Further results on edge irregularity strength of graphs

Muhammad Imran\textsuperscript{a,b}, Adnan Aslam\textsuperscript{c}, Sohail Zafar\textsuperscript{d}, Waqas Nazeer\textsuperscript{e}

\textsuperscript{a}Department of Mathematical Sciences, United Arab Emirates University, United Arab Emirates
\textsuperscript{b}Department of Mathematics, National University of Sciences and Technology, Islamabad, Pakistan
\textsuperscript{c}University of Engineering and Technology, Lahore, Pakistan (RCET)
\textsuperscript{d}Department of Mathematics, University of Management and Technology, Lahore, Pakistan
\textsuperscript{e}Division of Science and Technology, University of Education, Lahore, Pakistan

imrandhab@gmail.com, sohailahmad04@gmail.com, adnanaslam15@yahoo.com, waqaster@yahoo.com

Abstract

A vertex $k$-labelling $\phi : V(G) \rightarrow \{1, 2, \ldots, k\}$ is called irregular $k$-labeling of the graph $G$ if for every two different edges $e$ and $f$, there is $\phi(e) \neq \phi(f)$; where the weight of an edge is given by $e = xy \in E(G)$ is $\phi(xy) = \phi(x) + \phi(y)$. The minimum $k$ for which the graph $G$ has an edge irregular $k$-labelling is called edge irregularity strength of $G$, denoted by $es(G)$.

In the paper, we determine the exact value of the edge irregularity strength of caterpillars, $n$-star graphs, $(n, t)$-kite graphs, cycle chains and friendship graphs.

Keywords: irregular assignment, irregular total $k$-labeling, irregularity strength
Mathematics Subject Classification : 05C78
DOI: 10.19184/ijc.2017.1.2.5

1. Introduction and preliminary results

The graph labeling has caught the attention of many authors and many new labeling results appear every year. This popularity is not only due to the mathematical challenges of graph labeling, but also for the wide range of its application, for instance X-ray, crystallography, coding theory, radar, astronomy, circuit design, network design and communication design. Bloom and Golomb studied applications of graph labelings to other branches of science [10, 11].
All the graphs in this paper are finite, undirected and simple. For a graph $G$, the $V(G)$ and $E(G)$ denote the vertex set and edge set, respectively. A labeling of a graph $G$ is any mapping that sends some set of graph elements to a set of non-negative integers. If the domain is vertex set or the edge set, the labeling is called vertex labelings or edge labelings, respectively. Moreover, if the domain is $V(G) \cup E(G)$, then the labeling is called a total labeling. Thus for an edge $k$-labeling, $\phi : E(G) \longrightarrow \{1, 2, \ldots, k\}$, the associated weight of a vertex $x \in V(G)$ is

$$w_\phi(x) = \Sigma \phi(xy)$$

where the sum is taken over all the vertices $y$ adjacent to $x$.

Chartrand et al. in [6] introduced edge $k$-labeling of a graph $G$ such that $w_\phi(x) \neq w_\phi(y)$ for all vertices $x, y \in V(G)$ with $x \neq y$. Such labelings were called irregular assignments and the irregularity strength $s(G)$ of a graph $G$ is known as the minimum $k$ for which $G$ has irregular assignments using labels atmost $k$. Some results on irregularity strength $s(G)$ of a graph $G$ can be found in [1, 3, 6, 7, 8, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23].

Let $\phi$ be a vertex labeling of a graph $G$. Then we define the edge weight of $xy \in E(G)$ to be $w(xy) = \phi(x) + \phi(y)$. A vertex labeling $\phi : V(G) \rightarrow \{1, 2, \ldots, k\}$ is called $k$-labeling. Ali et al. in [2] introduced vertex $k$-labeling $\phi$ of a graph $G$ such that $w_\phi(e) \neq w_\phi(f)$ for every two different edges $e$ and $f$. Such a labeling were called an edge irregular $k$-labeling of the graph $G$. The minimum $k$ for which the graph $G$ has an edge irregular $k$-labeling is called the edge irregularity strength of $G$, denoted by $es(G)$.

They gives a lower bound of the parameter $es(G)$ and determine the exact values of the edge irregularity strength for several family of graphs namely, paths, stars, double stars and cartesian product of two paths.

**Theorem 1.1** ([2]). Let $G$ be simple graph with maximum degree $\Delta = \Delta(G)$. Then

$$es(G) \geq \max \left\{ \left\lceil \frac{|E(G)| + 1}{2} \right\rceil, \Delta(G) \right\}$$

In this paper, we determine the exact value of edge irregularity strength of we determine the exact value of the edge irregularity strength of caterpillars, $n$-star graphs, $(n, t)$-kite graphs, cycle chains and friendship graphs.

2. Main results

Let $P_n$ be a path on $n$ vertices and let $P_n(k)$ be the graph which is obtained by attaching $k$ edges to each vertex of $P_n$. Then $P_n(k)$ is a caterpillar graph. The vertex set $V(P_n(k))$ and edge set $E(P_n(k))$ of this caterpillar graph $P_n(k)$ are $V(P_n(k)) = \{u_i, u_{ij} : 1 \leq i \leq n; 1 \leq j \leq k\}$ and $E(P_n(k)) = \{u_iu_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_iu_{ij} : 1 \leq i \leq n; 1 \leq j \leq k\}$ respectively.

**Theorem 2.1.** Let $P_n(k)$ be the caterpillar graph. If $n$ is even, then $es(P_n(k)) = \frac{n(k+1)}{2}$.

**Proof.** Let $P_n(k)$ be a caterpillar graph. According to Theorem 1.1[2] we have that $es(P_n(k)) \leq \left\lceil \frac{n(k+1)}{2} \right\rceil = \frac{n(k+1)}{2}$. To prove the equality, it suffices to prove the existence of an edge irregular
Let $\phi_1 : V(P_n(k)) \rightarrow \{1, 2, \ldots, \frac{n(k+1)}{2}\}$ be vertex labeling such that
\[
\phi_1(u_i) = \begin{cases} 
\frac{(k+1)i}{2}, & i \equiv 0 \pmod{2}; \\
\frac{i(k+1)+1}{2}, & i \equiv 1 \pmod{2}.
\end{cases}
\]
\[
\phi_1(u_{ij}) = \begin{cases} 
\frac{1}{2}j - k + \frac{i(k+1)}{2}, & i \equiv 0 \pmod{2}; \\
\frac{1}{2}j + \frac{(i-1)(k+1)}{2}, & i \equiv 1 \pmod{2}.
\end{cases}
\]
Since $w_{\phi_1}(u_i u_{i+1}) = (k+1)i+1$ and $w_{\phi_1}(u_i u_{ij}) = k(i-1)+i+j$ for $1 \leq i \leq n$ and $1 \leq j \leq k$, the weights of the edges under the labeling $\phi_1$ successively attain values $2, 3, \ldots, n(k+1)$. We can see that all vertex labels are at most $\frac{n(k+1)}{2}$ and edge weights are distinct for all pairs of distinct edges. Therefore the labeling $\phi_1$ is suitable edge irregular $\frac{n(k+1)}{2}$ labeling. Hence $es(P_n(k)) = \frac{n(k+1)}{2}$.

The gluing together of identical cycles appears in various guises in the literature. But the construction of chains of cycles, with adjacent cycles sharing a single common vertex, is not prevalent. For this reason, we require the following definition. The graph $C_n^{m}$ results from attaching two n-cycles together at a single shared vertex. Continuing in this manner, we define $C_n^{3}$ by attaching a third n-cycle to one of the n-cycles of $C_n^{2}$ in a similar uniform manner so that the cycle containing two shared vertices consists of two identical $\frac{m}{2}$-paths. Recursively, the graph $C_n^{m}$ consists of a chain of $m$ consecutive n-cycles. We refer to each of the graphs in this family as a cycle chain.

**Theorem 2.2.** Let $C_n^{m}$ be cycle chain. If $n$ is even, then
\[
es(C_n^{m}) = \frac{mn}{2} + 1
\]

*Proof.* The vertices of $C_n^{m}$ are identified as follows. First, the shared vertices of cycles $C_i$ and $C_{i+1}$ are identified as $c_i$ for $1 \leq i \leq m - 1$. Also we identify a vertex of $C_1$ and $C_m$ to be $c_0$ and $c_m$ respectively in such a way that we have $\frac{n}{2} - 1$ vertices in between $c_0$ and $c_1$ and $c_{m-1}$ and $c_m$ on both sides. For $1 \leq i \leq m$, the remaining vertices are identified as $c_{i,1}, c_{i,2}, \ldots, c_{i,2^k-1}$ if we move clockwise from the vertex $c_{i-1}$ to the vertex $c_i$ and $c'_{i,1}, c'_{i,2}, \ldots, c'_{i,2^k-1}$ if we move anticlockwise from the vertex $c_{i-1}$ to the vertex $c_i$.

From Theorem 1.1 it follows that $es(C_n^{m}) \geq \left\lceil \frac{mn+1}{2} \right\rceil = \frac{mn}{2} + 1$. For the converse, we define a suitable edge irregular labeling $\phi_2 : V(C_n^{m}) \rightarrow \{1, 2, \ldots, \frac{mn}{2} + 1\}$ as follows:
\[
\phi_2(c_i) = 1 + \frac{n}{2}i \quad for \quad 0 \leq i \leq m
\]
The remaining vertices of $C_n^{m}$ are labeled depending on whether $\frac{n}{2} \equiv 0 \pmod{2}$ or $\frac{n}{2} \equiv 1 \pmod{2}$.

**Case I.** If $\frac{n}{2} \equiv 0 \pmod{2}$, then for $1 \leq j \leq \frac{n}{2} - 1$ and $1 \leq i \leq m$ we define $\phi_2$ as,
\[
\phi_2(c_{i,j}) = \begin{cases} 
\frac{j}{2} + 1 + \frac{n(i-1)}{2}, & j \equiv 0 \pmod{2}; \\
\frac{j}{2} + \frac{n(i-1)}{2}, & j \equiv 1 \pmod{2}.
\end{cases}
\]
and

\[ \phi_2(c_{i,j}) = j + 1 + \frac{n(i-1)}{2} \]

Since \( w_{\phi_2}(c_{i,c_{i+1}}) = 2 + ni \), \( w_{\phi_2}(c_{i,c_{i+1}}') = 3 + ni \) for \( 0 \leq i \leq m - 1 \) and \( w_{\phi_2}(c_{i,j},c_{i,j+1}) = 2j + 2 + n(i-1), w_{\phi_2}(c_{i,j}',c_{i,j+1}) = 2j + 3 + n(i-1) \) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq \frac{n}{2} - 1 \), so the edge weights are distinct for all pairs of distinct edges. Thus the vertex labeling \( \phi_2 \) is an \( \frac{mn}{2} + 1 \)-labeling.

**Case 2.** If \( \frac{n}{2} \equiv 1 \pmod{2} \), then for \( 1 \leq j \leq \frac{n}{2} - 1 \) and \( 1 \leq i \leq m \) we define \( \phi_2 \) as,

\[
\phi_2(c_{i,j}) = \begin{cases} 
  j + \frac{n(i-1)}{2}, & j \equiv 0 \pmod{2} \text{ and } i \equiv 1 \pmod{2}; \\
  j + 1 + \frac{n(i-1)}{2}, & j \equiv 1 \pmod{2} \text{ and } i \equiv 1 \pmod{2}; \\
  j + 1 + \frac{2n(i-1)}{2}, & j \not\equiv 1, j \equiv 1 \pmod{2} \text{ and } i \equiv 0 \pmod{2}; \\
  j + \frac{n(i-1)}{2}, & j \equiv 0 \pmod{2} \text{ and } i \equiv 0 \pmod{2}.
\end{cases}
\]

Now for \( 0 \leq i \leq m - 1 \), we have \( w_{\phi_2}(c_{i,c_{i+1}}) = 3 + ni \), \( w_{\phi_2}(c_{i,c_{i+1}}') = 2 + ni \) if \( i \equiv 0 \pmod{2} \) and \( w_{\phi_2}(c_{i,c_{i+1}}) = 1 + ni \), \( w_{\phi_2}(c_{i,c_{i+1}}') = 4 + ni \) if \( i \equiv 1 \pmod{2} \). Also for \( 1 \leq i \leq m \) and \( j \not\equiv 1 \), we have \( w_{\phi_2}(c_{i,j},c_{i,j+1}) = 2j + 2 + n(i-1), w_{\phi_2}(c_{i,j}',c_{i,j+1}) = 2j + 3 + n(i-1) \) if \( i \equiv 1 \pmod{2} \) and \( w_{\phi_2}(c_{i,j},c_{i,j+1}) = 2j + 2 + n(i-1), w_{\phi_2}(c_{i,j}',c_{i,j+1}) = 2j + 2 + n(i-1) \) if \( i \equiv 0 \pmod{2} \). It is not difficult to see that all vertex labels are at most \( \frac{mn}{2} + 1 \) and the weights of the edges are pairwise distinct. Thus the vertex labeling \( \phi_2 \) is an \( \frac{mn}{2} + 1 \)-labeling.

Truszczynski [4] defines a dragon as a graph obtained by joining a cycle graph \( C_n \) to a path \( P_t \) of length \( t \) with a bridge. Kim and park [19] call them \((n,t)-kites\). Next theorem gives the exact value of the edge irregularity strength for \((n,t)-kite\).

**Theorem 2.3.** Let \( G = (n,t)-kite \). Then

\[ es(G) = \left\lceil \frac{n + t + 1}{2} \right\rceil \]

**Proof.** Let \( G = (n,t)-kite \) graph, the vertex set of \( G \) is

\[ \{v_i|1 \leq i \leq n\} \cup \{u_i|1 \leq i \leq t\} \]

and the edge set of \( G \) is

\[ \{v_iv_{i+1}|1 \leq i \leq n-1\} \cup \{u_iu_{i+1}|1 \leq i \leq t-1\} \cup \{v_nv_1, u_1v_1\} \]
By Theorem 1.1 it follows that $es(G) = \lceil \frac{n+t+1}{2} \rceil$. For the converse, we define a vertex $\lceil \frac{n+t+1}{2} \rceil$-labeling $\phi_3$ as follows:

**Case 1.** If $n = 2k$ and $k \equiv 0 \pmod{2}$, then we define $\phi_3 : V(G) \to \{1, 2, \ldots, \lceil \frac{n+t+1}{2} \rceil \}$ as

\[
\phi_3(v_i) = \begin{cases} 
  k - i + 2, & 1 \leq i \leq k \text{ and } i \equiv 1 \pmod{2}; \\
  k - i + 1, & 1 \leq i \leq k \text{ and } i \equiv 0 \pmod{2}; \\
  i - k, & k + 1 \leq i \leq n.
\end{cases}
\]

\[
\phi_3(u_t) = \begin{cases} 
  k + i + 1, & i \equiv 1 \pmod{2}; \\
  k + \frac{i}{2} + 1, & i \equiv 0 \pmod{2}.
\end{cases}
\]

Since, $w_{\phi_3}(v_i v_{i+1}) = 2(k - i) + 2$ for $1 \leq i \leq k$, $w_{\phi_3}(v_i v_{i+1}) = 2(k - i) + 1$ for $k + 1 \leq i \leq n$ and $w_{\phi_3}(u_t u_{t+1}) = n + i + 2$, the weights of the edges under the labeling $\phi_3$ successively attain values $2, 3, \ldots, n + t + 1$. We can see that all vertex labels are at most $\lceil \frac{n+t+1}{2} \rceil$ and the edge weights are distinct for all pairs of distinct edges. Therefore the labeling $\phi_3$ is a suitable edge irregular $\lceil \frac{n+t+1}{2} \rceil$-labeling.

**Case 2.** If $n = 2k$ and $k \equiv 1 \pmod{2}$, then we define $\phi_3 : V(G) \to \{1, 2, \ldots, \lceil \frac{n+t+1}{2} \rceil \}$ as

\[
\phi_3(v_i) = \begin{cases} 
  k - i - 1, & 1 \leq i \leq k - 2; \\
  1, & i = k - 1; \\
  i - k + 3, & k \leq i \leq 2k - 3 \text{ and } i \equiv 1 \pmod{2}; \\
  i - k + 2, & k \leq i \leq 2k - 3 \text{ and } i \equiv 0 \pmod{2}; \\
  k + 1, & i = 2k - 2, 2k - 1; \\
  k - 1, & i = 2k.
\end{cases}
\]

\[
\phi_3(u_t) = \begin{cases} 
  k + i + 1, & i \equiv 1 \pmod{2}; \\
  k + \frac{i}{2} + 1, & i \equiv 0 \pmod{2}.
\end{cases}
\]

It is not difficult to see that all vertex labels are at most $\lceil \frac{n+t+1}{2} \rceil$ and the weights of the edges are pairwise distinct. Thus the function $\phi_3$ is the desired edge irregular $\lceil \frac{n+t+1}{2} \rceil$-labeling.

**Case 3.** If $n = 2k + 1$ and $k \equiv 0 \pmod{2}$, then we define $\phi_3 : V(G) \to \{1, 2, \ldots, \lceil \frac{n+t+1}{2} \rceil \}$ as

\[
\phi_3(v_i) = \begin{cases} 
  k - i + 2, & 1 \leq i \leq k \text{ and } i \equiv 1 \pmod{2}; \\
  k - i + 1, & 1 \leq i \leq k \text{ and } i \equiv 0 \pmod{2}; \\
  i - k, & k + 1 \leq i \leq n.
\end{cases}
\]

\[
\phi_3(u_t) = \begin{cases} 
  k + i + 1, & i \equiv 1 \pmod{2}; \\
  k + \frac{i}{2} + 1, & i \equiv 0 \pmod{2}.
\end{cases}
\]

From discussion of Case I, it is clear that $\phi_3$ is suitable $\lceil \frac{n+t+1}{2} \rceil$ - labeling.

**Case 4.** If $n = 2k$ and $k \equiv 1 \pmod{2}$, then we define $\phi_3 : V(G) \to \{1, 2, \ldots, \lceil \frac{n+t+1}{2} \rceil \}$ as

\[
\phi_3(v_i) = \begin{cases} 
  k - i - 1, & 1 \leq i \leq k - 2; \\
  1, & i = k - 1; \\
  i - k + 3, & k \leq i \leq 2k - 4 \text{ and } i \equiv 1 \pmod{2}; \\
  i - k + 2, & k \leq i \leq 2k - 4 \text{ and } i \equiv 0 \pmod{2}; \\
  3k - i, & 2k - 3 \leq i \leq n.
\end{cases}
\]
\[ \phi_3(u_i) = \begin{cases} k + \frac{i+3}{2}, & i \equiv 1 \pmod{2}; \\ k + \frac{i}{2} + 1, & i \equiv 0 \pmod{2}. \end{cases} \]

We can see that all vertex labels are at most \( \lceil \frac{n+i+1}{2} \rceil \) and the edge weights are distinct for all pairs of distinct edges. Therefore the labeling \( \phi_3 \) is a suitable edge irregular \( \lceil \frac{n+i+1}{2} \rceil \)-labeling. Hence, \( es(G) = \lceil \frac{n+i+1}{2} \rceil \).

In [7] Seoud and El Sakhawi introduced the following operation of graphs. The symmetric product \( G_1 \oplus G_2 \), of two graphs \( G_1 \) and \( G_2 \), is the graph having vertex set \( V(G_1) \times V(G_2) \) and edge set \( \{(u,v)(u',v') : uu' \in E(G_1) \text{ or } vv' \in E(G_2) \text{ but not both}\} \).

**Theorem 2.4.** Let \( G = P_n \oplus K_2^* \), where \( P_n \) is a path of order \( n \) and \( K_2^* \) is a null graph of order 2. Then

\[ es(G) = \left\lfloor \frac{4n-3}{2} \right\rfloor \]

**Proof.** Let \( G = P_n \oplus K_2^* \) be symmetric product of \( P_n \) and \( K_2^* \), the vertex set of \( G \) is \( V(G) = \{(x, y) | 1 \leq i \leq n, 1 \leq j \leq 2\} \) and the edge set of \( G \) is \( E(G) = \{(x_i, y_j)(x_{i+1}, y_j) \}

\( \cup \{(x_i, y_1)(x_{i+1}, y_2)\} \cup \{(x_i, y_2)(x_{i+1}, y_1)\} \) for \( 1 \leq i \leq n-1 \) and \( 1 \leq j \leq 2 \). So \( P_n \oplus K_2^* \) is a graph of order \( 2n \) and size \( 4n - 4 \). As \( \Delta(P_n \oplus K_2^*) = 4 \) then from Theorem 1.1 it follows that \( es(G) \geq \left\lfloor \frac{4n-3}{2} \right\rfloor \).

For the converse, we define a suitable edge irregular labeling \( \phi_4 : V(G) \rightarrow \{1, 2, \ldots, \left\lfloor \frac{4n-3}{2} \right\rfloor \} \) as follows:

\[ \phi_4(x_i, y_1) = 2i - 1 \]

\[ \phi_4(x_i, y_2) = \begin{cases} 2i - 2, & i \equiv 0 \pmod{2}; \\ 2i + 1, & i \equiv 1 \pmod{2}. \end{cases} \]

Since, \( w_{\phi_4}(x_i, y_1)(x_{i+1}, y_1) = 4i \), \( w_{\phi_4}(x_i, y_2)(x_{i+1}, y_2) = 4i + 1 \), \( w_{\phi_4}(x_i, y_1)(x_{i+1}, y_2) = 4i - 1 \) if \( i \equiv 1 \pmod{2} \), \( w_{\phi_4}(x_i, y_1)(x_{i+1}, y_2) = 4i + 2 \) if \( i \equiv 0 \pmod{2} \), \( w_{\phi_4}(x_i, y_2)(x_{i+1}, y_1) = 4i - 1 \) if \( i \equiv 0 \pmod{2} \) and \( w_{\phi_4}(x_i, y_2)(x_{i+1}, y_1) = 4i + 2 \) if \( i \equiv 1 \pmod{2} \), the weights of the edges under the labeling \( \phi_4 \) successively attain values \( 3, 4, \ldots, 4n - 2 \). We can see that all vertex labels are at most \( \left\lfloor \frac{4n-3}{2} \right\rfloor \) and the edge weights are distinct for all pairs of distinct edges. Therefore the labeling \( \phi_4 \) is a suitable edge irregular \( \left\lfloor \frac{4n-3}{2} \right\rfloor \)-labeling.

Let \( C_4^t \) denote the one-point union of \( t \) cycles of length 4. So \( C_4^t \) is a graph of order \( 3t + 1 \) and size \( 4t \). As \( \Delta(C_4^t) = 2t \) then from Theorem 1.1 it follows that \( es(G) \geq \left\lfloor \frac{4t+1}{2} \right\rfloor \). Next theorem gives the exact value of edge irregularity strength of \( C_4^t \).

**Theorem 2.5.** Let \( C_4^t \) be friendship graph, then

\[ es(C_4^t) = \left\lfloor \frac{4t+1}{2} \right\rfloor = 2t + 1 \]
Proof. The vertices of \( C^t_4 \) are identified as follows: the common vertex of each cycle is identified as \( u \). The remaining vertices of cycle \( C_i \) are identified as \( c_{i,1}, c_{i,2}, c_{i,3} \) if we complete the cycle moving clockwise from the vertex \( u \) to itself. Now, for \( 1 \leq i \leq t \) we construct the function \( \phi_5 : V(C^t_4) \to \{1, 2, \ldots, 2t+1\} \) as follows:

\[
\phi_5(u) = 2t + 1
\]

\[
\phi_5(c_{i,1}) = 2i - 1
\]

\[
\phi_5(c_{i,2}) = 1
\]

\[
\phi_5(c_{i,3}) = 2i
\]

One can see observe the labeling \( \phi_5 \) is an edge irregular \( 2t+1 \)-labeling, which implies the assertion.

Let \( T(n,k) \) be a graph obtained by connecting a vertex \( v \) to the central vertices of \( n \) copies of star on \( k \) vertices. In particular, \( n \) copies of star on \( k + 1 \) vertices shares a common single vertex \( v \). We call \( T(n,k) \) a \( n \)-star graph. The vertex set \( V(T(n,k)) \) and edge set \( E(T(n,k)) \) are \( V(T(n,k)) = \{v\} \cup \{u_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n; 1 \leq j \leq k - 1\} \) and \( E(T(n,k)) = \{vu_i : 1 \leq i \leq n\} \cup \{u_iu_{ij} : 1 \leq i \leq n; 1 \leq j \leq k - 1\} \) respectively. So \( T(n,k) \) is a graph of order \( nk + 1 \) and size \( nk \). As \( \Delta(T(n,k)) = \max\{n, k\} \) then from Theorem 1.1 it follows that \( es(T(n,k)) \geq \lceil \frac{nk+1}{2} \rceil \).

An exact value of the edge irregularity strength of \( n \)-star graph is given by the following theorem.

**Theorem 2.6.** Let \( G = T(n,k) \) be a \( n \)-star graph, then \( es(G) = \lceil \frac{nk+1}{2} \rceil \)

**Proof.** We define a suitable edge irregular labeling \( \phi_6 : V(T(n,k)) \to \{1, 2, \ldots, \lceil \frac{nk+1}{2} \rceil\} \) as follows:

\[
\phi_6(v) = \frac{nk+1}{2}
\]

The remaining vertices of \( T(n,k) \) are labeled depending on whether \( n \equiv 0 \pmod{2} \) or \( n \equiv 1 \pmod{2} \).

**Case 1.** If \( n \equiv 0 \pmod{2} \), then we define \( \phi_6 \) as,

\[
\phi_6(u_i) = \begin{cases} 
  i, & 1 \leq i \leq \frac{n}{2}; \\
  \frac{k(2i-n)}{2} + 1, & \frac{n}{2} < i \leq n.
\end{cases}
\]

\[
\phi_6(u_{ij}) = \begin{cases} 
  j - k + i(k - 2) + 2, & 1 \leq i \leq \frac{n}{2}; \\
  j - k + \frac{n(k-1)}{2}, & i = \frac{n}{2} + 1 \text{ and } 1 \leq j \leq \frac{n}{2}; \\
  j + \frac{k(n-2)}{2}, & \frac{n}{2} < k - 1, i = \frac{n}{2} + 1 \text{ and } \frac{n}{2} < j \leq n; \\
  j + \frac{k(n-2)}{2} + 1, & i > \frac{n}{2} + 1.
\end{cases}
\]

We can see that all vertex labels are at most \( \lceil \frac{nk+1}{2} \rceil = \frac{nk}{2} + 1 \) and edge weights are distinct for all pairs of distinct edges. Therefore the labeling \( \phi_6 \) is suitable edge irregular \( \frac{nk}{2} + 1 \) labeling.
Hence \( es(T(n, k)) = \frac{nk}{2} + 1 \)

**Case 2.1.** If \( n \equiv 1 \pmod{2} \) and \( n = k \), then we define \( \phi_6 \) as,

\[
\phi_6(u_i) = \begin{cases} 
  i, & 1 \leq i \leq \frac{n+1}{2}; \\
  \frac{i}{k(2i-n)+1}, & \frac{n+1}{2} < i \leq n.
\end{cases}
\]

\[
\phi_6(u_{ij}) = \begin{cases} 
  j - k + i(k - 2) + 2, & 1 \leq i \leq \frac{n+1}{2}; \\
  j + \frac{1}{(n+1)(n+2)} - \frac{5n+1}{2}, & \frac{n+1}{2} < i > \frac{n+1}{2}.
\end{cases}
\]

We can see that all vertex labels are at most \( \lceil \frac{nk+1}{2} \rceil = \frac{nk+1}{2} \) and edge weights are distinct for all pairs of distinct edges. Therefore the labeling \( \phi_6 \) is suitable edge irregular \( \frac{nk+1}{2} \) labeling. Hence \( es(T(n, k)) = \frac{nk+1}{2} \).

**Case 2.2.** If \( n \equiv 1 \pmod{2} \) and \( n > k \), then we define \( \phi_6 \) as,

\[
\phi_6(u_i) = \begin{cases} 
  i, & 1 \leq i \leq \frac{n+1}{2}; \\
  \frac{i}{k(2i-n)+1} - (n-i)k, & \frac{n+1}{2} < i \leq n.
\end{cases}
\]

\[
\phi_6(u_{ij}) = \begin{cases} 
  j - k + i(k - 2) + 2, & 1 \leq i \leq \frac{n+1}{2}; \\
  j + \frac{1}{(n+1)(2k-1)} - \frac{nk+1}{2}, & i = \frac{n+3}{2} \text{ and } j \leq \frac{n+k}{2} - k; \\
  j + \frac{1}{(n+1+k(n-1))} - \frac{n+k}{2}, & i = \frac{n+3}{2} \text{ and } j > \frac{n+k}{2} - k; \\
  j - k + \frac{2}{nk+1}, & i > \frac{n+1}{2}.
\end{cases}
\]

We can see that the labeling \( \phi_6 \) is an edge irregular \( \lceil \frac{nk+1}{2} \rceil \)-labeling.

**Case 2.3.** If \( n \equiv 1 \pmod{2} \) and \( n < k \), then we define \( \phi_6 \) as,

\[
\phi_6(u_i) = \begin{cases} 
  i, & 1 \leq i \leq \frac{n-1}{2}; \\
  \frac{i}{k(2i-n)+1} - (n-i)k, & \frac{n-1}{2} < i \leq n.
\end{cases}
\]

\[
\phi_6(u_{ij}) = \begin{cases} 
  j - k + i(k - 2) + 2, & 1 \leq i \leq \frac{n-1}{2}; \\
  j + \frac{1}{(n+1)(2k-1)} - \frac{nk+1}{2}, & i = \frac{n+1}{2} \text{ and } j \leq \frac{n+k}{2}; \\
  j + \frac{1}{(n+1+k(n-1))} - \frac{n+k}{2}, & i = \frac{n+1}{2} \text{ and } j > \frac{n+k}{2}; \\
  j - k + \frac{2}{nk+1}, & i > \frac{n+1}{2}.
\end{cases}
\]

We can see that all vertex labels are at most \( \lceil \frac{nk+1}{2} \rceil \) and edge weights are distinct for all pairs of distinct edges. Therefore the labeling \( \phi_6 \) is suitable edge irregular \( \lceil \frac{nk+1}{2} \rceil \) labeling. Hence \( es(T(n, k)) = \lceil \frac{nk+1}{2} \rceil \).

**Acknowledgement**

This research is supported by the Start Up Research Grant 2016 of United Arab Emirates University via Grant No. G00002233.
Further results on edge irregularity strength of graphs

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