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Family of Graphs with Partition Dimension Three

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Abstract

The characterization of all connected graphs of order $n \geq 3$ with partition dimension $2, n-1$ or n has been completely done. Additionally, all connected graphs of order $n \geq 9$ with partition dimension $n-2$ and graphs of order $n \geq 11$ with partition dimension $n-3$ have been characterized as well. However, the characterization of all connected graphs with partition dimension 3 is an open problem. In this paper, we construct many families of disconnected as well as connected graphs with partition dimension 3 by generalizing the concept of the partition dimension so that it can be applied to disconnected graphs.

Keywords: connected, disconnected, graph, partition dimension
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1. Introduction

Let $F = (V, E)$ be a connected graph. The *metric dimension* of a graph F is the minimum cardinality of the set $S \subseteq V(F)$ such that all vertices in F have distinct metric representations with respect to S. The *metric representation* of a vertex $v \in V(F)$ with respect to S is a vector consisting of all distances from v to every vertex in S. The idea of the metric dimension of a graph was introduced by Harary & Melter [7] and previously by Slater [17] using a different term, namely

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a locating set. In this paper, we adopt the terminology used by Harary & Melter [7]. There are many applications of the metric dimension of graphs such as in long range aids navigation [17], chemical compounds in chemistry [13], or in robot navigation in a network [6, 14]. For further results in the metric dimension of graphs see, for instances, [15, 2, 3].

To obtain a new insight into metric dimension of graphs, Chartrand et al. [4] introduced a new concept called a partition dimension of a graph. Let $\Lambda = \{A_1, A_2, \ldots, A_k\}$ be a partition of a connected graph F. The *partition representation* of a vertex $v \in V(F)$ with respect to Λ is defined as $r(v|\Lambda) = (d(v, A_1), d(v, A_2), \ldots, d(v, A_k))$, where $d(v, A_i)$ denotes the distance between the vertex v and A_i, i.e. $d(v, A_i) = \min\{d(v, w) : w \in A_i\}$. A partition Λ is a *resolving partition* of F if every two distinct vertices $v, w \in V(F)$ are resolved by A_i for integer $i \in [1, k]$, or in short $d(v, A_i) \neq d(w, A_i)$ for some $1 \leq i \leq k$. The minimum cardinality of a resolving partition Λ of F is called a *partition dimension* of F and it is denoted by $pd(F)$. In general, for a connected graph F, $pd(F) \leq \dim(F) + 1$, where $\dim(F)$ is the metric dimension of F.

Recently in [9, 10], Haryeni et al. generalized the definition of the partition dimension of a graph such that it can be also applied to disconnected graphs, as follows. Given any (not necessarily connected) graph $G = (V, E)$ and an ordered partition $\Lambda = \{A_1, A_2, \ldots, A_l\}$ of $V(G)$, where A_i is a *partition class* of Λ for each $1 \leq i \leq l$. If all distance $d(v, A_i)$ of a vertex v and A_i for each $1 \leq i \leq l$ are finite for all $v \in V(G)$, then we can define the *partition representation* of v under Λ as $r(v|\Lambda) = (d(v, A_1), d(v, A_2), \ldots, d(v, A_l))$. Thereby, each component of G must contain the same number of partition class. The partition Λ is a *resolving partition* of G if $r(v|\Lambda) \neq r(w|\Lambda)$ for any two distinct vertices $v, w \in V(G)$. The smallest cardinality of a resolving partition Λ of G is called a *partition dimension* of G. We use the notation $pd(G)$ or $pdd(G)$ for the partition dimension of connected or disconnected G, respectively. If a disconnected graph G has no resolving l-partition, then define $pdd(G) = \infty$.

Many results in determining the partition dimension of connected or disconnected graphs have been obtained. For connected cases, Chartrand et al. [5] characterized all connected graphs G of order $n \geq 3$ with the partition dimension $2, n$ or $n - 1$. The characterization is presented in the following theorem.

Theorem 1.1. *[5] For a connected graph G of order $n \geq 3$, then*

(1) $pd(G) = 2$ *if and only if* $G = P_n$.

(2) $pd(G) = n - 1$ *if and only if G is one of the graphs* $K_{1,n-1}, K_n - e$ *or* $K_1 + (K_1 \cup K_{n-2})$.

(3) $pd(G) = n$ *if and only if* $G = K_n$.

Furthermore, Tomescu [18] showed that there are 23 connected graphs G of order $n \geq 9$ having the partition dimension $n - 2$. Recently, the characterization of connected graphs with partition dimension $n - 3$ was studied by Baskoro et al. in [1] and Haryeni et al. in [12]. Up to now, the characterization of the connected graph G with $pd(G) = k$ for any integer $k \in [3, n - 4]$, is still an open problem. The study of the partition dimension of graphs is also considered for graph operations, such as corona product, Cartesian product, and strong product [16, 20, 19].

For disconnected graphs, there are still few results concerning their partition dimensions. Haryeni et al. in [10] established the upper and lower bounds of the partition dimension $pdd(G)$ of a disconnected graph G, if they are finite, as follows.

Theorem 1.2. *[10] Let $G = \bigcup_{i=1}^{m} G_i$. If $pdd(G) < \infty$, then $\max\{pd(G_i) : 1 \leq i \leq m\} \leq pdd(G) \leq \min\{|V(G_i)| : 1 \leq i \leq m\}$.*

By Theorem 1.1(1), the partition dimension of any graph other than a path is at least 3. For a disconnected graph containing a path as its component, Haryeni et al. in the same paper obtained the following results.

Theorem 1.3. *[10] Let P_{n_i} be a path on $n_i \geq 3$ vertices. Then,*

(1) *$pdd(\bigcup_{i=1}^{m} P_{n_i}) = 3$ for any integer $m \geq 2$, where $n_i \geq 3$ for all i and $n_i \neq n_j$ for all integer $i, j \in [1, m]$,*

(2) *for $n \geq 3$, $pdd(mP_n) = 3$ if and only if $2 \leq m \leq 3\lfloor \frac{n-1}{3} \rfloor$,*

(3) *$pdd(K_3 \cup mP_n) = 3$ if and only if $n \geq 4$ and $m \leq 3\lfloor \frac{n-2}{3} \rfloor$.*

In [8], Haryeni and Baskoro generate classes of graph obtained from union of path such that the partition dimension of these graphs equal to 3. Let $G = \bigcup_{i=1}^{m} P_{n_i}$ where $m \geq 2$, $n_1 \geq 3$ and $n_{i+1} = n_i + 1$ for all $i \in [1, m-1]$. Let the set of vertices and edges of G by

$$\begin{aligned} V(G) &= \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n_i\} \text{ and} \\ E(G) &= \{v_{i,j}v_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq n_i - 1\}, \end{aligned}$$

respectively. Let $G' = G \cup E_1 \cup E_2$ and $G \subseteq G'' \subseteq G'$, where E_1 and E_2 are two sets of additional edges connecting some vertices of F as follows.

$$\begin{aligned} E_1 &= \{v_{i,j}v_{i+1,j} : 1 \leq i \leq m-1, 1 \leq j \leq n_i\} \\ E_2 &= \{v_{i,j}v_{i+1,j+1} : 1 \leq i \leq m-1, 1 \leq j \leq n_i - 1\}. \end{aligned}$$

Furthermore, they gave the following results.

Theorem 1.4. *[8] Let $G' = G \cup E_1 \cup E_2$ and $G \subseteq G'' \subseteq G'$. Then, $pd(G'') = 3$.*

Theorem 1.5. *[8] Let $G' = G \cup E_1 \cup E_2$, $F \subseteq E(G)$ where $F = \{v_{i,j}v_{i,j+1} : 2 \leq i \leq m-1, 1 \leq j \leq n_i - 1\}$ and $F' \subseteq F$. Then, $pd(G' - F') = 3$.*

The partition dimension for disconnected graphs with two components also has been studied in [11]. Moreover, in [9] Haryeni et al. studied the partition dimensions of disjoint union of some homogenous graphs, namely stars, double stars and some cycles.

In this paper, we present further results of (disconnected) graphs having the partition dimension 3. In particular, we give the conditions such that for $G = \bigcup C_{m_i}$, then $pdd(G) = 3$. Furthermore, we construct new graphs G' from G by connecting some vertices of G such that the partition dimensions of G' still remain equal to 3. Next, we consider a new connected graph G'' obtained from G' by deleting some edges in $G(\subset G')$. We show that $pd(G'') = 3$. The set of all connected graphs G' and G'' contribute to a big family of connected graphs with partition dimension 3.

2. Main Results

In this section, we will consider the partition dimension of a graph containing cycles. Let $G = \bigcup_{i=1}^{t} C_{m_i}$ where $m_i \geq 3$ for all $1 \leq i \leq t$, $V(G) = \{v_{i,j} : 1 \leq i \leq t, 1 \leq j \leq m_i\}$ and $E(G) = \{v_{i,j}v_{i,j+1}, v_{i,1}v_{i,m_i} : 1 \leq i \leq t, 1 \leq j \leq m_i - 1\}$. Before presenting our main theorems, we recall the definition of a connected partition as follows.

Let G_i be a connected graph and $G = \bigcup_{i=1}^{m} G_i$. A partition $\Lambda = \{A_1, A_2, \ldots, A_k\}$ of $V(G)$ is a *connected partition* if every subgraph induced by $A_j \cap V(G_i)$ is connected for every integers $j \in [1, k]$ and $i \in [1, m]$. Such a partition Λ is called a *connected resolving partition* if Λ is a resolving partition which is connected.

In order to prove the main results, we need Lemma 2.1 which has been proven in [11].

Lemma 2.1. *[11] For $3 \leq k \leq n$, any connected k-partition of P_n or C_n is a resolving partition.*

Now, we give the partition dimension for $G = \bigcup_{i=1}^{t} C_{m_i}$ with certain length of m_i, as follows.

Theorem 2.1. *For any integer $t \geq 2$, let $G = \bigcup_{i=1}^{t} C_{m_i}$ where $m_1 \geq 3$ and $m_{i+1} \geq m_i + 3$ for any integer $i \in [1, t-1]$. Then, $pdd(G) = 3$.*

Proof. For $t \geq 2$, let $G = \bigcup_{i=1}^{t} C_{m_i}$ where $m_1 \geq 3$ and $m_{i+1} \geq m_i + 3$ for all $1 \leq i \leq t-1$. It is obvious that G is not a path, and thus $pdd(G) \geq 3$. To show that $pdd(G) \leq 3$, we define a partition $\Lambda = \{A_1, A_2, A_3\}$ of G induced by a function $f : V(G) \to \{1, 2, 3\}$ as follows.

$$f(v_{i,j}) = \begin{cases} 1, & \text{if } j = 1, 2, \ldots, i, \\ 2, & \text{if } j = i+1, i+2, \ldots, 2i, \\ 3, & \text{if } j = 2i+1, 2i+2, \ldots, m_i. \end{cases}$$

Note that $f(x) = k$ means $x \in A_k$. By using the definition of the function f, for integers $i \in [1, t]$, $p \in [1, i], q \in [i+1, 2i]$ and $r \in [2i+1, m_i]$, we have

$$d(v_{i,p}, A_k) = \begin{cases} 0, & \text{if } k = 1, \\ i - p + 1, & \text{if } k = 2, \\ p, & \text{if } k = 3, \end{cases}$$

$$d(v_{i,q}, A_k) = \begin{cases} 0, & \text{if } k = 2, \\ q - i, & \text{if } k = 1, \\ 2i - q + 1, & \text{if } k = 3, \end{cases}$$

$$d(v_{i,r}, A_k) = \begin{cases} 0, & \text{if } k = 3, \\ \min\{r - 2i, m_i - r + (i+1)\}, & \text{if } k = 2, \\ \min\{m_i - r + 1, r - i\} & \text{if } k = 1. \end{cases}$$

Now, we are going to show that Λ is a resolving partition of G. This means that for any two vertices $x, y \in V(G)$ in A_k for some $1 \leq k \leq 3$, we will show that $d(x, A_t) \neq d(y, A_t)$ for some t. Since Λ is a connected 3-partition, any two vertices x and y in $V(C_{m_i})$ for some i are resolved by some A_l where $l \neq k$ by Lemma 2.1. Furthermore, we assume that $x \in V(C_{m_i})$ and $y \in V(C_{m_j})$ where $1 \leq i < j \leq t$. Let $x = v_{i,a}$ and $y = v_{j,b}$ in A_1 where $1 \leq a \leq i$ and $1 \leq b \leq j$. If $a = b$, then $d(x, A_2) = i - a + 1 = i - b + 1 < j - b + 1 = d(y, A_2)$. Otherwise, $d(x, A_3) = a \neq b = d(y, A_3)$. Let $x = v_{i,a}$ and $y = v_{j,b}$ in A_2 where $i + 1 \leq a \leq 2i$

and $j+1 \leq b \leq 2j$. If $a-i = b-j$, then $d(x, A_3) = 2i-a+1 = i-b+j+1 < 2j-b+1 = d(y, A_3)$. Otherwise, $d(x, A_1) = a - i \neq b - j = d(y, A_1)$. Now we consider $x = v_{i,a}$ and $y = v_{j,b}$ in A_3 where $2i + 1 \leq a \leq m_i$ and $2j + 1 \leq b \leq m_j$. Let P^{t1} and P^{t2} be two path in C_{m_t} where its vertices are contined in the partition class A_1 and A_2, respectively. We distinguish four cases.

Case 1. $d(x, A_2) = a - 2i$ and $d(y, A_2) = b - 2j$.
Let $|A_1^t|, |A_2^t|$ and $|A_3^t|$ denote the number of vertices in the partition class A_1, A_2 and A_3 of the component C_{m_t}. From the definition of the partition Λ induced by the function f above, then $|A_1^t| = |A_2^t| = t$ and $|A_3^t| \geq t + 1$. Note that for any component $j > i$, $|A_l^j| \geq |A_l^i| + 1$ for each $l \in \{1, 2, 3\}$. Thus, for $d(x, A_2) = a - 2i$ and $d(y, A_2) = b - 2j$, the shortest paths passed by each vertex x and y to A_2 is directly through P^{i2} and P^{j2}, respectively. If $a - 2i = b - 2j$, then clearly that $d(x, A_1) < d(y, A_1)$ for any shortest path passed by each vertex x and y to the partition class A_1. Otherwise, $d(x, A_2) \neq d(y, A_2)$.

Case 2. $d(x, A_2) = a - 2i$ and $d(y, A_2) = m_j - b + (j + 1)$.
Therefore, the shortest path passed by y to A_1 is directly through P^{j1} or $d(y, A_1) = m_j - b + 1 = d(y, A_2) - j < d(y, A_2)$. If $a - 2i = m_j - b + (j + 1)$, then we consider two cases.

Subcase 2.1. If the shortest path passed by x to A_1 is through P^{i2}, then $d(x, A_1) = d(x, A_2) + i = d(y, A_2) + i > d(y, A_1)$.

Subcase 2.2. If the shortest path passed by x to A_1 is through P^{i1}, then $d(x, A_1) = m_i - a + 1 \geq a - 3i = m_j - b + (j + 1) - i > m_j - b + 1 = d(y, A_1)$.

Otherwise, $d(x, A_2) \neq d(y, A_2)$.

Case 3. $d(x, A_2) = m_i - a + (i + 1)$ and $d(y, A_2) = b - 2j$.
Therefore, the shortest path passed by x to A_1 is directly through P^{i1} or $d(x, A_1) = m_i - a + 1 = d(x, A_2) - i < d(x, A_2)$. If $m_i - a + (i + 1) = b - 2j$, then we consider two cases.

Subcase 2.1. If the shortest path passed by y to A_1 is through P^{j2}, then $d(y, A_1) = d(y, A_2) + j = d(x, A_2) + j < d(x, A_1)$.

Subcase 2.2. If the shortest path passed by y to A_1 is through P^{j1}, then $d(y, A_1) = m_j - b + 1 > m_j - j + i - b + 1 = m_j - 3(j - i) - b + 2j + 1 - 2i \geq m_i - b + 2j + 1 - 2i = a - 3i > m_i - a + 1 = d(x, A_1)$.

Otherwise, $d(x, A_2) \neq d(y, A_2)$.

Case 4. $d(x, A_2) = m_i - a + (i + 1)$ and $d(y, A_2) = m_j - b + (j + 1)$. If $m_i - a + (i + 1) = m_j - b + (j + 1)$, then clearly that $d(x, A_1) = m_i - a + 1$ and $d(y, A_1) = m_j - b + 1$. Furthermore, we have $d(x, A_1) = m_i - a + 1 = m_j - b + (j + 1) - i > m_j - b + 1 = d(y, A_1)$. Otherwise, $d(x, A_2) \neq d(y, A_2)$.

Therefore, $r(x|\Lambda) \neq r(y|\Lambda)$ for any two vertices $x, y \in V(G)$. This concludes the proof, namely $\Lambda = \{A_1, A_2, A_3\}$ is a resolving 3-partition of G. □

Now, we will define new graphs G_j, where $j = 1, 2, 3$, constructed from the graph $G = \bigcup_{i=1}^{t} C_{m_i}$ by adding new edges connecting some vertices in G. Note that $m_1 \geq 3$ and $m_{i+1} \geq m_i + 3$ for each $i \geq 1$. Let $V(G) = \{v_{i,j} : 1 \leq i \leq t, 1 \leq j \leq m_i\}$ and $E(G) = \{v_{i,j}v_{i,j+1}, v_{i,1}v_{i,m_i} : 1 \leq i \leq t, 1 \leq j \leq m_i - 1\}$. For all integer $i \in [1, t]$, let us define 7 sets of additional edges

$E_1, E_2, E_3, E_4, E_5, E_6$ and E_7, as follows.

$$
\begin{aligned}
E_1 &= \left\{v_{i,j}v_{i+1,j}, v_{i,j}v_{i+1,j+1} : 1 \le j \le \left\lfloor \frac{m_i}{3} \right\rfloor \right\}, \\
E_2 &= \left\{v_{i,j}v_{i+1,j+1}, v_{i,j}v_{i+1,j+2} : \left\lfloor \frac{m_i}{3} \right\rfloor + 1 \le j \le \left\lfloor \frac{2m_i}{3} \right\rfloor \right\}, \\
E_3 &= \left\{v_{i,j}v_{i+1,j+2}, v_{i,j}v_{i+1,j+3} : \left\lfloor \frac{2m_i}{3} \right\rfloor + 1 \le j \le m_i \right\}, \\
E_4 &= \{v_{i,\lfloor \frac{2m_i}{3} \rfloor + 1} v_{i+1,\lfloor \frac{2m_i}{3} \rfloor + 3}, v_{i,m_i}v_{i+1,m_{i+1}}\}, \\
E_5 &= \{v_{i,j}v_{i+1,j}, v_{i,j}v_{i+1,j+1} : 1 \le j \le i\}, \\
E_6 &= \{v_{i,j}v_{i+1,j+1}, v_{i,j}v_{i+1,j+2} : i+1 \le j \le 2i\}, \\
E_7 &= \{v_{i,2i+1}v_{i+1,2i+3}, v_{i,m_i}v_{i+1,m_{i+1}}\}.
\end{aligned}
$$

Now, we define:

(C-1) G_1 is a connected graph constructed from G by adding edges in $E_1 \cup E_2 \cup E_3$ provided $m_{i+1} = m_i + 3$ for all $i \ge 1$.

(C-2) G_2 is a connected graph constructed from G by adding edges in $E_1 \cup E_2 \cup E_4$ provided $m_{i+1} > m_i + 3$ for some $i \ge 1$.

(C-3) G_3 is a connected graph constructed from G by adding edges in $E_5 \cup E_6 \cup E_7$ provided $m_{i+1} \ge m_i + 3$ for all $i \ge 1$.

In the following theorem, we will show that each subgraph G'_j of G_j ($j \in \{1, 2, 3\}$) containing G has partition dimension 3.

Theorem 2.2. *For $t \ge 2$, $m_1 \ge 3$ and $m_{i+1} \ge m_i + 3$ for all $i \in [1, t]$, let $G = \bigcup_{i=1}^{t} C_{m_i}$, $G \subseteq G'_1 \subseteq G_1$, $G \subseteq G'_2 \subseteq G_2$ and $G \subseteq G'_3 \subseteq G_3$. Then, $pdd(G'_1) = pdd(G'_2) = pdd(G'_3) = 3$.*

Proof. Clearly, for each G'_j where $j = 1, 2, 3$, $pdd(G'_j) \ge 3$. Now, we will show that $pdd(G'_j) \le 3$. Let $V(G'_1) = V(G'_2) = V(G'_3) = V(G) = \{v_{i,j} : 1 \le i \le t, 1 \le j \le m_i\}$. We consider two cases.

Case 1. $G \subseteq G'_1 \subseteq G_1$ or $G \subseteq G'_2 \subseteq G_2$. Let $\Lambda_1 = \{A_1, A_2, A_3\}$ be a partition of G'_1 or G'_2 induced by the function $g : V(G'_1) \cup V(G'_2) \to \{1, 2, 3\}$ as follows.

$$
g(v_{i,j}) = \begin{cases} 1, & \text{if } j = 1, 2, \ldots, \lfloor \frac{m_i}{3} \rfloor, \\ 2, & \text{if } j = \lfloor \frac{m_i}{3} \rfloor + 1, \lfloor \frac{m_i}{3} \rfloor + 2, \ldots, \lfloor \frac{2m_i}{3} \rfloor, \\ 3, & \text{if } j = \lfloor \frac{2m_i}{3} \rfloor + 1, \lfloor \frac{2m_i}{3} \rfloor + 2, \ldots, m_i, \end{cases}
$$

where $g(x) = k$ means $x \in A_k$. By the definition of the function g, for integers $i \in [1, t]$, $p \in [1, \lfloor \frac{m_i}{3} \rfloor]$, $q \in [\lfloor \frac{m_i}{3} \rfloor + 1, \lfloor \frac{2m_i}{3} \rfloor]$ and $r \in [\lfloor \frac{2m_i}{3} \rfloor + 1, m_i]$, we have

$$
d(v_{i,p}, A_k) = \begin{cases} 0, & \text{if } k = 1, \\ \lfloor \frac{m_i}{3} \rfloor - p + 1, & \text{if } k = 2, \\ p, & \text{if } k = 3, \end{cases}
$$

$$d(v_{i,q}, A_k) = \begin{cases} 0, & \text{if } k = 2, \\ q - \lfloor \frac{m_i}{3} \rfloor & \text{if } k = 1, \\ \lfloor \frac{2m_i}{3} \rfloor - q + 1, & \text{if } k = 3, \end{cases}$$
$$d(v_{i,r}, A_k) = \begin{cases} 0, & \text{if } k = 3, \\ r - \lfloor \frac{2m_i}{3} \rfloor, & \text{if } k = 2, \\ m_i - r + 1, & \text{if } k = 1. \end{cases}$$

We consider any two vertices $x, y \in V(G'_1)$ or $x, y \in V(G'_2)$ in A_k for some $1 \leq k \leq 3$. For $x = v_{i,a}$ and $y = v_{i,b}$ in A_1 where $1 \leq a < b \leq \lfloor \frac{m_i}{3} \rfloor]$ or in A_2 where $\lfloor \frac{m_i}{3} \rfloor + 1 \leq a < b \leq \lfloor \frac{2m_i}{3} \rfloor$ or in A_3 where $\lfloor \frac{2m_i}{3} \rfloor + 1 \leq a < b \leq m_i$, clearly if $a < b$, then $d(x, A_2) = \lfloor \frac{m_i}{3} \rfloor - a + 1 > \lfloor \frac{m_i}{3} \rfloor - b + 1 = d(y, A_2)$, or $d(x, A_3) = \lfloor \frac{2m_i}{3} \rfloor - a + 1 > \lfloor \frac{2m_i}{3} \rfloor - b + 1 = d(y, A_3)$ or $d(x, A_1) = m_i - a + 1 > m_i - b + 1 = d(y, A_1)$, respectively. Now, assume that $x = v_{i,a}$ and $y = v_{j,b}$ where $1 \leq i < j \leq t$, so that $m_j \geq m_i + 3(j - i)$. We distinguish three subcases.

Subcase 1.1 $x = v_{i,a}$ and $y = v_{j,b}$ in A_1 where $1 \leq a \leq \lfloor \frac{m_i}{3} \rfloor$ and $1 \leq b \leq \lfloor \frac{m_j}{3} \rfloor$. Note that $\lfloor \frac{m_i}{3} \rfloor < \lfloor \frac{m_i}{3} + 1 \rfloor \leq \lfloor \frac{m_i}{3} + (j - i) \rfloor \leq \lfloor \frac{m_j}{3} \rfloor$. Hence if $a = b$, then $d(x, A_2) = \lfloor \frac{m_i}{3} \rfloor - a + 1 < \lfloor \frac{m_j}{3} \rfloor - b + 1 = d(y, A_2)$. Otherwise, $d(x, A_3) = a \neq b = d(y, A_3)$.

Subcase 1.2 $x = v_{i,a}$ and $y = v_{j,b}$ in A_2 where $\lfloor \frac{m_i}{3} \rfloor + 1 \leq a \leq \lfloor \frac{2m_i}{3} \rfloor]$ and $\lfloor \frac{m_j}{3} \rfloor + 1 \leq b \leq \lfloor \frac{2m_j}{3} \rfloor]$. This is easy to see that $\lfloor \frac{2m_i}{3} \rfloor - \lfloor \frac{m_i}{3} \rfloor = \lfloor \frac{m_i+1}{3} \rfloor$. Hence if $a - \lfloor \frac{m_i}{3} \rfloor = b - \lfloor \frac{m_j}{3} \rfloor$, then

$$\begin{aligned} d(x, A_3) &= \left\lfloor \frac{2m_i}{3} \right\rfloor - a + 1 \\ &= \left\lfloor \frac{2m_i}{3} \right\rfloor - \left[b - \left\lfloor \frac{m_j}{3} \right\rfloor + \left\lfloor \frac{m_i}{3} \right\rfloor \right] + 1 \\ &= \left\lfloor \frac{m_i + 1}{3} \right\rfloor - b + \left\lfloor \frac{m_j}{3} \right\rfloor + 1 \\ &\leq \left\lfloor \frac{m_j - 3(j - i) + 1}{3} \right\rfloor - b + \left\lfloor \frac{m_j}{3} \right\rfloor + 1 \\ &= \left\lfloor \frac{2m_j}{3} \right\rfloor - (j - i) - b + 1 \\ &< \left\lfloor \frac{2m_j}{3} \right\rfloor - b + 1 \\ &= d(y, A_3). \end{aligned}$$

Otherwise, $d(x, A_1) = a - \lfloor \frac{m_i}{3} \rfloor \neq b - \lfloor \frac{m_j}{3} \rfloor = d(y, A_1)$.

Subcase 1.3 $x = v_{i,a}$ and $y = v_{j,b}$ in A_3 where $\lfloor \frac{2m_i}{3} \rfloor + 1 \leq a \leq m_i$ and $\lfloor \frac{2m_j}{3} \rfloor + 1 \leq b \leq m_j$.

This is easy to see that $m_i - \lfloor \frac{2m_i}{3} \rfloor = \lceil \frac{m_i}{3} \rceil$. Hence if $a - \lfloor \frac{2m_i}{3} \rfloor = b - \lfloor \frac{2m_j}{3} \rfloor$, then

$$\begin{aligned} d(x, A_1) &= m_i - a + 1 \\ &= m_i - \left[b - \left\lfloor \frac{2m_j}{3} \right\rfloor + \left\lfloor \frac{2m_i}{3} \right\rfloor \right] + 1 \\ &= \left\lceil \frac{m_i}{3} \right\rceil - b + \left\lfloor \frac{2m_j}{3} \right\rfloor + 1 \\ &\leq \left\lceil \frac{m_j - 3(j-i)}{3} \right\rceil - b + \left\lfloor \frac{2m_j}{3} \right\rfloor + 1 \\ &= m_j - (j-i) - b + 1 \\ &< m_j - b + 1 \\ &= d(y, A_1). \end{aligned}$$

Otherwise, $d(x, A_2) = a - \lfloor \frac{2m_i}{3} \rfloor \neq b - \lfloor \frac{2m_j}{3} \rfloor = d(y, A_2)$.

This implies that $r(x|\Lambda_1) \neq r(y|\Lambda_1)$ for any two vertices $x, y \in V(G'_1)$ or $x, y \in V(G'_2)$ in A_k for $1 \leq k \leq 3$, and so Λ_1 is a resolving partition of G'_1 or G'_2.

Case 2. $G \subseteq G'_3 \subseteq G_3$. Let A, B and C be three subsets of $V(G'_3)$ where for all integers $i \in [1, t]$, $A = \{v_{i,a} : 1 \leq a \leq i\}$, $B = \{v_{i,b} : i+1 \leq b \leq 2i\}$ and $C = \{v_{i,c} : 2i+1 \leq c \leq m_i\}$. Let us consider the partition $\Lambda = \{A_1, A_2, A_3\}$ of $V(G)$ used in the proof of Theorem 2.1. Hence for some $1 \leq j < i \leq t$, we have

$$\begin{aligned} d(v_{i,a}, B) &= \min\{d(v_{i,a}, v_{i,i+1}), d(v_{i,a}, v_{j,j}) + 1\} = d(v_{i,a}, v_{i,i+1}) = i + 1 - a \\ &= d(v_{i,a}, A_2), \\ d(v_{i,a}, C) &= \min\{d(v_{i,a}, v_{i,m_i}), d(v_{i,a}, v_{j,1}) + 1\} = d(v_{i,a}, v_{i,m_i}) = a \\ &= d(v_{i,a}, A_3), \\ d(v_{i,b}, A) &= \min\{d(v_{i,b}, v_{i,i}), d(v_{i,b}, v_{j,j+1}) + 1\} = d(v_{i,b}, v_{i,i}) = b - i \\ &= d(v_{i,b}, A_1), \\ d(v_{i,b}, C) &= \min\{d(v_{i,b}, v_{i,2i+1}), d(v_{i,b}, v_{j,2j}) + 1\} = d(v_{i,b}, v_{i,2i+1}) = 2i + 1 - b \\ &= d(v_{i,b}, A_3), \\ d(v_{i,c}, A) &= \min\{d(v_{i,c}, v_{i,1}), d(v_{i,c}, v_{i,i}), d(v_{i,c}, v_{j,m_j}) + 1, d(v_{i,c}, v_{j,j+1}) + 1\} \\ &= \min\{d(v_{i,c}, v_{i,1}), d(v_{i,c}, v_{i,i})\} = \min\{m_i - c + 1, c - i\} \\ &= d(v_{i,c}, A_1), \\ d(v_{i,c}, B) &= \min\{d(v_{i,c}, v_{i,2i}), d(v_{i,c}, v_{i,i+1}), d(v_{i,c}, v_{j,2j+1}) + 1, d(v_{i,c}, v_{j,j}) + 1\} \\ &= \min\{d(v_{i,c}, v_{i,2i}), d(v_{i,c}, v_{i,i+1})\} = \min\{c - 2i, m_i - c + (i+1)\} \\ &= d(v_{i,c}, A_2). \end{aligned}$$

Note that $\Lambda = \{A_1, A_2, A_3\}$ is a resolving partition of G. Thus, we can define a partition $\Lambda_2 = \{A, B, C\}$ of $V(G'_3)$, such that there is a 1-1 correspondence between Λ and Λ_2 where $A = A_1$, $B = A_2$ and $C = A_3$. Therefore, for any two vertices $v_{i,j}, v_{k,l} \in V(G'_3)$ for integers $i, k \in [1, t]$, $j \in [1, m_i]$ and $l \in [1, m_k]$, then $r(v_{i,j}|\Lambda_2) = r(v_{i,j}|\Lambda) \neq r(v_{k,l}|\Lambda) = r(v_{k,l}|\Lambda_2)$. This concludes that Λ_2 is a resolving partition of $V(G'_3)$. □

The four graphs in Figure 1 give an illustration of the graphs provided for Theorem 2.2. These graphs are obtained from a disjoint union of 3 cycles having partition dimension 3. Figure 1 (a) represents $G_1 = G \cup E_1 \cup E_2 \cup E_3$ where $G = C_7 \cup C_{10} \cup C_{13}$. Furthermore, Figure 1 (b), (c) and (d) represent $G_2 = G \cup E_1 \cup E_2 \cup E_4$, $G_3 = G \cup E_5 \cup E_6 \cup E_7$ and $G'_3 \subset G_3$, respectively, where $G = C_7 \cup C_{10} \cup C_{15}$.

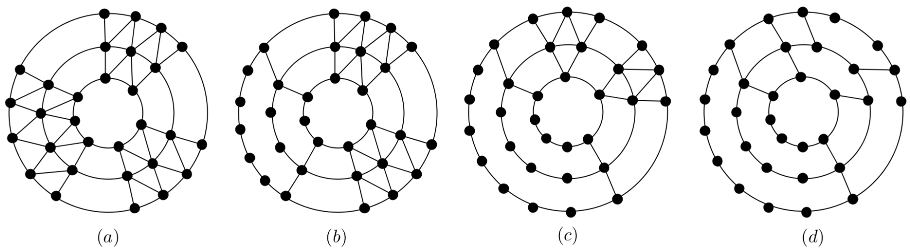

Figure 1. Graphs (a) G_1, (b) G_2, (c) G_3 and (d) $G'_3 \subset G_3$ where $pd(G_1) = pd(G_2) = pd(G_3) = pd(G'_3) = 3$.

In advance, for $G = \bigcup_{i=1}^{t} C_{m_i}$, G_1, G_2 and G_3 defined in **(C-1)**, **(C-2)** and **(C-3)**, respectively, let F_1, F_2 and F_3 be three subsets of $E(G)$ as follows.

$$
\begin{aligned}
F_1 &= \left\{v_{i,j}v_{i,j+1} : 2 \le i \le t,\ 1 \le j \le m_i - 1, j \ne \left\lfloor \frac{m_i}{3} \right\rfloor \text{ and } j \ne \left\lfloor \frac{2m_i}{3} \right\rfloor \right\}, \\
F_2 &= \left\{v_{i,j}v_{i,j+1} : 2 \le i \le t,\ 1 \le j \le \left\lfloor \frac{2m_i}{3} \right\rfloor - 1, j \ne \left\lfloor \frac{m_i}{3} \right\rfloor \right\}, \\
F_3 &= \{v_{i,j}v_{i,j+1} : 2 \le i \le t, 1 \le j \le 2i - 1, j \ne i\}.
\end{aligned}
$$

By eliminating some edges of F_i of G_i for $1 \le i \le 3$, we obtain some new connected graphs whose partition dimensions are 3.

Theorem 2.3. *If $F_i' \subseteq F_i$ for any integer $i \in [1, 3]$, then $pd(G_i - F_i') = 3$.*

Proof. For $i = 1, 2, 3$, let $H_i = G_i - F_i'$. Certainly, $pd(H_i) \ge 3$ for each $i = 1, 2, 3$. To show that $pd(H_i) \le 3$, we consider two cases.

Case 1. Graph H_1 or H_2.

Define a partition $\Lambda_1' = \{A_1', A_2', A_3'\}$ of H_1 or H_2 such that for all $i \ge 1$, $A_1' = \{v_{i,j} : 1 \le j \le \lfloor \frac{m_i}{3} \rfloor\}$, $A_2' = \{v_{i,j} : \lfloor \frac{m_i}{3} \rfloor + 1 \le j \le \lfloor \frac{2m_i}{3} \rfloor]\}$ and $A_3' = \{v_{i,j} : \lfloor \frac{2m_i}{3} \rfloor + 1 \le j \le m_i]\}$. By considering the resolving partition $\Lambda_1 = \{A_1, A_2, A_3\}$ of G_1' or G_2' used in the proof of Theorem 2.2 Case 1, then for integers $i \in [1, t]$, $a \in [1, \lfloor \frac{m_i}{3} \rfloor]$, $b \in [\lfloor \frac{m_i}{3} \rfloor + 1, \lfloor \frac{2m_i}{3} \rfloor]$ and $c \in [\lfloor \frac{2m_i}{3} \rfloor + 1, m_i]$, the following equalities hold for H_1 or H_2.

$$
\begin{aligned}
d(v_{i,a}, A_2') &= \min\{d(v_{i,a}, v_{j,\lfloor \frac{m_j}{3} \rfloor}) + 1 : j \le i\} = \left\lfloor \frac{m_i}{3} \right\rfloor + 1 - a = d(v_{i,a}, A_2), \\
d(v_{i,a}, A_3') &= \min\{d(v_{i,a}, v_{j,1}) + 1 : j \le i\} = a = d(v_{i,a}, A_3), \\
d(v_{i,b}, A_1') &= \min\{d(v_{i,b}, v_{j,\lfloor \frac{m_j}{3} \rfloor + 1}) + 1 : j \le i\} = b - \left\lfloor \frac{m_i}{3} \right\rfloor = d(v_{i,b}, A_1), \\
d(v_{i,b}, A_3') &= \min\{d(v_{i,b}, v_{j,\lfloor \frac{2m_j}{3} \rfloor}) + 1 : j \le i\} = \left\lfloor \frac{2m_i}{3} \right\rfloor + 1 - b = d(v_{i,b}, A_3), \\
d(v_{i,c}, A_1') &= \min\{d(v_{i,c}, v_{j,m_j}) + 1 : j \le i\} = m_i - c + 1 = d(v_{i,c}, A_1), \\
d(v_{i,c}, A_2') &= \min\{d(v_{i,c}, v_{j,\lfloor \frac{2m_j}{3} \rfloor + 1}) + 1 : j \le i\} = c - \left\lfloor \frac{2m_i}{3} \right\rfloor = d(v_{i,c}, A_2).
\end{aligned}
$$

Therefore, by taking a 1-1 correspondence between Λ_1' and Λ_1 of the proof of Theorem 2.2 Case 1, in which $A_i' = A_i$ for all $i = 1, 2, 3$, we can also conclude that Λ_1' is also a resolving partition of H_1 or H_2.

Case 2. Graph H_3.

Define the 3-partition $\Lambda_2' = \{A, B, C\}$ of H_3 where $A = \{v_{i,a} : 1 \le a \le i\}$, $B = \{v_{i,b} : i + 1 \le b \le 2i\}$ and $C = \{v_{i,c} : 2i + 1 \le c \le m_i\}$. By considering the resolving partition $\Lambda = \{A_1, A_2, A_3\}$ of G in the proof of Theorem 2.1, we can verify that these equalities hold for

H_3.

$$\begin{aligned}
d(v_{i,a}, B) &= \min\{d(v_{i,a}, v_{j,j}) + 1 : j \leq i\} = i + 1 - a = d(v_{i,a}, A_2),\\
d(v_{i,a}, C) &= \min\{d(v_{i,a}, v_{j,1}) + 1 : j \leq i\} = a = d(v_{i,a}, A_3),\\
d(v_{i,b}, A) &= \min\{d(v_{i,b}, v_{j,j+1}) + 1 : j \leq i\} = b - i = d(v_{i,b}, A_1),\\
d(v_{i,b}, C) &= \min\{d(v_{i,b}, v_{j,2j}) + 1 : j \leq i\} = 2i + 1 - b = d(v_{i,b}, A_3),\\
d(v_{i,c}, A) &= \min\{d(v_{i,c}, v_{i,1}), d(v_{i,c}, v_{i,i})\} = \min\{m_i - c + 1, c - i\} = d(v_{i,c}, A_1),\\
d(v_{i,c}, B) &= \min\{d(v_{i,c}, v_{i,2i}), d(v_{i,c}, v_{i,i+1}) = \min\{c - 2i, m_i - c + (i + 1)\}\\
&= d(v_{i,c}, A_2).
\end{aligned}$$

Therefore, by having a 1-1 correspondence between Λ_2' and Λ of the proof of Theorem 2.1 where $A = A_1$, $B = A_2$ dan $C = A_3$, we can also conclude that Λ_2' is also a resolving partition H_3. □

Figure 2 represents some graphs satisfying Theorem 2.3. In Figure 2 (a) and (b) we give an illustration of $H_1 = G_1 - F_1'$ and $H_1' \supset H_1$, respectively, where $G_1 = G \cup E_1 \cup E_2 \cup E_3$ and $G = C_7 \cup C_{10} \cup C_{13}$. Meanwhile in Figure 2 (c) and (d) we give an example of $H_2 = G_2 - F_2'$ and $H_3 = G_3 - F_3'$, respectively, where $G_2 = G \cup E_1 \cup E_2 \cup E_4$, $G_3 = G \cup E_5 \cup E_6 \cup E_7$ and $G = C_7 \cup C_{10} \cup C_{15}$.

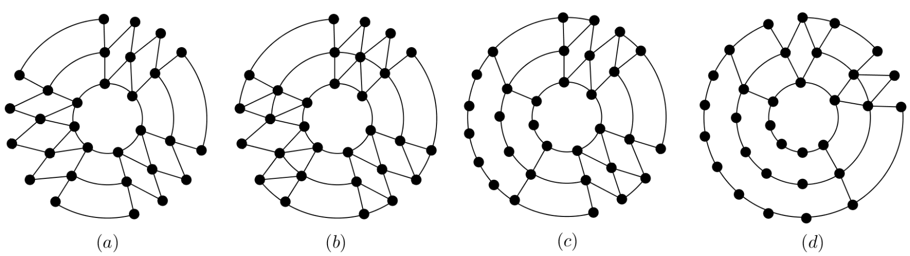

Figure 2. Graphs (a)H_1, (b)$H_1' \supset H_1$, (c)H_2 and (d)H_3 where $pd(H_1) = pd(H_1') = pd(H_2) = pd(H_3) = 3$.

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