Indonesian Journal of Combinatorics **8** (1) (2024), 13–25

INDONESIAN JOURNAL OF COMBINATORICS

On Ramsey numbers for trees versus fans of even order

Intan Sherlin[a], Suhadi Wido Saputro[b,c], Edy Tri Baskoro*[b,c], Finny Oktariani[b,c]

[a]*Doctoral Program of Mathematics,*
Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Indonesia
[b]*Combinatorial Mathematics Research Group,*
Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Indonesia
[c]*Collaborative Research Center for Graph Theory and Combinatorics, Indonesia*

30121002@mahasiswa.itb.ac.id, suhadi@itb.ac.id, *e3baskoro82@gmail.com, oktariani.f@itb.ac.id

Abstract

Given two graphs G and H. The graph Ramsey number $R(G, H)$ is the least natural number r such that, for every graph F on r vertices, either F contains a duplicate of G or $\overline{F}$ contains a duplicate of H. A vertex v is a dominating vertex in G if it is adjacent to every other vertices of G. A wheel graph W_m consists of one dominating vertex and m other vertices forming a cycle. A fan graph $F_{1,m}$ is formed from W_m by expelling one cycle-edge. In this paper, we consider the graph Ramsey number $R(T_n, F_{1,m})$ of a tree T_n versus a fan $F_{1,m}$. Li et al. (2016) initiated the study on $R(T_n, F_{1,m})$ when T_n is a star. Sherlin et al. (2023) continued the research for T_n that is not a star versus fan $F_{1,m}$ with even $m \leq 8$. The graph Ramsey numbers $R(T_n, F_{1,m})$ for odd $m \leq 8$ will be provided in this paper.

Keywords: Ramsey number, Ramsey number for trees versus fans, Ramsey number for trees versus fans of even order
Mathematics Subject Classification : 05D10, 05C55

1. Introduction

Let $G(V, E)$ be a simple graph. Let $G[X]$ be the maximal subgraph of G with vertex set X, i.e. the induced sub-graph of G by X, for $X \subseteq V(G)$. The *degree* $deg_G(x)$ of a vertex

Received: 31 October 2023, Revised: 22 December 2023, Accepted: 27 December 2023.

$x \in V(G)$ is defined as $|\{v \in V(G) : xv \in E(G)\}|$. The highest degree of G is represented by $\Delta(G) = \max\{\deg(v) : v \in V(G)\}$, whereas the smallest degree of G is represented by $\delta(G) = \min\{\deg(x) : x \in V(G)\}$. We denote the neighbors of x in S as $N_S(x) = \{s \in S : sv \in E(G)\}$, and $E(S,T)$ as the set of the edges between S and T for every $x \in V(G)$ and $S, T \subseteq V(G)$.

Let a *tree* T_n be an acyclic connected graph of order n, and a *star* S_n be a bipartite graph $K_{1,n-1}$. We refer to a multipartite graph as *k-partite* if it has k subsets of vertices $V_1, V_2, \ldots, V_k$. When two vertices in a k-partite graph are neighbors if and only if they belong to two distinct subsets, the graph is said to be a complete k-partite. $K_{n_1,n_2,\ldots,n_k}$ is a complete k-partite graph with $|V_i| = n_i$ for every $i = 1, 2, \ldots, k$, while $K_{m\times n}$ is a complete m-partite graph with $|V_i| = n$ for every $i = 1, 2, \ldots, m$. The notation $S_n(l, m)$ refers to a tree of order n that is derived by subdividing each of the l selected edges of $S_{n-m\times l}$ m times. Similarly, $S_n(l)$ represents a tree of order n that is formed by adding an edge to connect the centers of S_l and S_{n-l}. These follow the notations and definitions in [4]. The hub vertex of $S_n(l, m)$ is the center of $S_{n-m\times l}$, while the hub vertex of $S_n(l)$ is the center of S_{n-l}. An illustration of star graphs, tree graphs, and some subdivisions of trees can be seen in Figure 1.

A vertex v is a dominating vertex in G if it is adjacent to every other vertex of G. A wheel graph W_m consists of one dominating vertex and m other vertices forming a cycle C_m. All edges of the cycles in the wheel are called *cycle-edges* or rims. Other edges of the wheel are called *spoke-edges* or spokes. We use notation tW_m to describe a graph with t copies of wheel W_m. A fan graph $F_{1,m}$ is formed from a wheel W_m by removing one cycle-edge. We illustrate how some fan graphs can be drawn as a subgraph of a wheel graph in Figure 2.

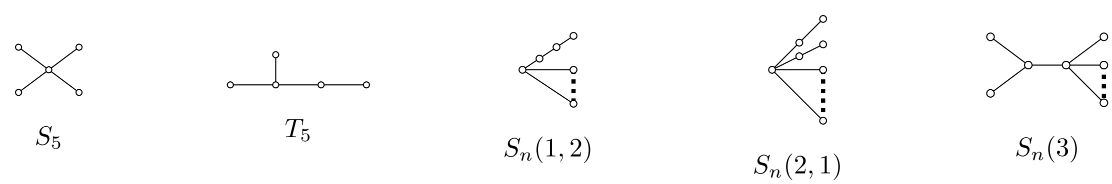

Figure 1: Star graph, tree graph, and some subdivision of tree graph

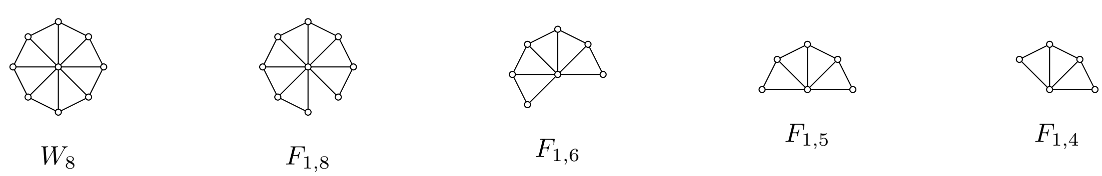

Figure 2: Some fan graphs drawn as a subgraph of a wheel W_8

The Ramsey number $R(G, H)$ for a graph F of order r given two graphs G and H is the smallest natural number r by which each of the following two conditions is satisfied: F contains a duplicate of G or $\overline{F}$ contains a duplicate of H. In terms of coloring, $R(G, H) = r$ states that every edge coloring with two colors of the complete graph K_r of order r will always contain a monochromatic G or H as a subgraph. A graph F on order r is (G, H, r)-*good*, or simply called (G, H)-*good*, if $G \not\subseteq F$ and $H \not\subseteq \overline{F}$. Therefore, $R(G, H) \geq r + 1$ if a (G, H)-*good* graph F of order r could be identified. In other terminology, F is (G, H)–*free* if F contains no G and $\overline{F}$

contains no H. Thus we have an equivalent definition of the Ramsey number $R(G,H)$ which is the smallest positive integer r such that there is no (G,H)–free graph of order n. Some known values of classical Ramsey number $r(n,m)$ and graph Ramsey number $R(G,H)$ can be found in the dynamic survey of Radziszowski [24], whereas its applications were collected by Rosta in [25].

Let $\chi(H)$ denote the chromatic number of H, and $c(G)$ represent the order of the largest component of G. The relationship between $R(G,H)$, $\chi(H)$, and $c(G)$ was established by Chvátal and Harary [9] in the theorem below.

Theorem 1.1. *[9]* $R(G,H) \geq (\chi(H)-1)(c(G)-1)+1.$

Burr [22] then refines this lower bound by introducing the term $s(H)$ as the *chromatic surplus* of H, i.e. minimum cardinality of color classes taken over all proper $\chi(H)$-coloring of H), as follows.

Theorem 1.2. *[22]* $R(G,H) \geq (\chi(H)-1)(c(G)-1)+s(H).$

The graph G is called H–good when the above equality holds. The goodness of some G and H were mentioned by Sudarsana in [23], as follows: C_n is W_m-good (Surahmat et al., 2006), P_n is W_m-good for even m and $n \geq m-1 \geq 3$ (Chen et al., 2005), P_n is tW_4-good for $n \geq 15t^2-4t+2, t \geq 1$ (Sudarsana, 2014), and C_n is tK_m-good (Sudarsana, 2016). It has also been conjectured that C_n is K_m-good for $n \geq m \geq 3$, except for $n = m = 3$. Meanwhile, Chen et al. [7] established that S_n is not W_6-good for $n \geq 3$.

Now, let us consider G as a tree T_n and H as a subgraph of a wheel W_m. In case $H = W_m$, if m is even, then by Theorem 1.1 we have $R(T_n,W_m) \geq 2n-1$, otherwise $R(T_n,W_m) \geq 3n-2$. However, determining an exact value of $R(T_n,W_m)$ for any positive integers $n,m \geq 3$ is hard. Therefore, some authors investigated the graph Ramsey number of T_n versus W_m for a certain class of trees or a fix value of m. We wrote some of their results in the next subsection.

1.1. Earlier Studies on $R(T_n,W_m)$

Surahmat et al. [12] established the Ramsey number for star versus wheel W_4 and W_5, while Baskoro et al. [1] provided the Ramsey numbers for T_n which is not a star versus W_4 and W_5. Their results were summarized in Theorem 1.3.

Theorem 1.3. *[12]*

- $R(S_n,W_4) = \begin{cases} 2n+1, & n \text{ even} \\ 2n+3, & n \text{ odd} \end{cases} \quad ; n \geq 4$
- $R(T_n,W_4) = 2n-1,\ n \geq 4$ *if* T_n *is not a star*
- $R(T_n,W_5) = 3n-2,\ n \geq 3$ *for every* T_n

As mentioned in [7], T_n will be isomorphic to either S_n, $S_n(1,1)$, $S_n(1,2)$, $S_n(2,1)$, or $S_n(3)$ if we examine any T_n with $\Delta(T_n) \geq n-3$. Chen et al. [7] provided the Ramsey numbers $R(T_n,W_6)$ and $R(T_n,W_7)$ with $\Delta(T_n) \geq n-3$ in the following theorem.

Theorem 1.4. *[4, 7, 8]*

- $R(S_n, W_6) = 2n + 1, n \geq 5$
- $R(S_n(1,1), W_6) = 2n, n \geq 5$
- $R(S_n(1,2), W_6) = \begin{cases} 2n & , n \equiv 0 \ (mod\ 3) \\ 2n-1, & n \not\equiv 0 \ (mod\ 3) \end{cases} \quad ; n \geq 5$
- $R(S_n(2,1), W_6) = 2n - 1, n \geq 5$
- $R(S_n(3), W_6) = 2n - 1, n \geq 5$
- $R(T_n, W_7) = 3n - 2, n \geq 6$

Zhang et. al. [23, 24] determined the Ramsey number $R(S_n, W_8)$, while Hafidh and Baskoro [10] determined $R(T_n, W_8)$ for T_n other than a star with $\Delta(T_n) \geq n - 3$. Their results can be seen in the following theorem.

Theorem 1.5. *[17, 18, 10]*

- $R(S_n, W_8) = \begin{cases} 2n+2 & , n \text{ even} \\ 2n+1 & , n \text{ odd} \end{cases} \quad ; n \geq 5$
- $R(S_n(1,1), W_8) = \begin{cases} 2n & , n \text{ even} \\ 2n+1, & n \text{ odd} \end{cases} \quad ; n \geq 5$
- $R(S_n(1,2), W_8) = \begin{cases} 2n & , n \not\equiv 3 \ (mod\ 4) \\ 2n+1, & n \equiv 3 \ (mod\ 4) \end{cases} \quad ; n \geq 8$
- $R(S_n(2,1), W_8) = \begin{cases} 2n-1, & n \text{ odd} \\ 2n & , n \text{ even} \end{cases} \quad ; n \geq 8$
- $R(S_n(3), W_8) = \begin{cases} 2n-1, & n \text{ odd} \\ 2n & , n \text{ even} \end{cases} \quad ; n \geq 8.$

In [19], Li et. al. have determined the Ramsey number $R(S_n, F_{1,m})$ as follows:

Theorem 1.6. *[19] Let $m, n \geq 2$ be a natural number with $m \leq 2n - 1$.*

$$R(S_n, F_{1,m}) = \begin{cases} 2n + \lfloor m/2 \rfloor - 3 & , n \text{ and } m \text{ even} \\ 2n + \lfloor m/2 \rfloor - 2 & , \text{otherwise} \end{cases}$$

Some exact values of $R(S_n, F_{1,m})$ derived from Theorem 1.6 are presented in Corollary 1.1.

Corollary 1.1. *[19] Let $n, m \geq 2$ be natural number with $m \leq 2n-1$. The Ramsey number for star S_n of order n versus fan $F_{1,m}$, $2 \leq m \leq 8$ are as follows:*

- $R(S_n, F_{1,2}) = \begin{cases} 2n-2 & , n \text{ even} \\ 2n-1 & , n \text{ odd} \end{cases}$
- $R(S_n, F_{1,3}) = 2n-1$
- $R(S_n, F_{1,4}) = \begin{cases} 2n-1 & , n \text{ even} \\ 2n & , n \text{ odd} \end{cases}$
- $R(S_n, F_{1,5}) = 2n$
- $R(S_n, F_{1,6}) = \begin{cases} 2n & , n \text{ even} \\ 2n+1 & , n \text{ odd} \end{cases}$
- $R(S_n, F_{1,7}) = 2n+1$
- $R(S_n, F_{1,8}) = \begin{cases} 2n+1 & , n \text{ even} \\ 2n+2 & , n \text{ odd} \end{cases}$

Sherlin et. al. (2023) provided the following theorem which is useful to prove some results in this paper.

Theorem 1.7. *[20] If G, H_1, and H_2 be graphs with $H_2 \subseteq H_1$, then $R(G, H_2) \leq R(G, H_1)$.*

Proof. Let $R(G, H_1) = n$ and F be any graph with n vertices. If $G \not\subseteq F$, then $H_1 \subseteq \overline{F}$. Since $H_2 \subseteq H_1$ and $H_1 \subseteq \overline{F}$, then $H_2 \subseteq \overline{F}$. Therefore, $R(G, H_2) \leq n = R(G, H_1)$. □

1.2. Earlier Results on $R(T_n, F_{1,m})$

In this section, we write some results from Sherlin et. al. (2023) on $R(T_n, F_{1,m})$ for a tree T_n of order n other than a star versus fan $F_{1,m}$ with even $m \leq 8$.

Theorem 1.8. *[20] $R(T_n, F_{1,4}) = 2n-1, n \geq 4$.*

Proof. Since $F_{1,4} \subseteq W_4$ and $R(T_n, W_4) = 2n-1$, from Theorem 1.7 and 1.3 we get $R(T_n, F_{1,4}) \leq R(T_n, W_4) = 2n-1$. By Theorem 1.1, we know that $R(T_n, F_{1,4}) \geq 2n-1$. Therefore, $R(T_n, F_{1,4}) = 2n-1$.

□

Theorem 1.9. *[20] The resulting Ramsey number $R(T_n, F_{1,6})$ for a tree T_n of order n, excluding stars, with $\Delta(T_n) \geq n-3$ is as follows:*

- $R(S_n(1,1), F_{1,6}) = 2n, n \geq 5$

- $R(S_n(1,2), F_{1,6}) = \begin{cases} 2n & , n \equiv 0 \ (mod\ 3) \\ 2n-1, & n \not\equiv 0 \ (mod\ 3) \end{cases} \quad ; n \geq 5$
- $R(S_n(2,1), F_{1,6}) = 2n-1, n \geq 5$
- $R(S_n(3), F_{1,6}) = 2n-1, n \geq 5$

Theorem 1.10. *[20] The resulting Ramsey number $R(T_n, F_{1,8})$ for a tree T_n of order n, excluding stars, with $\Delta(T_n) \geq n-3$ is as follows:*

- $R(S_n(1,1), F_{1,8}) = \begin{cases} 2n & , \textit{even } n \\ 2n+1, & \textit{odd } n \end{cases} \quad ; n \geq 5$
- $R(S_n(1,2), F_{1,8}) = \begin{cases} 2n & , n \not\equiv 3 \pmod 4 \\ 2n+1, & n \equiv 3 \pmod 4) \end{cases} \quad ; n \geq 8$
- $R(S_n(2,1), F_{1,8}) = \begin{cases} 2n-1, & \textit{odd } n \\ 2n & , \textit{even } n \end{cases} \quad ; n \geq 8$
- $R(S_n(3), F_{1,8}) \quad = \begin{cases} 2n-1, & \textit{odd } n \\ 2n & , \textit{even } n \end{cases} \quad ; n \geq 8.$

The proofs of the two theorems above have been given in [20] thus were excluded in this paper.

2. Main Results

Let T_n be a tree of order n which is not a star and $\Delta(T_n) \geq n-3$. We will provide an exact value of $R(T_n, F_{1,m})$ for odd $m \leq 7$.

Theorem 2.1. $R(T_n, F_{1,3}) = 2n-1$, $n \geq 4$.

Proof. This theorem is a direct consequence of Theorems 1.1, 1.7, and 1.8. □

Theorem 2.2. $R(T_n, F_{1,5}) = 2n-1$, *for all* T_n, $n \geq 5$

Proof. Let $n \geq 5$. Consider $F = 2K_{n-1}$. It is obvious that $F \not\supseteq T_n$ and $\overline{F} \not\supseteq F_{1,5}$. So, $R(T_n, F_{1,5}) > 2n-2 > 2n-1$. To show the upper bound, consider graph F of order $2n-1$ and F does not contain T_n. We will show that $\overline{F}$ contains $F_{1,5}$. From Corollary 1.1, we have $R(S_{n-1}, F_{1,5}) = 2n-2$. Since $2n-2 > 2n-1$, then $F \supseteq S_{n-1}$ or $\overline{F} \supseteq F_{1,5}$. Suppose that $F \supseteq S_{n-1}$ with u_1 as the center (see Figure 3). Let $U = V(S_{n-1}) = \{u_1, ..., u_{n-1}\}$, and $W = V(F-U) = \{w_1, ..., w_n\}$. Since $F \not\supseteq T_n$, we know that $E(U, W) = \emptyset$ and W consists of two components W_1 and W_2. Suppose that W_1 is the smallest component of W, then there is at least one vertex w_1 in W_1. Since $n \geq 5$ and $|W_2| \geq |W_1|$, then there are at least three vertices w_2, w_3, w_4 in W_2. The induced graph in the complement by $\{u_2, u_3, u_4, w_1, w_2, w_3\}$, namely $\overline{F}[w_1, w_2, w_3, u_2, u_3, u_4]$, contains $F_{1,5}$ with w_1 as the dominating vertex connected to the path $u_2w_2u_3w_3u_4$. Therefore, $\overline{F} \supseteq F_{1,5}$.

□

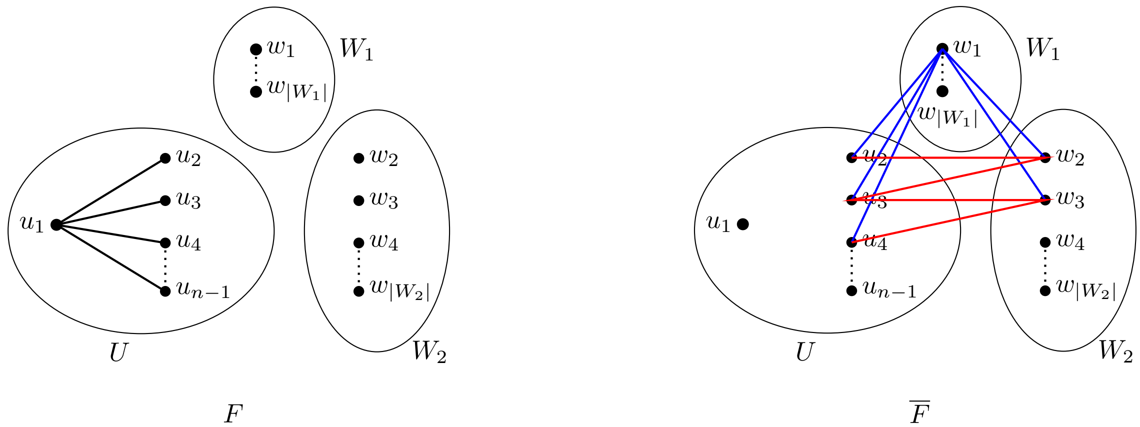

Figure 3: S_{n-1} in F and $F_{1,5}$ in $\overline{F}$

Theorem below will give $R(T_n, F_{1,7})$ for tree $S_n(1,1)$ versus fan $F_{1,7}$.

Theorem 2.3. $R(S_n(1,1), F_{1,7}) = 2n, n \geq 6$

Proof. Let $n \geq 6$. For even n, $R(S_n(1,1), F_{1,7}) = 2n$ as a direct consequence of Theorems 1.7 and 1.10. Now let n be odd. From Theorem 1.9, we have $2n \leq R(S_n(1,1), F_{1,7})$. We will prove that $R(S_n(1,1), F_{1,7}) \leq 2n$. Let F be a graph of order $2n$ and F does not contain $S_n(1,1)$. We will show that $\overline{F}$ contains $F_{1,7}$. From Corollary 1.1, we know that $R(S_{n-1}, F_{1,7}) = 2n - 1$. Since $2n > 2n - 1$, then either $F \supseteq S_{n-1}$ or $\overline{F} \supseteq F_{1,7}$. Suppose that F contains S_{n-1} with u_1 as the center (see Figure 4). Let $U = V(S_{n-1}) - \{u_1\} = \{u_2, ..., u_{n-1}\}$, and $W = V(F) - (U \cup \{u_1\}) = \{w_1, ..., w_n, w_{n+1}\}$. Since F does not contain $S_n(1,1)$, then every vertex in U cannot be adjacent to every vertex in W, thus $E(U, W) = \emptyset$.

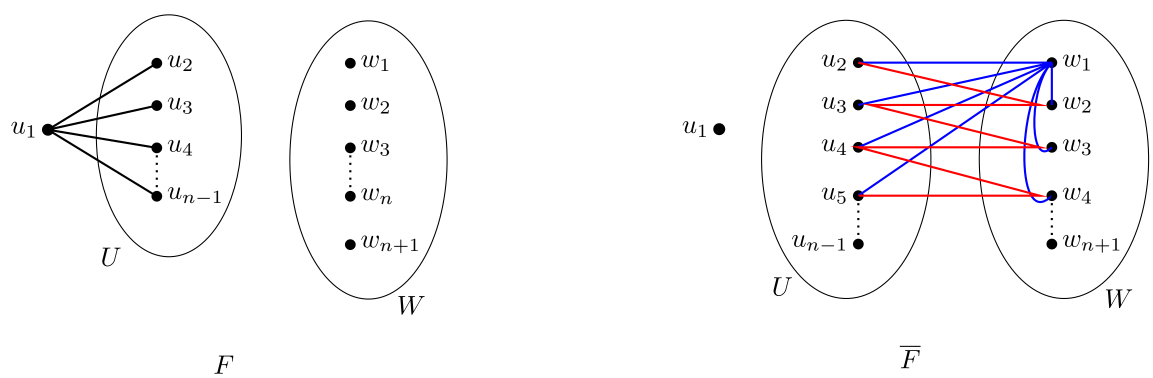

Figure 4: S_{n-1} in F and $F_{1,7}$ in $\overline{F}$

Case 1: $\delta(F[W]) \leq n - 3$. Let $w_1 \in V(F[W])$ be a vertex with $deg_{F[W]}(w_1) = \delta(F[W]) \leq n - 3$. As $|W| = n + 1$, we know that $deg_{\overline{F}[W]}(w_1) \geq n + 1 - 1 - (n - 3) = 3$. Let $\{w_2, w_3, w_4\} \subseteq N_{\overline{F}[W]}(w_1)$. Then the induced subgraph in the complement by $\{w_1, w_2, w_3, w_4, u_2, u_3, u_4, u_5\}$, namely $\overline{F}[w_1, w_2, w_3, w_4, u_2, u_3, u_4, u_5]$, contains $F_{1,7}$ with w_1 as the dominating vertex connected to path $u_2w_2u_3w_3u_4w_4u_5$. Therefore, $\overline{F}$ contains $F_{1,7}$. Figure 4 shows an $F_{1,7}$ in $\overline{F}$ with red and blue edges, in which P_7 is shown in red.

Case 2: $\delta(F[W]) \geq n-2$. It means that every vertex of $F[W]$ has a degree of at least $n-2$. W.l.o.g, let $N_{F[W]}(w_1) = \{w_2, ..., w_{n-1}\}$. Since $\delta(F[W]) \geq n-2$ and $n \geq 5$, w_n has at three or more neighbours in $F[W]$. If $\{w_1, w_{n+1}\} \subseteq N_{F[W]}(w_n)$, then w_n has one or more neighbour in $N_{F[W]}(w_1)$ forming an $S_n(1,1)$ in F. This is a contradiction.

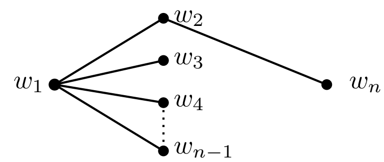

Figure 5: Illustration of Case 2: $S_n(1,1)$ in F

As a conclusion, there is no graph F of order $2n$ that does not contain $S_n(1,1)$ and $\overline{F}$ does not contain $F_{1,7}$. Hence, $R(S_n(1,1), F_{1,7}) \leq 2n$ for $n \geq 5$.

□

Next, to determine $R(S_n(1,2), F_{1,7})$, we are going to apply the lemma below.

Lemma 2.1. *Let G be a graph of order $2n$ containing an $S_n(1,1)$, with $n \geq 8$. If $\overline{G}$ does not contain $F_{1,7}$, then G contains $S_n(1,2)$.*

Proof. For $n \geq 8$, let G be a graph of order $2n$ which contains an $S_n(1,1)$. Let $V(S_n(1,1)) = \{u_0, u_1, ..., u_{n-1}\}$ with u_0 as the center of $S_n(1,1)$ in G and $u_0u_1, u_1u_{n-1} \in E(S_n(1,1))$. Let $U = V(S_n(1,1)) - \{u_0, u_1, u_{n-1}\} = \{u_2, ..., u_{n-2}\}$ and $W = V(G) - V(S_n(1,1)) = \{w_1, ..., w_n\}$. We are going to prove that if $\overline{G}$ does not contain $F_{1,7}$ then G will contain $S_n(1,2)$ by contraposition. Suppose that G does not contain $S_n(1,2)$, then u_{n-1} cannot be adjacent to every other vertex except u_0 and u_1, and every vertex in W can only have at most one neighbor in U. Thus, $N(u_{n-1}) \subseteq \{u_0, u_1\}$ and $|N_{G[U]}(w)| \leq 1, \forall w \in W$.

Case 1: There is a vertex u_2 in $G[U]$ with three or more neighbors in $G[W]$, w.l.o.g. $\{w_1, w_2, w_3\} \subseteq N_{G[W]}(u_2)$. Since $|N_{G[U]}(w)| \leq 1, \forall w \in W$, then we have u_3, u_4, u_5, and u_6 which are not the neighbors of w_1, w_2, and w_3 in G. The induced subgraph in the complement by vertex set $\{w_1, w_2, w_3, u_3, u_4, u_5, u_6, u_{n-1}\}$, namely $\overline{G}[w_1, w_2, w_3, u_3, u_4, u_5, u_6, u_{n-1}]$, will contain $F_{1,7}$ with u_{n-1} as the center connected to path $u_3w_1u_4w_2u_5w_3u_6$. Therefore, $\overline{G}$ contains $F_{1,7}$. Figure 6 shows that $F_{1,7}$ in $\overline{G}$ has red and blue edges, with a P_7 shown in red.

Case 2: Every vertex in $G[U]$ has maximum two neighbors in $G[W]$. Suppose that we have two vertices in $G[U]$ where each has two neighbors in $G[W]$, w.l.o.g. $N_{G[W]}(u_2) = \{w_1, w_2\}$ and $N_{G[W]}(u_3) = \{w_3, w_4\}$. Since $|N_{G[U]}(w)| \leq 1, \forall w \in W$, then w_1, w_2, w_3, and w_4 are not the neighbors of u_4. Therefore, the induced subgraph $\overline{G}[w_1, w_2, w_3, w_4, u_2, u_3, u_4, u_{n-1}]$ contains $F_{1,7}$ with u_{n-1} as the dominating vertex connected to path $w_1u_3w_2u_4w_3u_2w_4$ as shown in Figure 7 (left).

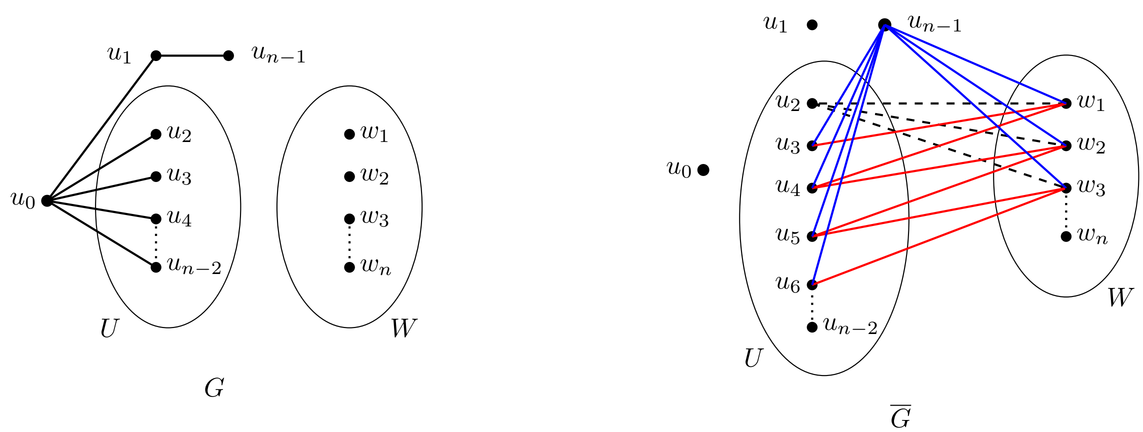

Figure 6: $S_n(1,1)$ in G and $F_{1,7}$ in $\overline{G}$

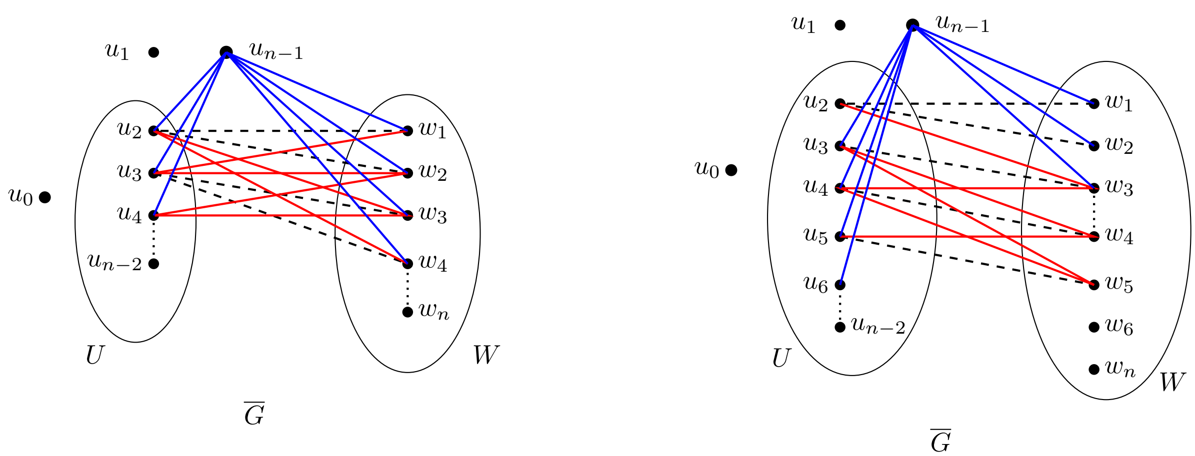

Figure 7: Illustration of Lemma 2 Case 2

As a result, there should be at most one vertex in $G[U]$ which has two neighbors in $G[W]$, w.l.o.g. $N_{G[W]}(u_2) = \{w_1, w_2\}$. If $N_{G[W]}(u_i) \neq \emptyset$, suppose that $w_i \in N_{G[W]}(u_i)$ for $i \in \{3, 4, ..., n-2\}$. Since $|N_{G[U]}(w)| \leq 1, \forall w \in W$, then u_3, u_4, and u_5 each has at most one neighbor in W. The induced subgraph $\overline{G}[w_3, w_4, w_5, u_2, u_3, u_4, u_5, u_{n-1}]$ contains $F_{1,7}$ with u_{n-1} as the dominating vertex connected to path $u_2w_3u_4w_5u_3w_4u_5$ as shown in Figure 7 (right).

□

Theorem below gives the Ramsey number $R(S_n(1,2), F_{1,7})$ for $n \geq 8$.

Theorem 2.4. $R(S_n(1,2), F_{1,7}) = 2n,\ n \geq 8$.

Proof. Let $n \geq 8$.
Case 1: $n \equiv 0 \pmod 3$ and $n \not\equiv 3 \pmod 4$. $R(S_n(1,2), F_{1,7}) = 2n$ is a direct consequence of Theorems 1.7, 1.9 and 1.10.
Case 2: $n \not\equiv 0 \pmod 3$ and $n \not\equiv 3 \pmod 4$. Theorems 1.7 and 1.10 shows that $R(S_n(1,2), F_{1,7}) \leq$

$2n$. We will now show that $R(S_n(1,2), F_{1,7}) \geq 2n$. Let $n = 3k + i, 0 \leq i \leq 2, k \geq 1$. Consider graph $F = \overline{H} \cup K_{n-1}$, with

$$H = \Big(\frac{n-(3+i)}{3}\Big) C_3 \cup C_{3+i}.$$

We know that $K_{n-1} \not\supseteq S_n(1,2)$. Therefore, if there is an $S_n(1,2)$ in F, then it must be contained in $\overline{H}$. Choose a vertex u in $\overline{H}$ to be the center of $S_n(1,2)$ with degree $n-2$. Regardless of our choice of u, it is impossible to create a path P_4 from u since there are only two independent vertices left. Therefore, F does not contain $S_n(1,2)$. Now we will show that $\overline{F}$ does not contain $F_{1,7}$ by contradiction. Let us assume that $\overline{F}$ contains an $F_{1,7}$. If the center of $F_{1,7}$ is in $\overline{K_{n-1}}$, then we need 7 vertices to form P_7 in H. This is not possible since H is a union of C_3, C_4, or C_5. If the center of $F_{1,7}$ is in H, we cannot form an alternating path P_7 from a vertex in $\overline{K_{n-1}}$ to $N_{\overline{F}}(u)$ in H. Therefore, $\overline{F}$ does not contain $F_{1,7}$. As the order of F is $2n-1$, then $R(S_n(1,2), F_{1,7}) > 2n-1 \geq 2n$. Therefore, $R(S_n(1,2), F_{1,7}) = 2n$.
Case 3: $n \equiv 0 \pmod 3$ and $n \equiv 3 \pmod 4$. Theorems 1.7 and 1.9 shows that $R(S_n(1,2), F_{1,7}) \geq 2n$. We will now show that $R(S_n(1,2), F_{1,7}) \leq 2n$. Let F be a graph of order $2n$ and assume that $F_{1,7} \not\subseteq \overline{F}$. Since $R(S_n(1,1), F_{1,7}) = 2n$ from Theorem 2.3, then $S_n(1,1) \subseteq F$. Lemma 2.1 says that F will contain $S_n(1,2)$. Thus, $R(S_n(1,2), F_{1,7}) \leq 2n$. As a result, $R(S_n(1,2), F_{1,7}) = 2n$.
Case 4: $n \not\equiv 0 \pmod 3$ and $n \equiv 3 \pmod 4$. The same good graph as in Case 2 and similar proof as in Case 3 can be used to show that $R(S_n(1,2), F_{1,7}) \geq 2n$ and $R(S_n(1,2), F_{1,7}) \leq 2n$ respectively. □

To determine the Ramsey numbers $R(S_n(2,1), F_{1,7})$ and $R(S_n(3), F_{1,7})$, we are going to utilize the following lemma.

Lemma 2.2. *Let G be a graph of order $2n$ and G contains $S_n(1,1)$, with $n \geq 7$. If $\overline{G}$ does not contain $F_{1,7}$, then G contains $S_n(2,1)$ and $S_n(3)$.*

Proof. Let G be a graph of order $2n$ containing $S_n(1,1)$. Let $V(S_n(1,1)) = \{u_0, u_1, ..., u_{n-1}\}$ with u_0 as the center of $S_n(1,1)$ in G and $u_0u_1, u_1u_{n-1} \in E(S_n(1,1))$. Let $U = V(S_n(1,1)) - \{u_0, u_1, u_{n-1}\} = \{u_2, ..., u_{n-2}\}$ and $W = V(G) - V(S_n(1,1)) = \{w_1, ..., w_n\}$. First, we will prove that if $\overline{G}$ does not contain $F_{1,7}$, then G contains $S_n(2,1)$ by contraposition. Suppose that G does not contain $S_n(2,1)$, then each vertex in U can not be adjacent to any other vertex in U and any vertex in W, and so $E(U,W) = \emptyset$ and $G[U] = \overline{K_{n-3}}$. As a consequence, the induced subgraph $\overline{G}[w_1, w_2, w_3, w_4, u_2, u_3, u_4, u_5]$ will contain $F_{1,7}$ with u_2 as the dominating vertex connected to path $w_1u_3w_2u_4w_3u_5w_4$ as shown in Figure 8 (left). Therefore, $\overline{G}$ contains $F_{1,7}$.

Second, we will prove that G contains $S_n(3)$. Suppose on the contrary that G does not contain $S_n(3)$, then $N_G(u_1) = \{u_0, u_{n-1}\}$, which means that the only neighbors of u_1 are u_0 and u_{n-1}. Furthermore, $|N_{G[W]}(u_i)| \leq 1$ for $i := 1$ to $n-2$. Thus, $|E(U,W)| \leq n-3$. If $N_{G[W]}(u_i) \neq \emptyset$, suppose that $w_i = N_{G[W]}(u_i)$ for $i \in \{2, ..., n-2\}$. There will be at least three vertices w_1, w_{n-1}, w_n which have no neighbor in U. As a consequence, the induced subgraph $\overline{G}[w_1, w_{n-1}, w_n, u_1, u_2, u_3, u_4, u_5]$ contains $F_{1,7}$ with u_1 as the dominating vertex connected to path $u_2w_{n-1}u_3w_nu_4w_1u_5$ as shown in Figure 8 (right). Therefore, $\overline{G}$ contains $F_{1,7}$.

□

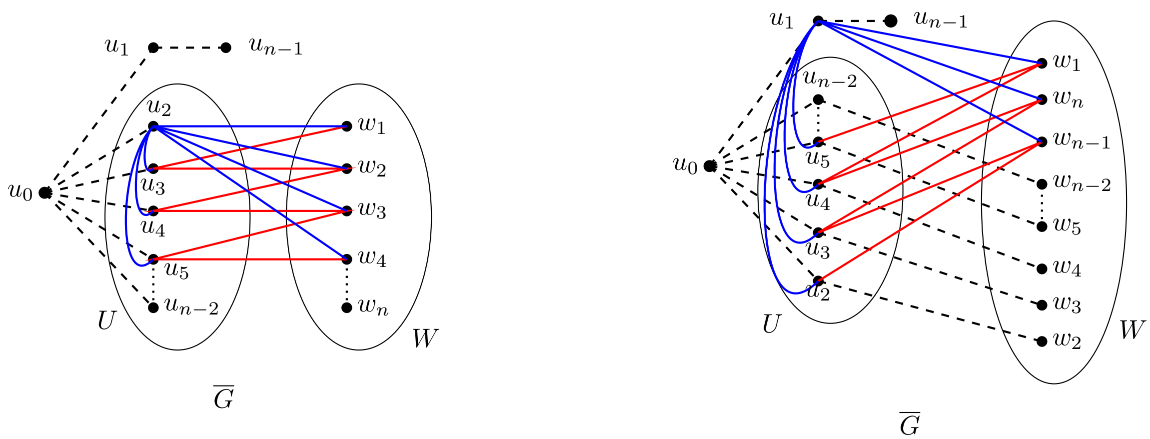

Figure 8: $F_{1,7}$ is in $\overline{G}$ if $S_n(2,1)$ and $S_n(3)$ is not in G

Theorem below will give the remaining values of $R(T_n, F_{1,7})$ for a tree T_n which is not a star with $\Delta(T_n) \geq n-3$ versus fan $F_{1,7}$.

Theorem 2.5.

- $R(S_n(2,1), F_{1,7}) = \begin{cases} 2n-1, & \textit{odd } n \\ 2n \quad, & \textit{even } n \end{cases} \quad ; n \geq 8$
- $R(S_n(3), F_{1,7}) = \begin{cases} 2n-1, & \textit{odd } n \\ 2n \quad, & \textit{even } n \end{cases} \quad ; n \geq 8.$

Proof. For odd $n \geq 8$, $R(S_n(2,1), F_{1,7}) = R(S_n(3), F_{1,7}) = 2n-1$ are direct consequences of Theorems 1.1, 1.7, 1.9, and 1.10.
For even $n \geq 8$, the upper bound is a direct consequence of Theorem 1.10, while the lower bound is proved using Lemma 2.2.

□

3. Conclusion

In this paper we have determined $R(T_n, F_{1,m})$ for tree T_n which is not a star with $\Delta(T_n) \geq n-3$ versus fan $F_{1,m}$ with odd $m \leq 8$, as follows.

1. $R(T_n, F_{1,3}) = 2n-1, n \geq 4$.
2. $R(T_n, F_{1,5}) = 2n-1, n \geq 6$.
3. $R(T_n, F_{1,7}) = \begin{cases} 2n-1, & T_n = \{S_n(2,1), S_n(3)\} \text{ and odd } n \\ 2n \quad, & \text{otherwise} \end{cases} \quad ; n \geq 8.$

Acknowledgement

We appreciate the valuable feedback provided by anonymous referees for this study. The authors have no relevant financial or non-financial interests to disclose.

References

[1] E.T. Baskoro, Surahmat, S.M. Nababan and M. Miller, On Ramsey number for tree versus wheel of five and six vertices, *Graphs Combin.* **18** (2002), 717–721.

[2] J.A. Bondy and U.S.R. Murty, Graph Theory, Springer, (2008).

[3] J.A. Bondy, Pancyclic graphs, *J. Combin. Theory, Ser. B*, **11** (1971), 80–84.

[4] Y.J. Chen, Y.Q. Zhang and K.M. Zhang, The Ramsey number $R(T_n, W_6)$ for $\Delta(T_n) \geq n-3$, *Appl. Math. Lett.*, **17** (2004), 281–285.

[5] Y.J. Chen, Y.Q. Zhang and K.M. Zhang, The Ramsey number $R(T_n, W_6)$ for small n, *Util. Math.*, **67** (2005).

[6] Y.J. Chen, Y.Q. Zhang and K.M. Zhang, The Ramsey number $R(T_n, W_6)$ for T_n without certain deletable sets, *J. Syst. Sci. Compl.*, **18** (2005) 95-101.

[7] Y.J. Chen, Y.Q. Zhang and K.M. Zhang, The Ramsey numbers of stars versus wheels, *European J. Combin.*, **25** (2004), 1067–1075.

[8] Y.J. Chen, Y.Q. Zhang and K.M. Zhang, The Ramsey numbers of trees versus W_6 or W_7, *European J. Combin.*, **27** (2006) 558-564.

[9] V. Chvátal and F. Harary, Generalized Ramsey theory for graphs, III: Small off-diagonal numbers, *Pacific J. Math.* **41** (1972), 335–345.

[10] Yusuf Hafidh, Edy Tri Baskoro, The Ramsey number for tree versus wheel with odd order, *Bull. Malays. Math. Soc.* **44** (4) (2021), 2151–2160.

[11] Hasmawati, E.T. Baskoro and H. Assiyatun, Star-wheel Ramsey numbers, *J. Combin. Math. Combin. Comput.*, **55** (2005), 123–128.

[12] Surahmat and E.T. Baskoro, On the Ramsey number of a path or a star versus W_4 or W_5, *Proceeding of the* 12^{th} *Australian Workshop on Combinatorial Algorithms*, Bandung, Indonesia, July 14-17 (2001), 174–179.

[13] Surahmat, E.T. Baskoro and H.J. Broersma, The Ramsey numbers of large stars-like trees versus large odd wheels, *J. Combin. Math. Combin. Comput.* **65** (2008), 153–162.

[14] Y. Zhang, H. Broersma and Y. Chen, A remark on star-C_4 and wheel-C_4 Ramsey numbers, *Electron. J. Graph Theory Appl.* **2** (2) (2014), 110–114.

[15] Y.Q. Zhang, H. Broersma and Y.J. Chen, On fan-wheel and tree-wheel Ramsey numbers, *Discrete Math.*, **339** (2016), 2284–2287.

[16] Y. Zhang and Y. Chen, The Ramsey numbers of fans versus a complete graph of order five, *Electron. J. Graph Theory Appl.* **2** (1) (2014), 66–69.

[17] Y.Q. Zhang, Y.J. Chen and K. Zhang, The Ramsey number for stars of even order versus a wheel of order nine, *European J. Combin.*, **29** (2008), 1744–1754.

[18] Y.Q. Zhang, T.C.E. Cheng and Y.J. Chen, The Ramsey number for stars of odd order versus a wheel of order nine, *Discrete Math. Algorithms Appl.*, **1** (2009), 413–436.

[19] B. Li, Y.Q. Zhang, H. Broersma, An exact formula for all star-kipas Ramsey numbers, *Graphs Combin.*, **33** (2017), 141–148.

[20] I. Sherlin, S.W. Saputro, E.T. Baskoro, On Ramsey number for tree versus fan graph of odd order, *Proceeding of ICMNS 2023*, *accepted*.

[21] L. Gerencsér and A. Gyárfás, On Ramsey-type problems, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math*, **10** (1967), 167–170.

[22] S. A. Burr, Ramsey numbers involving graphs with long suspended paths, *J. the London Math. Soc.*, **2.3** (1981), 405-413.

[23] I. W. Sudarsana, A note on the Ramsey number for cycle with respect to multiple copies of wheels, *Electron. J. Graph Theory Appl.*, **9** (2) (2021), 561–566.

[24] S. Radziszowski, Small ramsey numbers, *Electron. J. Combin.*, **1000** (2021) (DS1–Aug).

[25] V. Rosta, Ramsey theory applications, *Electron. J. Combin.*, **1000** (2004) (DS13–Dec).