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Further results on edge irregularity strength of graphs

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Abstract

A vertex k-labelling $\phi : V(G) \longrightarrow \{1, 2, \ldots, k\}$ is called irregular k-labeling of the graph G if for every two different edges e and f, there is $w_\phi(e) \neq w_\phi(f)$; where the weight of an edge is given by $e = xy \in E(G)$ is $w_\phi(xy) = \phi(x) + \phi(y)$. The minimum k for which the graph G has an edge irregular k-labelling is called *edge irregularity strength* of G, denoted by $es(G)$.
In the paper, we determine the exact value of the edge irregularity strength of caterpillars, n-star graphs, (n, t)-kite graphs, cycle chains and friendship graphs.

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1. Introduction and preliminary results

The graph labeling has caught the attention of many authors and many new labeling results appear every year. This popularity is not only due to the mathematical challenges of graph labeling, but also for the wide range of its application, for instance X-ray, crystallography, coding theory, radar, astronomy, circuit design, network design and communication design. Bloom and Golomb studied applications of graph labelings to other branches of science [10, 11].

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All the graphs in this paper are finite, undirected and simple. For a graph G, the $V(G)$ and $E(G)$ denote the vertex set and edge set, respectively. A labeling of a graph G is any mapping that sends some set of graph elements to a set of non-negative integers. If the domain is vertex set or the edge set, the labeling is called *vertex labelings* or *edge labelings*, respectively. Moreover, if the domain is $V(G) \cup E(G)$, then the labeling is called a *total labeling*. Thus for an edge k-labeling, $\phi : E(G) \longrightarrow \{1, 2, \dots, k\}$, the associated weight of a vertex $x \in V(G)$ is

$$w_\phi(x) = \Sigma\phi(xy)$$

where the sum is taken over all the vertices y adjacent to x.
Chartrand et al. in [6] introduced edge $k-$ labeling of a graph G such that $w_\phi(x) \neq w_\phi(y)$ for all vertices $x, y \in V(G)$ with $x \neq y$. Such labelings were called *irregular assignments* and the *irregularity strength* $s(G)$ of a graph G is known as the minimum k for which G has irregular assignments using labels atmost k. Some results on irregularity strength $s(G)$ of a graph G can be found in [1, 3, 6, 7, 8, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23].
Let ϕ be a vertex labeling of a graph G. Then we define the edge weight of $xy \in E(G)$ to be $w(xy) = \phi(x) + \phi(y)$. A vertex labeling $\phi : V(G) \to \{1, 2, \dots, k\}$ is called $k-$ labeling. Ali et al. in [2] introduced vertex $k-$ labeling ϕ of a graph G such that $w_\phi(e) \neq w_\phi(f)$ for every two different edges e and f. Such a labeling were called an edge irregular $k-$ labeling of the graph G. The minimum k for which the graph G has an edge irregular $k-$ labeling is called the edge irregularity strength of G, denoted by $es(G)$.
They gives a lower bound of the parameter $es(G)$ and determine the exact values of the edge irregularity strength for several family of graphs namely, paths, stars, double stars and cartesian product of two paths.

Theorem 1.1 ([2])**.** *Let G be simple graph with maximum degree $\Delta = \Delta(G)$. Then*

$$es(G) \geq \max\big\{\lceil\frac{|E(G)| + 1}{2}\rceil, \Delta(G)\big\}$$

In this paper, we we determine the exact value of edge irregularity strength of we determine the exact value of the edge irregularity strength of caterpillars, n-star graphs, (n, t)-kite graphs, cycle chains and friendship graphs.

2. Main results

Let P_n be a path on n vertices and let $P_n(k)$ be the graph which is obtained by attaching k edges to each vertex of P_n. Then $P_n(k)$ is a caterpillar graph. The vertex set $V(P_n(k))$ and edge set $E(P_n(k))$ of this caterpillar graph $P_n(k)$ are $V(P_n(k)) = \{u_i, u_{ij} : 1 \leq i \leq n; 1 \leq j \leq k\}$ and $E(P_n(k)) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iu_{ij} : 1 \leq i \leq n; 1 \leq j \leq k\}$ respectively.

Theorem 2.1. *Let $P_n(k)$ be the caterpillar graph. If n is even, then $es(P_n(k)) = \frac{n(k+1)}{2}$.*

Proof. Let $P_n(k)$ be a caterpillar graph. According to Theorem 1.1[2] we have that $es(P_n(k)) \leq \lceil\frac{n(k+1)}{2}\rceil = \frac{n(k+1)}{2}$. To prove the equality, it suffices to prove the existence of an edge irregular

$\frac{n(k+1)}{2}$ labeling.
Let $\phi_1 : V(P_n(k)) \to \{1, 2, \ldots, \frac{n(k+1)}{2}\}$ be vertex labeling such that

$$\phi_1(u_i) = \begin{cases} \frac{(k+1)i}{2}, & i \equiv 0 \pmod 2; \\ \frac{i(k+1)-k+1}{2}, & i \equiv 1 \pmod 2. \end{cases}$$

$$\phi_1(u_{ij}) = \begin{cases} j - k + \frac{i(k+1)}{2}, & i \equiv 0 \pmod 2; \\ j + \frac{(i-1)(k+1)}{2}, & i \equiv 1 \pmod 2. \end{cases}$$

Since $w_{\phi_1}(u_i u_{i+1}) = (k+1)i+1$ and $w_{\phi_1}(u_i u_{ij}) = k(i-1)+i+j$ for $1 \leq i \leq n$ and $1 \leq j \leq k$, the weights of the edges under the labeling ϕ_1 successively attain values $2, 3, \ldots, n(k+1)$. We can see that all vertex labels are at most $\frac{n(k+1)}{2}$ and edge weights are distinct for all pairs of distinct edges. Therefore the labeling ϕ_1 is suitable edge irregular $\frac{n(k+1)}{2}$ labeling. Hence $es(P_n(k)) = \frac{n(k+1)}{2}$. □

The gluing together of identical cycles appears in various guises in the literature. But the construction of chains of cycles, with adjacent cycles sharing a single common vertex, is not prevalent. For this reason, we require the following definition. The graph C_n^2 results from attaching two n-cycles together at a single shared vertex. Continuing in this manner, we define C_n^3 by attaching a third n-cycle to one of the n-cycles of C_n^2 in a similar uniform manner so that the cycle containing two shared vertices consists of two identical $\frac{n}{2}$-paths. Recursively, the graph C_n^m consists of a chain of m consecutive n-cycles. We refer to each of the graphs in this family as a cycle chain.

Theorem 2.2. *Let C_n^m be cycle chain. If n is even, then*

$$es(C_n^m) = \frac{mn}{2} + 1$$

Proof. The vertices of C_n^m are identified as follows. First, the shared vertices of cycles C_i and C_{i+1} are identified as c_i for $1 \leq i \leq m-1$. Also we identify a vertex of C_1 and C_m to be c_0 and c_m respectively in such a way that we have $\frac{n}{2} - 1$ vertices in between c_0 and c_1 and c_{m-1} and c_m on both sides . For $1 \leq i \leq m$, the remaining vertices are identified as $c_{i,1}, c_{i,2}, \ldots, c_{i,\frac{n}{2}-1}$ if we move clockwise from the vertex c_{i-1} to the vertex c_i and $c'_{i,1}, c'_{i,2}, \ldots, c'_{i,\frac{n}{2}-1}$ if we move anticlockwise from the vertex c_{i-1} to the vertex c_i.
From Theorem 1.1 it follows that $es(C_n^m) \geq \left\lceil \frac{mn+1}{2} \right\rceil = \frac{mn}{2} + 1$. For the converse, we define a suitable edge irregular labeling $\phi_2 : V(C_n^m) \to \{1, 2, \ldots, \frac{mn}{2} + 1\}$ as follows:

$$\phi_2(c_i) = 1 + \frac{n}{2} i \quad for \quad 0 \leq i \leq m$$

The remaining vertices of C_n^m are labeled depending on whether $\frac{n}{2} \equiv 0 \pmod 2$ or $\frac{n}{2} \equiv 1 \pmod 2$.
Case I. If $\frac{n}{2} \equiv 0 mod(2)$, then for $1 \leq j \leq \frac{n}{2} - 1$ and $1 \leq i \leq m$ we define ϕ_2 as,

$$\phi_2(c_{i,j}) = \begin{cases} j + 1 + \frac{n(i-1)}{2}, & j \equiv 0 \pmod 2 \\ j + \frac{n(i-1)}{2}, & j \equiv 1 \pmod 2. \end{cases}$$

and

$$\phi_2(c'_{i,j}) \;=\; j+1+\frac{n(i-1)}{2}$$

Since $w_{\phi_2}(c_ic_{i+1,1}) = 2+ni$, $w_{\phi_2}(c_ic'_{i+1,1}) = 3+ni$ for $0 \leq i \leq m-1$ and $w_{\phi_2}(c_{i,j}c_{i,j+1}) = 2j+2+n(i-1)$, $w_{\phi_2}(c'_{i,j}c'_{i,j+1}) = 2j+3+n(i-1)$ for $1 \leq i \leq m$ and $1 \leq j \leq \frac{n}{2}-1$, so the edge weights are distinct for all pairs of distinct edges. Thus the vertex labeling ϕ_2 is an $\frac{mn}{2}+1$-labeling.

Case 2. If $\frac{n}{2} \equiv 1 \pmod 2$, then for $1 \leq j \leq \frac{n}{2}-1$ and $1 \leq i \leq m$ we define ϕ_2 as,

$$\phi_2(c_{i,j}) \;=\; \begin{cases} j+\frac{n(i-1)}{2}, & j \equiv 0 \pmod 2, \text{ and } i \equiv 1 \pmod 2; \\ j+1+\frac{n(i-1)}{2}, & j \equiv 1 \pmod 2 \text{ and } i \equiv 1 \pmod 2; \\ j+1+\frac{n(i-1)}{2}, & j \neq 1 \text{ and } i \equiv 0 \pmod 2; \\ \frac{n(i-1)}{2}, & j = 1 \text{ and } i \equiv 0 \pmod 2. \end{cases}$$

and

$$\phi_2(c'_{i,j}) = \begin{cases} 2j+\frac{n(i-1)}{2}, & j \equiv 0 \pmod 2 \text{ and } i \equiv 1 \pmod 2; \\ j+\frac{n(i-1)}{2}, & j \equiv 1 \pmod 2 \text{ and } i \equiv 1 \pmod 2; \\ j+1+\frac{n(i-1)}{2}, & j \neq 1, j \equiv 1 \pmod 2 \text{ and } i \equiv 0 \pmod 2; \\ 3+\frac{n(i-1)}{2}, & j = 1 \text{ and } i \equiv 0 \pmod 2; \\ j+\frac{n(i-1)}{2}, & j \equiv 0 \pmod 2 \text{ and } i \equiv 0 \pmod 2. \end{cases}$$

Now for $0 \leq i \leq m-1$, we have $w_{\phi_2}(c_ic_{i+1,1}) = 3+ni$, $w_{\phi_2}(c_ic'_{i+1,1}) = 2+ni$ if $i \equiv 0 \pmod 2$ and $w_{\phi_2}(c_ic_{i+1,1}) = 1+ni$, $w_{\phi_2}(c_ic'_{i+1,1}) = 4+ni$ if $i \equiv 1 \pmod 2$. Also for $1 \leq i \leq m$ and $j \neq 1$, we have $w_{\phi_2}(c_{i,j}c_{i,j+1}) = 2j+2+n(i-1)$, $w_{\phi_2}(c'_{i,j}c'_{i,j+1}) = 2j+3+n(i-1)$ if $i \equiv 1 \pmod 2$ and $w_{\phi_2}(c_{i,j}c_{i,j+1}) = 2j+3+n(i-1)$, $w_{\phi_2}(c'_{i,j}c'_{i,j+1}) = 2j+2+n(i-1)$ if $i \equiv 0 \pmod 2$. It is not difficult to see that all vertex labels are at most $\frac{mn}{2}+1$ and the weights of the edges are pairwise distinct. Thus the vertex labeling ϕ_2 is an $\frac{mn}{2}+1$-labeling. □

Truszczynski [4] defines a dragon as a graph obtained by joining a cycle graph C_n to a path P_t of length t with a bridge. Kim and park [19] call them $(n,t)-$ kites. Next theorem gives the exact value of the edge irregularity strength for $(n,t)-$ kite.

Theorem 2.3. *Let $G = (n,t)-$ kite. Then*

$$es(G) = \lceil \frac{n+t+1}{2} \rceil$$

Proof. Let $G = (n,t)-$ kite graph, the vertex set of G is

$$\{v_i|1 \leq i \leq n\} \cup \{u_i|1 \leq i \leq t\}$$

and the edge set of G is

$$\{v_iv_{i+1}|1 \leq i \leq n-1\} \cup \{u_iu_{i+1}|1 \leq i \leq t-1\} \cup \{v_nv_1, u_1v_1\}$$

By Theorem 1.1 it follows that $es(G) = \lceil \frac{n+t+1}{2} \rceil$. For the converse, we define a vertex $\lceil \frac{n+t+1}{2} \rceil$ -labeling ϕ_3 as follows:
Case 1. If $n = 2k$ and $k \equiv 0 \pmod 2$, then we define $\phi_3 : V(G) \to \{1, 2, \dots, \lceil \frac{n+t+1}{2} \rceil\}$ as

$$\phi_3(v_i) = \begin{cases} k - i + 2, & 1 \le i \le k \text{ and } i \equiv 1 \pmod 2; \\ k - i + 1, & 1 \le i \le k \text{ and } i \equiv 0 \pmod 2; \\ i - k, & k + 1 \le i \le n. \end{cases}$$

$$\phi_3(u_i) = \begin{cases} k + \frac{i+1}{2}, & i \equiv 1 \pmod 2; \\ k + \frac{i}{2} + 1, & i \equiv 0 \pmod 2)). \end{cases}$$

Since, $w_{\phi_3}(v_i v_{i+1}) = 2(k-i)+2$ for $1 \le i \le k$, $w_{\phi_2}(v_i v_{i+1}) = 2(i-k)+1$ for $k+1 \le i \le n$ and $w_{\phi_3}(u_i u_{i+1}) = n + i + 2$, the weights of the edges under the labeling ϕ_3 successively attain values $2, 3, \dots, n + t + 1$. We can see that all vertex labels are at most $\lceil \frac{n+t+1}{2} \rceil$ and the edge weights are distinct for all pairs of distinct edges. Therefore the labeling ϕ_3 is a suitable edge irregular $\lceil \frac{n+t+1}{2} \rceil$-labeling.
Case 2. If $n = 2k$ and $k \equiv 1 \pmod 2$, then we define $\phi_3 : V(G) \to \{1, 2, \dots, \lceil \frac{n+t+1}{2} \rceil\}$ as

$$\phi_3(v_i) = \begin{cases} k - i - 1, & 1 \le i \le k - 2; \\ 1, & i = k - 1; \\ i - k + 3, & k \le i \le 2k - 3 \text{ and } i \equiv 1 \pmod 2; \\ i - k + 2, & k \le i \le 2k - 3 \text{ and } i \equiv 0 \pmod 2; \\ k + 1, & i = 2k - 2, 2k - 1; \\ k - 1, & i = 2k. \end{cases}$$

$$\phi_3(u_i) = \begin{cases} k + \frac{i+1}{2}, & i \equiv 1 \pmod 2; \\ k + \frac{i}{2} + 1, & i \equiv 0 \pmod 2. \end{cases}$$

It is not difficult to see that all vertex labels are at most $\lceil \frac{n+t+1}{2} \rceil$ and the weights of the edges are pairwise distinct. Thus the function ϕ_3 is the desired edge irregular $\lceil \frac{n+t+1}{2} \rceil$-labeling.
Case 3. If $n = 2k + 1$ and $k \equiv 0 \pmod 2$, then we define $\phi_3 : V(G) \to \{1, 2, \dots, \lceil \frac{n+t+1}{2} \rceil\}$ as

$$\phi_3(v_i) = \begin{cases} k - i + 2, & 1 \le i \le k \text{ and } i \equiv 1 \pmod 2; \\ k - i + 1, & 1 \le i \le k \text{ and } i \equiv 0 \pmod 2; \\ i - k, & k + 1 \le i \le n. \end{cases}$$

$$\phi_3(u_i) = \begin{cases} k + \frac{i+3}{2}, & i \equiv 1 \pmod 2; \\ k + \frac{i}{2} + 1, & i \equiv 0 \pmod 2. \end{cases}$$

From discussion of Case I, it is clear that ϕ_3 is suitable $\lceil \frac{n+t+1}{2} \rceil-$ labeling.
Case 4. If $n = 2k$ and $k \equiv 1 \pmod 2$, then we define $\phi_3 : V(G) \to \{1, 2, \dots, \lceil \frac{n+t+1}{2} \rceil\}$ as

$$\phi_3(v_i) = \begin{cases} k - i - 1, & 1 \le i \le k - 2; \\ 1, & i = k - 1; \\ i - k + 3, & k \le i \le 2k - 4 \text{ and } i \equiv 1 \pmod 2; \\ i - k + 2, & k \le i \le 2k - 4 \text{ and } i \equiv 0 \pmod 2; \\ 3k - i, & 2k - 3 \le i \le n. \end{cases}$$

$$\phi_3(u_i) = \begin{cases} k + \frac{i+3}{2}, & i \equiv 1 \pmod 2; \\ k + \frac{i}{2} + 1, & i \equiv 0 \pmod 2. \end{cases}$$

We can see that all vertex labels are at most $\lceil \frac{n+t+1}{2} \rceil$ and the edge weights are distinct for all pairs of distinct edges. Therefore the labeling ϕ_3 is a suitable edge irregular $\lceil \frac{n+t+1}{2} \rceil$-labeling. Hence, $es(G) = \lceil \frac{n+t+1}{2} \rceil$. □

In [7] Seoud and El Sakhawi introduced the following operation of graphs. The symmetric product $G_1 \oplus G_2$, of two graphs G_1 and G_2, is the graph having vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u,v)(u', v' : uu' \in E(G_1) \quad \text{or} \quad vv' \in E(G_2) \quad \text{but not both}\}$.

Theorem 2.4. *Let $G = P_n \oplus K_2^*$, where P_n is a path of order n and K_2^* is a null graph of order 2. Then*

$$es(G) = \lceil \frac{4n-3}{2} \rceil$$

Proof. Let $G = P_n \oplus K_2^*$ be symmetric product of P_n and K_2^*, the vertex set of G is $V(G) = \{(x_i, y_j) | 1 \le i \le n, 1 \le j \le 2\}$ and the edge set of G is $E(G) = \{(x_i, y_j)(x_{i+1}, y_j)\}$ $\cup\{(x_i, y_1)(x_{i+1}, y_2)\}$ $\cup\{(x_i, y_2)(x_{i+1}, y_1)\}$ for $1 \le i \le n-1$ and $1 \le j \le 2$. So $P_n \oplus K_2^*$ is a graph of order $2n$ and size $4n-4$. As $\triangle(P_n \oplus K_2^*) = 4$ then from Theorem 1.1 it follows that $es(G) \ge \lceil \frac{4n-3}{2} \rceil$.
For the converse, we define a suitable edge irregular labeling $\phi_4 : V(G) \to \{1, 2, \ldots, \lceil \frac{4n-3}{2} \rceil\}$ as follows:

$$\phi_4(x_i, y_1) \quad = \quad 2i - 1$$

$$\phi_4(x_i, y_2) = \begin{cases} 2i - 2, & i \equiv 0 \pmod 2; \\ 2i + 1, & i \equiv 1 \pmod 2. \end{cases}$$

Since, $w_{\phi_4}(x_i, y_1)(x_{i+1}, y_1) = 4i$, $w_{\phi_4}(x_i, y_2)(x_{i+1}, y_2) = 4i+1$, $w_{\phi_4}(x_i, y_1)(x_{i+1}, y_2) = 4i-1$ if $i \equiv 1 \pmod 2$, $w_{\phi_4}(x_i, y_1)(x_{i+1}, y_2) = 4i+2$ if $i \equiv 0 \pmod 2$, $w_{\phi_4}(x_i, y_2)(x_{i+1}, y_1) = 4i-1$ if $i \equiv 0 \pmod 2$ and $w_{\phi_4}(x_i, y_2)(x_{i+1}, y_1) = 4i+2$ if $i \equiv 1 \pmod 2$, the weights of the edges under the labeling ϕ_4 successively attain values $3, 4, \ldots, 4n-2$. We can see that all vertex labels are at most $\lceil \frac{4n-3}{2} \rceil$ and the edge weights are distinct for all pairs of distinct edges. Therefore the labeling ϕ_4 is a suitable edge irregular $\lceil \frac{4n-3}{2} \rceil$-labeling. □

Let C_4^t denote the one-point union of t cycles of length 4. So C_4^t is a graph of order $3t+1$ and size $4t$. As $\triangle(C_4^t) = 2t$ then from Theorem 1.1 it follows that $es(G) \ge \lceil \frac{4t+1}{2} \rceil$. Next theorem gives the exact value of edge irregularity strength of C_4^t

Theorem 2.5. *Let C_4^t be friendship graph, then*

$$es(C_4^t) = \lceil \frac{4t+1}{2} \rceil = 2t + 1$$

Proof. The vertices of C_4^t are identified as follows: the common vertex of each cycle is identified as u. The remaining vertices of cycle C_i are identified as $c_{i,1}$, $c_{i,2}$, $c_{i,3}$ if we complete the cycle moving clockwise from the vertex u to itself. Now, for $1 \leq i \leq t$ we construct the function $\phi_5 : V(C_4^t) \rightarrow \{1, 2, \dots, 2t+1\}$ as follows:

$$\phi_5(u) = 2t+1$$

$$\begin{aligned}\phi_5(c_{i,1}) &= 2i-1\\ \phi_5(c_{i,2}) &= 1\\ \phi_5(c_{i,3}) &= 2i\end{aligned}$$

One can see observe the labeling ϕ_5 is an edge irregular $2t+1$-labeling, which implies the assertion. □

Let $T(n,k)$ be a graph obtained by connecting a vertex v to the central vertices of n copies of star on k vertices. In particular, n copies of star on $k+1$ vertices shares a common single vertex v. We call $T(n,k)$ a $n-$ star graph. The vertex set $V(T(n,k))$ and edge set $E(T(n,k))$ are $V(T(n,k)) = \{v\} \cup \{u_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n; 1 \leq j \leq k-1\}$ and $E((T(n,k)) = \{vu_i : 1 \leq i \leq n\} \cup \{u_iu_{ij} : 1 \leq i \leq n; 1 \leq j \leq k-1\}$ respectively. So $T(n,k)$ is a graph of order $nk+1$ and size nk. As $\triangle(T(n,k)) = \max\{n,k\}$ then from Theorem 1.1 it follows that $es(T(n,k)) \geq \lceil \frac{nk+1}{2} \rceil$.
An exact value of the edge irregularity strength of n-star graph is given by the following theorem.

Theorem 2.6. *Let $G = T(n,k)$ be $n-$ star graph, then $es(G) = \lceil \frac{nk+1}{2} \rceil$*

Proof. We define a suitable edge irregular labeling $\phi_6 : V(T(n,k)) \rightarrow \{1, 2, \dots, \lceil \frac{nk+1}{2} \rceil\}$ as follows:

$$\phi_6(v) = \lceil \frac{nk+1}{2} \rceil$$

The remaining vertices of $T(n,k)$ are labeled depending on whether $n \equiv 0 \pmod 2$ or $n \equiv 1 \pmod 2$.
Case 1. If $n \equiv 0 \pmod 2$, then we define ϕ_6 as,

$$\phi_6(u_i) = \begin{cases} i, & 1 \leq i \leq \frac{n}{2}; \\ \frac{k(2i-n)}{2} + 1, & \frac{n}{2} < i \leq n. \end{cases}$$

$$\phi_6(u_{ij}) = \begin{cases} j - k + i(k-2) + 2, & 1 \leq i \leq \frac{n}{2}; \\ j - k + \frac{n(k-1)}{2}, & i = \frac{n}{2} + 1 \text{ and } 1 \leq j \leq \frac{n}{2}; \\ j + \frac{k(n-2)}{2}, & \frac{n}{2} < k-1, i = \frac{n}{2} + 1 \text{ and } \frac{n}{2} < j \leq n; \\ j + \frac{k(n-2)}{2} + 1, & i > \frac{n}{2} + 1. \end{cases}$$

We can see that all vertex labels are at most $\lceil \frac{nk+1}{2} \rceil = \frac{nk}{2} + 1$ and edge weights are distinct for all pairs of distinct edges. Therefore the labeling ϕ_6 is suitable edge irregular $\frac{nk}{2} + 1$ labeling.

Hence $es(T(n,k)) = \frac{nk}{2} + 1$
Case 2.1. If $n \equiv 1 \pmod 2$ and $n = k$, then we define ϕ_6 as,

$$\phi_6(u_i) = \begin{cases} i, & 1 \le i \le \frac{n+1}{2}; \\ \frac{k(2i-n)+1}{2}, & \frac{n+1}{2} < i \le n. \end{cases}$$

$$\phi_6(u_{ij}) = \begin{cases} j - k + i(k-2) + 2, & 1 \le i \le \frac{n+1}{2}; \\ j + \frac{(n+1)(n+2)}{2} - \frac{5n+1}{2}, & i > \frac{n+1}{2}. \end{cases}$$

We can see that all vertex labels are at most $\lceil \frac{nk+1}{2} \rceil = \frac{nk+1}{2}$ and edge weights are distinct for all pairs of distinct edges. Therefore the labeling ϕ_6 is suitable edge irregular $\frac{nk+1}{2}$ labeling. Hence $es(T(n,k)) = \frac{nk+1}{2}$.
Case 2.2. If $n \equiv 1 \pmod 2$ and $n > k$, then we define ϕ_6 as,

$$\phi_6(u_i) = \begin{cases} i, & 1 \le i \le \frac{n+1}{2}; \\ \lceil \frac{nk+1}{2} \rceil - (n-i)k, & \frac{n+1}{2} < i \le n. \end{cases}$$

$$\phi_6(u_{ij}) = \begin{cases} j - k + i(k-2) + 2, & 1 \le i \le \frac{n+1}{2}; \\ j + \frac{(n-1)(2k-1)}{2} - \lceil \frac{nk+1}{2} \rceil, & i = \frac{n+3}{2} \text{ and } j \le \lceil \frac{n+k}{2} \rceil - k; \\ j + \frac{n+1+k(n-1)}{2} - \lceil \frac{n+k}{2} \rceil, & i = \frac{n+3}{2} \text{ and } j > \lceil \frac{n+k}{2} \rceil - k; \\ j - k + \lceil \frac{nk+1}{2} \rceil, & i > \frac{n+3}{2}. \end{cases}$$

We can see that the labeling ϕ_6 is an edge irregular $\lceil \frac{nk+1}{2} \rceil$-labeling.
Case 2.3. If $n \equiv 1 \pmod 2$ and $n < k$, then we define ϕ_6 as,

$$\phi_6(u_i) = \begin{cases} i, & 1 \le i \le \frac{n-1}{2}; \\ \lceil \frac{nk+1}{2} \rceil - (n-i)k, & \frac{n-1}{2} < i \le n. \end{cases}$$

$$\phi_6(u_{ij}) = \begin{cases} j - k + i(k-2) + 2, & 1 \le i \le \frac{n-1}{2}; \\ j + 1 + \frac{(n-1)(2k-1)}{2} - \lceil \frac{nk+1}{2} \rceil, & i = \frac{n+1}{2} \text{ and } j < \lceil \frac{n+k}{2} \rceil; \\ j + \frac{n+1+k(n-1)}{2} - \lceil \frac{n+k}{2} \rceil, & i = \frac{n+1}{2} \text{ and } j \ge \lceil \frac{n+k}{2} \rceil; \\ j - k + \lceil \frac{nk+1}{2} \rceil, & i > \frac{n+1}{2}. \end{cases}$$

We can see that all vertex labels are at most $\lceil \frac{nk+1}{2} \rceil$ and edge weights are distinct for all pairs of distinct edges. Therefore the labeling ϕ_6 is suitable edge irregular $\lceil \frac{nk+1}{2} \rceil$ labeling. Hence $es(T(n,k)) = \lceil \frac{nk+1}{2} \rceil$. □

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References

[1] Ahmad A., On vertex irregular total labelings of convex polytope graphs, *Utilitas Math.*, **89** (2012), 69–78.

[2] A. Ahmad, O. B. Saeed Al-Mushayt, M. Baca, On edge irregularity strength of graphs, *Appl. Math. Comput.*, **243** (2014), 607-610.

[3] W. C. Chen, H. I. Lu, and Y. N. Yeh, Operations of interlaced trees and graceful trees, *Southeast Asian Bull. Math.*, **21** (1997), 337–348.

[4] M. Truszczynski, Graceful unicyclic graphs, *Demonstatio Math.*, **17** (1984), 377–387.

[5] S. R. Kim and J. Y. Park, On super edge-magic graphs, *Ars Combin.*, **81** (2006), 113–127.

[6] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz, F. Saba, Irregular networks, *Congr. Numer.*, **64** (1988), 187-192.

[7] M. A. Seoud and E. El Sakhawi, On variations of graceful labelings, *Ars Combin.*, **87** (2008), 127–138.

[8] A. Ahmad, A. Q. Baig, M. Imran, On super edge-magicness of graphs, *Utilitas Math.*, **89** (2012), 373–380.

[9] M. Baca, Y. Lin, F. A. Muntaner-Batle, Super edge-antimagic labeling of path like trees, *Utilitas Math,*to appear.

[10] G. S. Bloom, S. W. Golomb, Applications of undirected graphs, *Proc. IEEE* **65** (1977), 562-570.

[11] G. S. Bloom G. Sand Golomb S. W., Numbered complete graphs, unusual rules, and assorted applications, *Theory and Applications of Graphs*, Lecture Notes in Math 642. Springer-Verlag (1978), 53-65.

[12] H. Enomoto, A. S. Llado. T. Nakamigawa, and G.R ingle, Super edge-magic graphs, *SUT J.Math*, **34** (1980),105–109.

[13] A. Q. Baig, A. Ahmad, E. T. Baskoro and R. Simanjuntak, On the super edge-magic graphs, *SUT J. Math*, **86** (2001), 147–159.

[14] R. M. Figueroa, R. Ichishima and F. A. Muntaner-Batle, The place of super edge-magic labeling among other classes of labeling, *Discrete Math.* **231** (2001), 153–168.

[15] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* 18 #DS6, http://www.combinatorics.org/surveys/ds6.pdf (2010).

[16] M. Hussain, E. T. Baskoro, K. Ali, On super antimagic total labeling of Harary graph, *Ars combin.*, to appear.

[17] M. Hussain, E. T. Baskoro, Slamin, On super edge-magic total labeling of banana trees, *Utilitas Math.*, **79** (2009), 243-251.

[18] M. Javed, M. Hussain, K. Ali, K. H. Dar, On super edge-magic total labeling of banana on w-trees, *Utilitas Math.*, **86** (2011), 183-191.

[19] S.-R. Kim and J. Y. Park, On super edge-magic graphs, *Ars Combin.*, **81** (2006), 113–127.

[20] A. Kotzig and A. Rosa, Magic valuaton of finite graphs, *Canad. Math. Bull.* 13(4) (1970), 451–461.

[21] J. Y. Park, J. H. Choi and J-H. Bae, On super edge-magic labeling of some graphs, *Bull. Korean Math. Soc.,* **45** (2008), No.1, 11-21.

[22] W. D. Wallis, Magic Graphs, Birkhäuser, Boston, 2001.

[23] W. D. Wallis, E. T. Baskoro, M. Miller, and Slamin, Edge-magic total labelings, *Australas. J. Combin.,* **22** (2000), 177–190.