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Local inclusive distance antimagic coloring of graphs

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Abstract

For a simple graph G, a bijection $f : V(G) \to [1, |V(G)|]$ is called as a *local inclusive distance antimagic (LIDA) labeling* of G if $w(u) \neq w(v)$ for every two adjacent vertices $u, v \in V(G)$ with $w(u) = \sum_{x \in N[u]} f(x)$. A graph G is said to be local inclusive distance antimagic (LIDA) graph if it admits a LIDA labeling. The function w induced by f also can be seen as a proper vertex coloring of G. The *local inclusive distance antimagic (LIDA) chromatic number* of G, denoted by $\chi_{lida}(G)$, is the minimum number of colors taken over all proper vertex colorings induced by LIDA labelings of G. In this paper, we study a LIDA labeling of simple graph. We provide some basic properties of LIDA labeling for any simple graphs. The LIDA chromatic number of certain multipartite graphs, double stars, subdivision of graphs and join product of graphs with K_1 are also investigated. We present an upper bound for graphs obtained from subdivision of super edge-magic total graphs. Furthermore, we present some new open problems.

Keywords: local distance antimagic labeling, local inclusive distance antimagic labeling, vertex coloring
Mathematics Subject Classification : 05C15, 05C78

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1. Introduction

We assume that all graphs in this paper are undirected, simple, and finite. For any two integers a and b, the notation $[a, b]$ refers to the set of all integers from a to b, inclusive. A *neighborhood*

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of a vertex v, denoted by $N(v)$, is a set consisting of every vertices adjacent to v. In addition, let $N[v] = N(v) \cup \{v\}$. Two vertices u, v are said to be *false twins* (or *true twins*) if $N(u) = N(v)$ (or $N[u] = N[v]$). A vertex v of a graph G is called as a *dominating vertex* in G if v is adjacent to every other vertex in G.

Let f be a bijection function defined as $f : V(G) \cup E(G) \to [1, |V(G)| + |E(G)|]$. In 1970, Kotzig and Rosa [13] introduced the definition that the function f is called as an *edge-magic total labeling* if there exists a positive integer m such that for every $xy \in E(G)$, $f(x) + f(xy) + f(y) = m$. We say that the integer m as a *magic constant* of f. In case of $\{f(v) \mid v \in V(G)\} = [1, |V(G)|]$, then we said f as the *super edge-magic total labeling* [8]. The graph G which admits an (super) edge-magic total labeling is called as *(super) edge-magic* graph. Some studies on (super) edge-magic graphs can be seen in [6, 9, 10, 14, 20].

By using the similar idea of the edge-magic total labeling, Miller et al. [15] then defined *the distance magic labeling* in 2003. This labeling is a bijective labeling $f : V(G) \to [1, |V(G)|]$ such that every vertex $v \in V(G)$ satisfies $w(v) = \sum_{u \in N(v)} f(u) = m$. We say $w(v)$ as the weight of vertex v with respect to labeling f. The graph G which admits a distance magic labeling is called *distance magic graph*. Several graphs known to be distance magic are certain complete multipartites graphs [2, 15], direct product of two graphs [1], and circulant graphs [3].

Later in 2013, Kamatchi and Arumugam [12] introduced the notion of *distance antimagic labeling* by remove the magicness property of distance magic labeling. In this labeling version, every two distinct vertices u and v in G satisfies $w(u) \neq w(v)$. A graph G is called *distance antimagic graph* if G admits a distance antimagic labeling. Some results on distance antimagic labeling can be found in [4, 19, 21].

Furthermore, many authors also studied some variants of distance antimagic labeling. Dafik et al. [5] introduced a labeling by using the similar idea of distance antimagic labeling, which is called *inclusive distance antimagic labeling*. In this variant of distance antimagic labeling, the weight of a vertex u is defined as $w(u) = f(u) + \sum_{x \in N(u)} f(x)$. Simanjuntak et al. [18] then generalized this concept and introduced D-antimagic labeling. For $D \subseteq [0, diam(G)]$, a *D-antimagic labeling* of a graph G is a bijection $f : V(G) \to [1, |V(G)|]$ such that $w(v) = \sum_{x \in N_D(v)}$ is distinct for each vertex v, where $N_D(v) = \{x \in V(G) \mid d(x, v) \in D\}$. Note that, the D-antimagic labeling is equivalent to distance antimagic labeling and inclusive distance antimagic labeling if $D = \{1\}$ and $D = \{0, 1\}$, respectively.

Divya and Yamini [7] investigate another variant of distance antimagic labeling, namely *local distance antimagic labeling*. In this version of distance antimagic labeling, every two adjacent vertices u and v in G satisfies $w(u) \neq w(v)$. Note that, any local antimagic labeling induces a proper vertex coloring of G where the vertex v is assigned the color $w(v)$. The *local distance antimagic chromatic number*, denoted by $\chi_{lda}(G)$, is the minimum number of colors taken over all colorings induced by local distance antimagic labelings of G. Handa et al. [11] obtained the local distance antimagic labelings for several families of graphs including paths, cycles, wheels, friendship graphs, complete multipartite graphs, and some special types of the caterpillars. The local distance antimagic chromatic number of union of stars and double stars are studied in [17]. Meanwhile, graphs having local distance antimagic chromatic number of 2 have been investigated by Priyadharsini and Nalliah [16].

In this paper, we introduce the inclusive version of local distance antimagic labeling. Let f be a bijection $f : V(G) \to [1, |V(G)|]$ and for $v \in V(G)$, $w(v) = \sum_{x \in N[v]} f(x)$ be a weight of v of G with respect to f. The function f is called as a *local inclusive distance antimagic (LIDA) labeling* of G if every two adjacent vertices u and v satisfies $w(u) \neq w(v)$. If a graph G admits a LIDA labeling, then G is said to be a *local inclusive distance antimagic (LIDA) graph*. Note that the weight function w induced by f also can be seen as a proper vertex coloring of G. The *local inclusive distance antimagic (LIDA) chromatic number* of G, denoted by $\chi_{lida}(G)$, is the minimum number of colors taken over all proper vertex colorings induced by LIDA labelings of G. If a graph G does not admit any LIDA labelings, then we say $\chi_{lida}(G) = \infty$. Since the weight of two adjacent vertices must be different, it is obvious that for a LIDA graph G, we have $\chi_{lida}(G) \geq 2$.

An illustration of LIDA labeling can be seen in Figure 1 below. The black numbers represent the label of vertex, whereas the blue and red numbers represent the weight of vertex. We can see that the labeling induced two colors of weight. Therefore, the LIDA chromatic number of the graph is two.

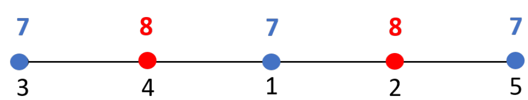

Figure 1: LIDA labeling of P_5.

In this paper, we study the LIDA labeling of simple graph. We provide some basic properties of LIDA labeling, such as bounds on the parameter, sufficient conditions of two vertices having distinct weights, and bounds of the weight of any given vertex in a graph. We also investigate the LIDA chromatic number of several classes of graphs such as complete multipartite graph without true twins, subdivided stars $S(K_{1,n})$, and double stars $S_{n,n}$. Moreover, the LIDA chromatic number of the join product graphs $G + K_1$ and $G + \overline{K_p}$ are also determined. A small connection of super edge-magic total graphs with LIDA chromatic number is also presented along with some open problems.

2. Some Properties of LIDA Labeling

In this section, we provide some basic results of LIDA labeling. First, we present a sufficient condition such that a graph does not admit any LIDA labelings.

Proposition 2.1. *Let G be a graph having true twins. Then $\chi_{lida}(G) = \infty$.*

Proof. Let u and v be a true twins in G and f be a bijection $f : V(G) \to [1, |V(G)|]$. Since $N[u] = N[v]$, we have $w(u) = \sum_{x \in N[u]} f(x) = \sum_{x \in N[v]} f(x) = w(v)$. Because u and v are adjacent, we obtain that f is not a LIDA labeling. □

Corollary 2.1. *For $n \geq 2$, let K_n be a complete graph. Then $\chi_{lida}(K_n) = \infty$.*

Next, let us consider a LIDA graph G. Since LIDA labeling induces a proper vertex coloring, it is obvious that we have the following proposition.

Proposition 2.2. *Let G be a LIDA graph. Then $2 \le \chi(G) \le \chi_{lida}(G) \le |V(G)|$.*

In the next proposition, we provide some properties of the weight of vertices induced by a LIDA labeling of graph. In particular, we give some conditions such that two certain distinct vertices have different weights.

Proposition 2.3. *Let G be a LIDA graph. Let u and v be two distinct vertices in G. If w is a weight induced by a LIDA labeling of G, then we have the following two conditions.*

(a) If $N[u] \subset N[v]$, then $w(u) < w(v)$.
(b) If u and v are false twins, then $w(u) \neq w(v)$.

Proof. Let f be a LIDA labeling of G which induces the weights w. If $N[u] \subset N[v]$, then it is easy to verify that

$$w(u) = \sum_{x \in N[u]} f(x) < \sum_{x \in N[v]} f(x) = w(v).$$

Meanwhile, if u and v are false twins, which means $N(u) = N(v)$, then we have

$$\sum_{x \in N(u)} f(x) = \sum_{x \in N(v)} f(x).$$

Since f is a bijection, then we have $f(u) \neq f(v)$. It implies that

$$w(u) = f(u) + \sum_{x \in N(u)} f(x) = f(u) + \sum_{x \in N(v)} f(x) \neq f(v) + \sum_{x \in N(v)} f(x) = w(v).$$

□

Now let G be a LIDA graph. For $v \in V(G)$, let $w(v)$ be the weight of vertex v induced by a LIDA labeling of G. In proposition below, we provide the possibility values of $w(v)$ for any LIDA labelings. In particular, we give the lower and upper bounds of $w(v)$ for any vertices v of G.

Proposition 2.4. *Let G be a LIDA graph of order n. Let f be a LIDA labeling of G which induces the weighting function $w(v) = \sum_{x \in N[v]} f(x)$ for $v \in V(G)$. If $\deg(v) = d$, then*

$$\frac{1}{2}(d+1)(d+2) \le w(v) \le \frac{1}{2}(d+1)(2n-d).$$

Furthermore, the lower bound is attained if and only if $f(N[v]) = [1, d+1]$, and the upper bound is attained if and only if $f(N[v]) = [n-d, n]$.

Proof. Let $v \in V(G)$. First, we prove the lower bound. Since $[1, d+1]$ is the set of $d+1$ smallest values of $[1, n]$, then the value of $w(v)$ must be at least the sum of all values in $[1, d+1]$, i.e., $w(v) \ge \sum_{i=1}^{d+1} i = \frac{1}{2}(d+1)(d+2)$. It is easy to see that the equality holds if and only if $f(N[v]) = [1, d+1]$.

Next, we prove the upper bound. Since $[n-d, n]$ is the set of $d+1$ greatest values of $[1, n]$, then the value of $w(v)$ must not exceed the sum of all values in $[n-d, n]$, i.e., $w(v) \le \sum_{i=n-d}^{n} i = \frac{1}{2}(d+1)(2n-d)$. It is easy to see that the equality holds if and only if $f(N[v]) = [n-d, n]$. Therefore, the proof is complete. □

3. LIDA Chromatic Number of Certain Classes of Graph

Now, this section will be divided in two subsections. The first subsection deals with several classes of graphs, and the later deals with certain graph operations.

In Theorem 3.1, we present a class of graph G which has LIDA chromatic number equals to its order.

Theorem 3.1. *Let G be a complete multipartite graph which has at most 1 dominating vertex. We have $\chi_{lida}(G) = |V(G)|$.*

Proof. Let G be a multipartite graph with a partition $V(G) = \cup_{i=1}^{p} V_p$ for some $p \in \mathbb{N}$ such that $|V_1|, |V_2|, ..., |V_p|$ forms a non-decreasing sequence and $n = |V(G)|$. For every $v \in V(G)$, let $w^*(v) = \sum_{x \in V(G)} f(x) - w(v)$. Since $\sum_{x \in V(G)} f(x)$ is constant, it can be seen that $w(v)$ is distinct for every vertex v if and only if $w^*(v)$ is distinct for every vertex v.

First, we will prove $\chi_{lida}(G) \geq |V(G)|$. Let f be a LIDA labeling of G. Since two vertices in different partition are adjacent, then they must be colored different. Hence, we are left to check vertices in the same partition. Consider a partition $V_i \subset V(G)$ with $|V_i| \geq 2$. It follows that $w^*(v) = \sum_{x \in V_i} f(x) - f(v)$ for every $v \in V_i$. Since $\sum_{x \in V_i} f(x)$ is constant, then

$$\begin{aligned} f \text{ is injective } &\implies f(u) \neq f(v), && \text{for } u, v \in V_i \\ &\implies \sum_{x \in V_i} f(x) - f(u) \neq \sum_{x \in V_i} f(x) - f(v), && \text{for } u, v \in V_i \\ &\implies w^*(u) \neq w^*(v), && \text{for } u, v \in V_i. \end{aligned}$$

This implies all vertices in G must have different weights. Consequently, $\chi_{lida}(G) \geq |V(G)|$.

To show $\chi_{lida}(G) \leq |V(G)|$, let $V(G) = \{v_1, v_2, ..., v_n\}$ such that $\deg(v_1), \deg(v_2), ..., \deg(v_n)$ forms a non-increasing sequence. Define a labeling f with

$$f(v_i) = i, \qquad i \in [1, n]$$

We already know that the injectivity of f leads to $w^*(u) \neq w^*(v)$ for u, v in a same partition. Then, consider two partitions V_i, V_j in G with $i < j$ and $u \in V_i, v \in V_j$. Since $\deg(u) \leq \deg(v)$ and $f(x) < f(y)$ for every $x \in V_i, y \in V_j, i < j$, it follows that $w^*(u) < w^*(v)$. This implies that every vertex v in $V(G)$ has distinct weights. Therefore, f is LIDA labeling of G and $\chi_{lida}(G) \leq |V(G)|$. □

We give an example of a LIDA coloring of $K_{2,2,4}$ with their respective $w^*(v)$ for every vertex $v \in V(K_{2,2,4})$ which induces $\chi_{lida}(K_{2,2,4}) = 8$ in Figure 2.

If a graph G of order n has $\chi_{lida}(G) = n$, then the graph G is inclusive distance antimagic. Hence, a lot of complete multipartite graphs are inclusive distance antimagic.

Corollary 3.1. *If G is a complete multipartite graph which has at most 1 dominating vertex, then G is inclusive distance antimagic.*

One simple subclass which satisfy the condition in Theorem 3.1 are stars. Hence, the following is also true.

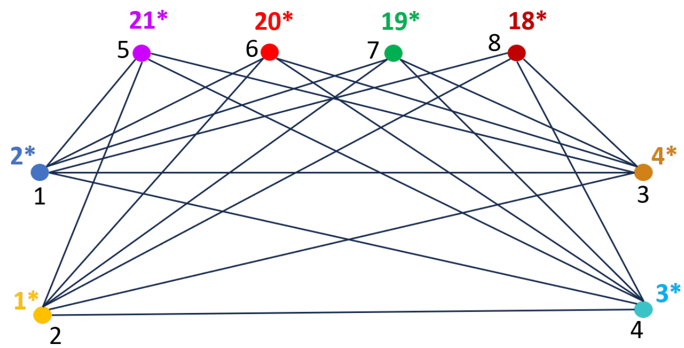

Figure 2: LIDA coloring of $K_{2,2,4}$ with $\chi_{lida}(K_{2,2,4}) = 8$.

Corollary 3.2. *For $n \geq 3$, we have $\chi_{lida}(K_{1,n}) = n + 1$.*

Since $\chi(K_{1,n}) = 2$, then $\chi_{lida}(K_{1,n}) - \chi(K_{1,n})$ can be arbitrarily large. Hence, the following problem is posed.

Problem 1. *Characterize graphs G which have $\chi_{lida}(G) = \chi(G)$.*

Let $S(G)$ be a subdivision of a graph G, that is a graph obtained from splitting every edges in G into two edges that are adjacent to new vertices. The following theorem gives the LIDA chromatic number of a subdivided star graph.

Theorem 3.2. *For $n \geq 3$, we have $\chi_{lida}(S(K_{1,n})) = 3$.*

Proof. Let $V(S(K_{1,n})) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and $E(S(K_{1,n})) = \{v_0v_i : i \in [1, n]\} \cup \{v_iu_i : i \in [1, n]\}$. First, we prove the upper bound. For $n = 3$, it is easy to verify that the vertex labeling of $S(K_{1,3})$ shown in Figure 3 is a LIDA labeling of $S(K_{1,3})$ which induces a coloring of 3 colors: 11, 10, and 9. Thus, $\chi_{lida}(S(K_{1,n})) \leq 3$ for $n = 3$.

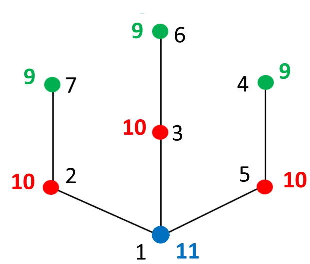

Figure 3: LIDA coloring of $S(K_{1,3})$ which induces a coloring of three colors.

Now, we assume that $n > 3$. We define a vertex labeling $f : V(S(K_{1,n})) \rightarrow [1, 2n + 1]$ as follows:

$$f(v_i) = i + 1, \qquad i \in [0, n],$$
$$f(u_i) = 2n + 2 - i, \qquad i \in [1, n].$$

It is easy to see that f is bijective, and the vertex coloring w induced by f is as follows:

$$w(v_0) = \sum_{i=0}^{n} f(v_i) = \sum_{i=0}^{n} (i+1) = \frac{1}{2}(n+1)(n+2),$$
$$w(v_i) = f(v_0) + f(v_i) + f(u_i) = 1 + (i+1) + (2n+2-i) = 2n+4, \quad i \in [1,n],$$
$$w(u_i) = f(v_i) + f(u_i) = (i+1) + (2n+2-i) = 2n+3, \quad i \in [1,n].$$

Since $n > 3$, then for all $i \in [1, n]$, we have

$$w(v_0) = \frac{1}{2}(n+1)(n+2) > w(v_i) = 2n+4 > w(u_i) = 2n+3.$$

Thus, every pair of adjacent vertices have distinct colors where there are three colors: $\frac{1}{2}(n+1)(n+2)$, $2n+4$, and $2n+3$. So, f is a LIDA labeling of $S(K_{1,n})$ which induces a coloring of three colors. Therefore, $\chi_{lida}(S(K_{1,n})) \leq 3$.

Now, to prove the lower bound, assume to the contrary that $\chi_{lida}(S(K_{1,n})) \leq 2$. This implies that there exists a LIDA labeling f of $S(K_{1,n})$ such that $w(v_0) = w(u_i)$ for every $i \in [1, n]$. By Proposition 2.4 and the fact that $n \geq 3$, we have

$$w(v_0) \geq \frac{(n+1)(n+2)}{2} \geq \frac{4(n+2)}{2} \geq 2n+4.$$

Meanwhile, since $n \cdot w(u_1) = \sum_{i=1}^{n} w(u_i)$, it follows that

$$\begin{aligned} w(u_1) &= \frac{1}{n} \sum_{i=1}^{n} w(u_i) \\ &= \frac{1}{n}\Bigg(\sum_{x \in V(S(K_{1,n}))} f(x) - f(v_0) \Bigg) \\ &\leq \frac{1}{n}\Bigg(\frac{(2n+1)(2n+2)}{2} - 1 \Bigg) \\ &\leq 2n+3. \end{aligned}$$

Consequently, $w(v_0) > w(u_1)$, this leads to a contradiction. Therefore, $\chi_{lida}(S(K_{1,n})) \geq 3$. □

For example, we present $S(K_{1,5})$ with its LIDA coloring in Figure 4.

Next, we can provide a lower bound for any graphs G with some pendants.

Proposition 3.1. *Let G be a graph, and $p(v)$ be a number of vertices which are pendants and adjacent to v. We have*

$$\chi_{lida}(G) \geq 1 + \max_{v \in V(G)} \{p(v)\}$$

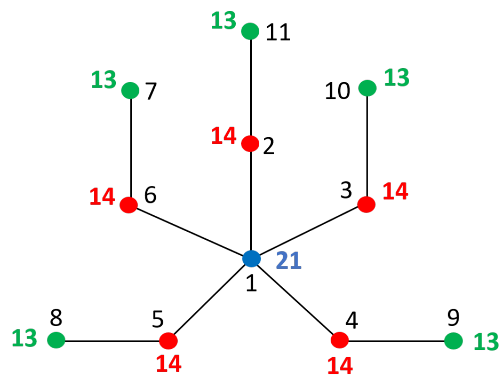

Figure 4: LIDA coloring of $S(K_{1,5})$ with $\chi_{lida}(S(K_{1,5})) = 3$.

Proof. It is obvious for a graph G with no pendants, so assume G has some pendants. Let f be a LIDA labeling of G and u be a vertex of G which is adjacent to some pendants $x_1, x_2, ..., x_k$ for some positive integer k. By Proposition 2.3(b), we have

$$w(x_i) \neq w(x_j)$$

for every distinct $i, j \in [1, k]$. In addition, by Proposition 2.3(a), it follows that

$$w(u) > w(x_i)$$

for any $i \in [1, k]$. Since we must have at least $1 + p(u)$ distinct weights in $N[u]$, this implies

$$\chi_{lida}(G) \geq 1 + p(u).$$

Finally, since u is chosen randomly, we have

$$\chi_{lida}(G) \geq 1 + \max_{v \in V(G)} \{p(v)\}.$$

□

The preceding proposition is sharp, since $\chi_{lida}(K_{1,n}) = n + 1 = 1 + p(c)$ where c is the center of $K_{1,n}$.

Next, for positive integers n and m, a *double star* $S_{n,m}$ is a graph obtained by connecting the center vertices of two star graphs $K_{1,n}$ and $K_{1,m}$. Thus, the graph $S_{n,m}$ has $n + m + 2$ vertices and $n + m + 1$ edges. The following theorem gives the LIDA chromatic number of $S_{n,n}$.

Theorem 3.3. *For $n \geq 1$, we have $\chi_{lida}(S_{n,n}) = n + 2$.*

Proof. Let $V(S_{n,n}) = X \cup Y \cup \{x_0, y_0\}$ where $X = \{x_i : i \in [1, n]\}$ and $Y = \{y_j : j \in [1, n]\}$, and $E(S_{n,n}) = \{x_0y_0\} \cup \{x_0x_i : i \in [1, n]\} \cup \{y_0y_i : i \in [1, n]\}$. Firstly, we prove the upper bound. We define the vertex labeling $f : V(S_{n,n}) \to [1, 2n+2]$ as follows:

$$f(x_i) = \begin{cases} 1, & i = 0; \\ 2i+2, & i \in [1,n]. \end{cases}$$

$$f(y_i) = \begin{cases} 2, & i = 0; \\ 2i+1, & i \in [1,n]. \end{cases}$$

It is clear that f is bijective, and the vertex coloring w induced by f is as follow:

$$\begin{aligned} w(x_0) &= f(x_0) + f(y_0) + \sum_{i=1}^{n} f(x_i) = 3 + 2 + \sum_{i=1}^{n} 2i = n^2 + 3n + 3, \\ w(y_0) &= f(x_0) + f(y_0) + \sum_{i=1}^{n} f(y_i) = 3 + 1 + \sum_{i=1}^{n} 2i = n^2 + 2n + 3, \\ w(x_i) &= f(x_0) + f(x_i) = 1 + 2i + 2 = 3 + 2i, \quad \text{for } i \in [1,n], \\ w(y_i) &= f(y_0) + f(y_i) = 2 + 2i + 1 = 3 + 2i, \quad \text{for } i \in [1,n]. \end{aligned}$$

Since $n \geq 1$, then we have

$$w(x_0) = n^2 + 3n + 3 > w(y_0) = n^2 + 2n + 3 > w(x_n) = w(y_n) = 3 + 2n$$

It is clear that $3 + 2i$ for $i \in [1,n]$ are all distinct. So, f is a LIDA labeling of $S_{n,n}$ which induces $n+2$ colors. Therefore, $\chi_{lida}(S_{n,n}) \leq n+2$.

Now, we prove the lower bound. From Proposition 3.1, we have $p(x_0) = p(y_0) = n$, hence $\chi_{lida}(S_{n,n}) \geq n+1$. Now, we prove that there is no LIDA labeling which induces a coloring of $n+1$ colors. It can be verified computationally that this is the case for $n = 1, 2$. Now, we assume that $n \geq 3$. Suppose to the contrary that there exists a LIDA labeling f which induces a coloring w with $n+1$ colors. Since all vertices in X are false twins and x_0 is adjacent to all vertices in X, then all $n+1$ vertices in $\{x_0\} \cup X$ must have distinct colors. Thus, each vertex corresponds to one of the $n+1$ colors of w. Same argument applies to the vertices of $\{y_0\} \cup Y$. These observations imply that there exists an $x \in X$ such that $w(x) = w(y_0)$. By giving the smallest labels $[1, n+1]$ to $\{y_0\} \cup Y$, we obtain

$$f(x) = w(x) - f(x_0) = w(y_0) - f(x_0) = f(y_0) + \sum_{j=1}^{n} f(y_j) \geq \frac{1}{2}(n+1)(n+2).$$

Since $f(x) \leq 2n+2$, then we have $\frac{1}{2}(n+1)(n+2) \leq 2n+2$ which is impossible for $n \geq 3$. Therefore, $\chi_{lida}(S_{n,n}) \geq n+2$. □

In Figure 5, we can see $S_{4,4}$ with $\chi_{lida}(S_{4,4}) = 6$.

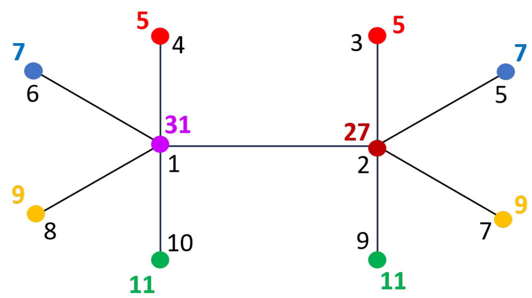

Figure 5: LIDA coloring of $S_{4,4}$ with $\chi_{lida}(S_{4,4}) = 6$.

3.1. Some graph operations

We begin by giving a relationship between $\chi_{lida}(G)$ and $\chi_{lida}(G + K_1)$ for a LIDA graph G which has no dominating vertex. Note that if G has a dominating vertex, then $G + K_1$ will have two dominating vertices which constitute true twins, hence it is not LIDA. Later on, we will prove a similar result for $G + \overline{K_p}$ for large enough $p \geq 2$. Note that for any graph G and positive integer $p \geq 2$, the graph $G + K_p$ is not LIDA since it has true twins.

Theorem 3.4. *Let G be a LIDA graph without a dominating vertex. Then we have*

$$\chi_{lida}(G + K_1) \leq \chi_{lida}(G) + 1.$$

Proof. Let $n = |V(G)|$, $V(G+K_1) = V(G) \cup \{u\}$, and $E(G+K_1) = E(G) \cup \{vu : v \in V(G)\}$. Let $\chi_{lida}(G) = \chi$, so there exist a LIDA labeling $f : V(G) \to [1, n]$ of G and the induced coloring w_f with χ colors. First, we prove the upper bound. We define a vertex labeling $g : V(G + K_1) \to [1, n + 1]$ as follows:

$$g(x) = \begin{cases} f(x), & x \in V(G); \\ n+1, & x = u. \end{cases}$$

It is clear that g is bijective since f is bijective, and the vertex coloring w_g induced by g is as follows:

$$w_g(x) = w_f(x) + g(u) = w_f(x) + (n+1), \quad x \in V(G),$$
$$w_g(u) = g(u) + \sum_{x \in V(G)} f(x) = \frac{1}{2}(n+1)(n+2).$$

Observe that for any two adjacent vertices $x, y \in V(G)$, $w_f(x) \neq w_f(y)$ implies $w_g(x) \neq w_g(y)$ since f is a LIDA labeling of G. Futhermore, since G has no dominating vertex, then for every $x \in V(G)$, we have $N[x] \subset V(G)$, and hence $w_f(x) = \sum_{y \in N[x]} f(y) < \sum_{y \in V(G)} f(y) = \frac{1}{2}n(n+1)$, so we have

$$w_g(x) = w_f(x) + (n+1) < \frac{1}{2}n(n+1) + (n+1) = \frac{1}{2}(n+1)(n+2) = w_g(u).$$

Thus, every two adjacent vertices in $G + K_1$ have distinct colors. Observe that all vertices in G still have χ colors, and u has a color which is distinct from all the colors of $V(G)$. So, g is a LIDA labeling of $G+K_1$ which induces a coloring w_g of $\chi+1$ colors. Therefore, $\chi_{lida}(G+K_1) \leq \chi+1$, and the proof is complete. □

Observe that the upper bound given in Theorem 3.4 is strict since it is attained by, for instance, $G \cong C_6$: $\chi_{lida}(C_6) = 2$ and $\chi_{lida}(W_6) = 3$ (where $W_6 \cong C_6 + K_1$). We present the coloring of C_6 and W_6 in Figure 6.

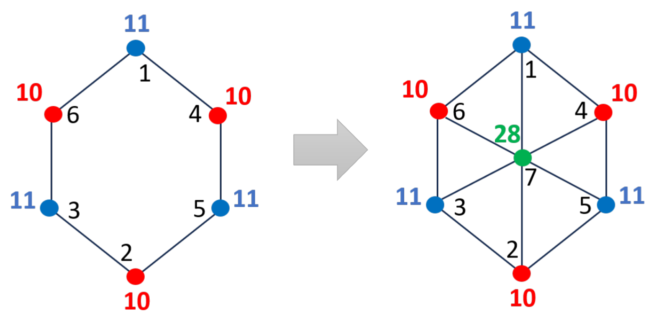

Figure 6: LIDA coloring of C_6 and $W_6 \cong C_6 + K_1$.

On the other hand, this upper bound is not attained by $G \cong C_7$ since $\chi_{lida}(C_7) = \chi_{lida}(C_7 + K_1) = 5$. This observation leads to the following problem.

Problem 2. *Characterize graphs G satisfying $\chi_{lida}(G + K_1) = \chi_{lida}(G) + 1$.*

Next, we present the local inclusive distance antimagic chromatic number of $G+\overline{K_p}$ for $p \geq 2$.

Theorem 3.5. *Let G be a graph of order n and minimum degree δ, and let $p \geq 2$ be a positive integer. If*

$$p > \frac{1}{2}[-2n + 1 + \sqrt{8n^2 + 8n + 1 - 4(\delta + 1)(\delta + 2)}],$$

then $\chi_{lida}(G + \overline{K_p}) \leq \chi_{lida}(G) + p$.

Proof. Let $n = |V(G)|$, $V(G + \overline{K_p}) = V(G) \cup U$ where $U = \{u_1, u_2, \ldots, u_p\}$, and $E(G + \overline{K_p}) = E(G) \cup \{vu : v \in V(G), u \in U\}$. Let $\chi_{lida}(G) = \chi$, so there exist a LIDA labeling $f : V(G) \to [1, n]$ of G and the induced coloring w_f with χ colors. We define a vertex labeling $g : V(G + \overline{K_p}) \to [1, n + p]$ as follows:

$$g(x) = \begin{cases} f(x), & x \in V(G); \\ n + j, & x = u_j, \quad j \in [1, p]. \end{cases}$$

It is clear that g is bijective since f is bijective onto $[1,n]$, and the vertex coloring w_g induced by g is as follows:

$$w_g(x) = w_f(x) + \sum_{j=1}^{p}(n+j) = w_f(x) + np + \frac{1}{2}p(p+1), \qquad x \in V(G),$$
$$w_g(u_j) = g(u_j) + \sum_{x\in V(G)} f(x) = j + \frac{1}{2}n(n+3), \qquad j \in [1,p].$$

Observe that the assumption for p implies $(\delta+1)(\delta+2) > n(n+3) - p(p+2n-1)$. Thus, from Proposition 2.4, we have

$$\begin{aligned}\min_{x\in V(G)} w_g(x) &= \min_{x\in V(G)} w_f(x) + np + \frac{1}{2}p(p+1)\\ &\geq \frac{1}{2}(\delta+1)(\delta+2) + np + \frac{1}{2}p(p+1)\\ &> \frac{1}{2}[n(n+3) - p(p+2n-1)] + np + \frac{1}{2}p(p+1)\\ &= p + \frac{1}{2}n(n+3)\\ &= \max_{u\in U} w_g(u).\end{aligned}$$

This observation implies that $V(G)$ and U share no common color since the minimum color in $V(G)$ is greater than the maximum color in U. Furthermore, observe that all vertices in $V(G)$ still have χ colors where the colors of two adjacent vertices are distinct, and all vertices in U have p additional colors which are distinct to the colors in $V(G)$ as mentioned before. Thus, every two adjacent vertices have distinct colors, so g is a LIDA labeling of $G+\overline{K_p}$ which induces a coloring of $\chi+p$ colors. Therefore, $\chi_{lida}(G+\overline{K_p}) \leq \chi + p = \chi_{lida}(G) + p$. □

As an example, consider $G \cong P_5 + \overline{K_4}$. Since $|V(P_5)| = 5$ and $\delta(P_5) = 1$, we have

$$\begin{aligned}\frac{1}{2}[-2(5)+1+\sqrt{8(5)^2+8(5)+1-4(1+1)(1+2)}] &= \frac{1}{2}(-9+\sqrt{217}),\\ &\leq \frac{1}{2}(6),\\ &\leq 3.\end{aligned}$$

Since $p = 4 > 3$, it follows that $\chi_{lida}(G) \leq \chi_{lida}(P_5) + 4 = 6$. Furthermore, the four false twins in G have unique colors due to Proposition 2.3(b) and adjacent to the others. Since the rest of the vertices must have at least two colors, we have shown that $\chi_{lida}(G) \geq 6$. As a result, $\chi_{lida}(G) = 6$. We present an illustration of the graph in Figure 7.

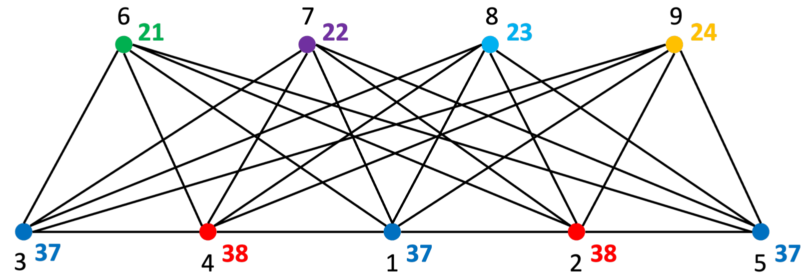

Figure 7: LIDA coloring of $G \cong P_5 + \overline{K_4}$ where $\chi_{lida}(G) = 6$.

Moreover, we can utilize super edge-magic total graphs to construct a graph with bounded LIDA chromatic number.

Theorem 3.6. *Let G be a super edge-magic total graph with an order n. Then, $\chi_{lida}(S(G)) \leq n + 1$.*

Proof. Let G be a graph with an order n and g be a super edge-magic total labeling of G with a magic constant m. Define a labeling of f with

$$f(v) = g(v), \qquad \text{for } v \in V(G) \cup E(G).$$

Let U and X be the set of vertices of $S(G)$ with $U = V(G)$ and $X = E(G)$. Since g is a super edge-magic total labeling, then $f(u) \leq n$ and $f(x) \geq n + 1$ for every $u \in U, x \in X$. By the construction of G, it can be seen that $\deg(x) = 2$ for every $x \in X$. Now, to prove that f is a LIDA labeling, it is sufficient to show that $f(u) \neq f(x)$ for every adjacent $u \in U, x \in X$.

If $\deg(u) = 1$, then clearly $N[u] \subset N[x]$. By Proposition 2.3(a), $w(u) < w(x)$. Hence, we can assume $\deg(u) \geq 2$. By giving the least labels to u, it follows that

$$w(u) \geq f(u) + f(x) + n + 1$$

Meanwhile, if we give x the largest labels, then

$$w(x) \leq f(u) + f(x) + n$$

This implies $w(x) < w(u)$ for every adjacent $u \in U, x \in X$. Therefore, f is a LIDA labeling. Now, since for every $x \in X$, $w(x) = m$, then f induces $n+1$ weights. Equivalently, $\chi_{lida}(S(G)) \leq n + 1$. □

In the application of Theorem 3.6, a LIDA coloring of a graph may be even lower than the given upper bound. In Figure 8, since $3P_3$ is super edge-magic total, an illustration of yielding upper bound of $\chi_{lida}(3P_5) \leq 6$ is given.

Trivially, P_2 is a super edge-magic total graph and it is not hard to check that $S(P_2) \cong P_3$ and $\chi_{lida}(P_3) = 3$. We are interested to see whether there exists other kind of graphs which satisfy the equality in Theorem 3.6.

Problem 3. *What kind of super edge-magic total graphs G satisfying $\chi_{lida}(S(G)) = n + 1$?*

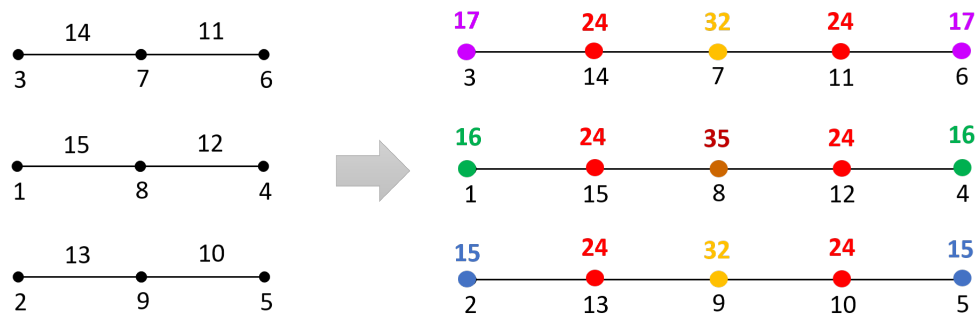

Figure 8: $3P_3$ is super edge-magic total which implies $\chi_{lida}(3P_5) \leq 6$.

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