their corresponding replicas in the second

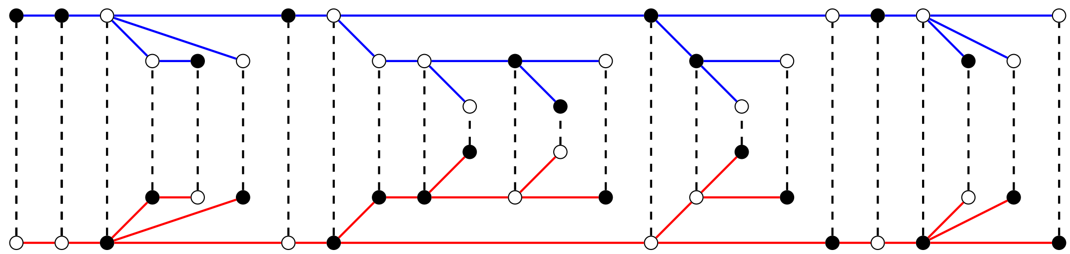

Figure 8. Zonal labeling of the Cartesian product $T \times P_2$, where T is a tree of order 24

copy. If the vertices on the boundary of the exterior zone determined by each copy of G lay on a straight line with the chords represented by arcs, drawn above this line on the first copy and below this line on the second copy, we obtain a plane representation of $G \times P_2$, where the boundary of the exterior zone is the cycle C_4, being its label 0 in $\mathbb{Z}_3$. Therefore, $G \times P_2$ is zonal. □

In Figure 9 we show an example of a zonal labeling for a graph $G \times P_2$, where G is an outerplanar graph of order 8 and size 11. The drawing of $G \times P_2$ follows the directions given in the proof of Proposition 4.2.

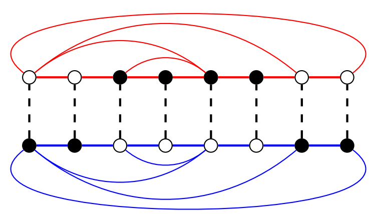

Figure 9. Zonal labeling of the Cartesian product of an outerplanar graph of order 8 and P_2

The same technique used to obtain the zonal labeling of the previous Cartesian products can be used when G is a unicyclic graph. This can be seen on the graph in Figure 8; connecting the end-vertices of the longest paths on each copy of T, i.e., the paths where the branches of each tree are attached, a unicyclic graph is obtained from each copy of T. In the concrete case of the graph in Figure 8, the cycle obtained by adding the new edge has order 10 and its label is 0 in $\mathbb{Z}_3$. Because of the similarity with the two previous results, we state with no proof the fact that $G \times P_2$ is zonal when G is unicyclic.

Proposition 4.3. *If G is a unicyclic graph, then $G \times P_2$ is zonal.*

As we mentioned in the introduction, the prism $D_n = C_n \times P_2$ is zonal. This graph fits in the category of cubic graphs studied in [1]. The technique used to prove that $G \times P_2$ is zonal, can be extended to demonstrate that the plane graph $C_n \times P_m$ is zonal.

Proposition 4.4. *The graph $C_n \times P_m$ is zonal for every $m \geq 2$.*

Proof. The fact that $C_n \times P_2$ is zonal was proven in [1]. Assume that $m > 2$. For each $i \in \{1, 2, \ldots, m\}$, let $v_1^i, v_2^i, \ldots, v_n^i$ be the consecutive vertices of the ith copy of C_n used to build $C_n \times P_m$, where v_j^i is connected to v_j^{i+1}. If ℓ is any zonal labeling of C_n, then for every $1 \leq i \leq m-1$, the copy C^i of C_n is labeled using ℓ while the copy C^{i+1} is labeled using the complementary labeling $\overline{\ell}$. Thus, the zones with boundary C_n have a label that is 0 in $\mathbb{Z}_3$. Any zone with boundary C_4, that is, determined by the vertices $v_j^i, v_{j+1}^i, v_{j+1}^{i+1}$, and v_j^{i+1}, has two vertex labels equal to 1 and two vertex labels equal to 2, regardless of the labelling ℓ, then its label is 0 in $\mathbb{Z}_3$. Therefore, $C_n \times P_m$ is zonal. □

5. Bracelets and Zonal Cycles

A *binary string* of length n is a sequence of n elements of $\mathbb{Z}_2$; usually, each of the entries of such a sequence is called *bead*. Let S_1 and S_2 be two of these strings, S_1 and S_2 are said to be *equivalent* if S_2 can be obtained by rotating the beads of S_1. A *necklace* is any of the elements of the equivalence classes induced by this relation. If, in addition, we admit that S_1 and S_2 are equivalent when S_2 is a reflection of S_1, then a necklace is called a *bracelet*. Thus, in our context, a bracelet is a circular arrangement of n beads of two different colors, i.e., the elements of $\mathbb{Z}_2$. The number of different types of necklaces and bracelets has been widely studied. The On-line Encyclopedia of Integer Sequences includes more than 1200 entries associated to necklaces and/or bracelets.

In a zonal labeling of the cycle C_n, we may understand the vertex labels as two different colors; thus, each zonal labeling of C_n corresponds to a bracelet, where the number 1 in $\mathbb{Z}_3$ is the number 1 in $\mathbb{Z}_2$ and the number 2 in $\mathbb{Z}_3$ is the number 0 in $\mathbb{Z}_2$.

Suppose that the cycle C_n has been zonally labeled. Recall that r denotes the amount of vertices of C_n labeled 1, where $0 \leq r \leq n$; as we mentioned in the introduction,

$$r \equiv \begin{cases} 0 \pmod 3 & \text{if } n \equiv 0 \pmod 3, \\ 2 \pmod 3 & \text{if } n \equiv 1 \pmod 3, \\ 1 \pmod 3 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

In other terms, $n + r \equiv 0 \pmod 3$. For a fixed value of $n \geq 3$, let $R = \{r : n + r \equiv 0 \pmod 3\}$. Sequence A052307 in OEIS gives the number $T(n, r)$ of bracelets with n beads, r of which are white (or 1) and $n - r$ are black (or 2). Consequently, $T(n, r)$ is the number of zonal labelings of C_n that use exactly r times the label 1, where

$$T(n,r) = \frac{1}{2}\Big(C\big(\lfloor \tfrac{n}{2} \rfloor - (r \bmod 2)(1 - (n \bmod 2)), \lfloor \tfrac{r}{2} \rfloor\big) + \tfrac{1}{n}\textstyle\sum_{d \in D} \phi(d) C(\tfrac{n}{d}, \tfrac{r}{d})\Big),$$

being $D = \{d : d \big| \gcd(n, r)\}$, $\phi(d)$ is Euler totient function, and $C(p, q)$ denotes the binomial coefficient. With all these facts, the proof of the following result is straightforward.

Proposition 5.1. *The number of zonal labelings of the cycle C_n is*

$$z(n) = \sum_{r \in R} T(n, r).$$

In Table 3 we show the values of $z(n)$, for $3 \leq n \leq 32$.

n	$z(n)$	n	$z(n)$	n	$z(n)$	n	$z(n)$	n	$z(n)$
3	2	9	16	15	410	21	16,992	27	831,256
4	2	10	26	16	750	22	32,303	28	1,602,026
5	2	11	42	17	1370	23	61,470	29	3,090,926
6	5	12	76	18	2565	24	117,574	30	5,973,644
7	6	13	126	19	4770	25	225,062	31	11,556,534
8	10	14	229	20	9004	26	432,286	32	22,386,350

Table 3. Number of zonal labelings of C_n

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