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Numbers of Weights of Convex Quadrilaterals in Weighted Point Sets

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Abstract

Let $\mathcal{P}_n$ denote the family of sets of points in general position in the plane each of which is assigned a different number, called a weight, in $\{1, 2, \ldots, n\}$. For $P \in \mathcal{P}_n$ and a polygon Q with vertices in P, we define the weight of Q as the sum of the weights of its vertices and denote by $W_k(P)$ the set of weights of *convex* k-gons with vertices in $P \in \mathcal{P}_n$. Let $f_k(n) = \min_{P \in \mathcal{P}_n} |W_k(P)|$. It is shown in [10] that $n - 5 \leq f_4(n) \leq 2n - 9$ for $n \geq 7$. In this paper, we show that $f_4(n) \geq 4n/3 - 7$.

Keywords: point set, weight, convex quadrilateral
Mathematics Subject Classification : 52C05, 52C10

1. Introduction

Let P be a set of points in the plane. The points of P are said to be *in general position* if no three of them are collinear. Let $\mathcal{P}$ denote the family of sets of points in general position in the plane. We say that P *contains* a convex k-gon if P contains k points that are vertices of a convex k-gon. A convex polygon with vertex set $Q(\subseteq P)$ will be simply referred to as a *convex polygon Q (of P)*.

In the winter of 1932/33, E. Klein found that

$$\text{any point set } P \in \mathcal{P} \text{ with } |P| \geq 5 \text{ contains a convex quadrilateral.} \tag{1}$$

Erdős and Szekeres [5] proved that for any integer $k \geq 3$, there is an integer $N(k)$ such that any point set $P \in \mathcal{P}$ with $|P| \geq N(k)$ contains a convex k-gon. They also conjectured that

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$N(k) = 2^{k-2} + 1$ and later proved $N(k) \geq 2^{k-2} + 1$ in [6]. At this time, it has been shown that $N(4) = 5$, $N(5) = 9$ [9], and, with the aid of a computer, $N(6) = 17$ [11]. In 1984, Erdős [4] asked the minimum number of convex k-gons contained in $P \in \mathcal{P}$ with $|P| = n$. In particular, for $k = 4$, this problem is equivalent to the problem of determining the *rectilinear crossing number* of K_n and has been studied for a long time [1, 2, 3, 7, 8].

A point set $P \in \mathcal{P}$ is called a *weighted point set* if each point is assigned a number called a *weight*. An *n-set* is a set with n elements. In this paper, we often consider 4-subsets of sets of points or integers. We denote by $\mathcal{P}_n$ the family of *weighted point sets* with n elements each of which receives a different weight in $\{1, 2, \ldots, n\}$. For a point p, let $w(p)$ denote the weight of p. For a weighted point set $P \in \mathcal{P}_n$, we let

$$w(P) = \sum_{p \in P} w(p).$$

The *weight of a polygon* (not necessarily a convex one) with vertex set P is also defined by $w(P)$.

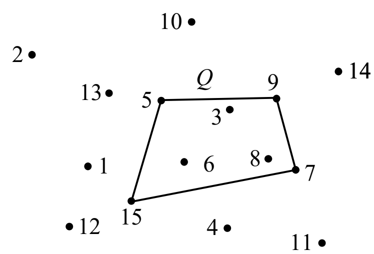


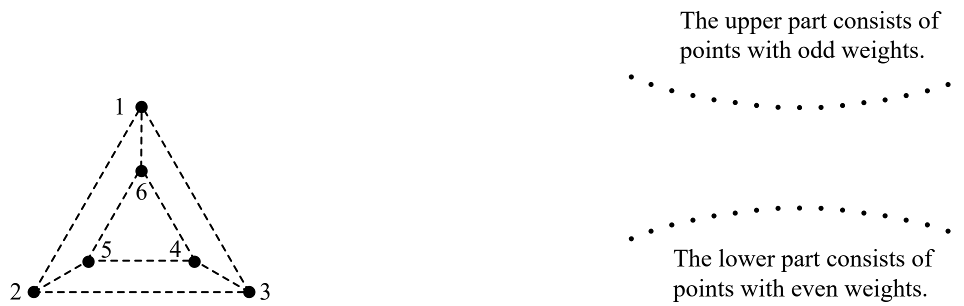
Figure 1. An example of $P \in \mathcal{P}_{15}$ and a convex quadrilateral Q with $w(Q) = 5 + 15 + 7 + 9 = 36$.

For a weighted point set P, let $W_k(P)$ denote the set of weights of *convex* k-gons of P. For positive integers n and $k \geq 3$, we define $f_k(n)$ by

$$f_k(n) = \min_{P \in \mathcal{P}_n} |W_k(P)|.$$

Obviously, for any $P \in \mathcal{P}_n$ with $n \geq 3$, and for any integer k between $1 + 2 + 3 = 6$ and $(n-2) + (n-1) + n = 3n - 3$, there exists a triangle T with $w(T) = k$. Thus we have $|W_3(P)| = 3n - 8$ for any $P \in \mathcal{P}_n$, and hence $|f_3(n)| = 3n - 8$ for $n \geq 3$. However, for $n \geq 4$, convex quadrilaterals of $P \in \mathcal{P}_n$ do not necessarily have all integers between $1 + 2 + 3 + 4 = 10$ and $(n-3) + (n-2) + (n-1) + n = 4n - 6$. For example, if $P \in \mathcal{P}_4$ consists of three vertices of a triangle and a point in its interior, P does not contain a convex quadrilateral, and hence $|W_4(P)| = 0$. Thus $f_4(4) = 0$. The point set $P \in \mathcal{P}_6$ shown in Figure 2 (a) contains three convex quadrilaterals, but they all have the same weight 14, which implies that $f_4(6) \leq 1$. Furthermore, the point set shown in Figure 2 (b) contains no convex quadrilateral with an odd weight. Based on these facts, it is shown in [10] that

$$f_4(n) = 0 \text{ for } n \leq 4, \quad f_4(5) = f_4(6) = 1 \tag{2}$$



(a) All convex quadrilaterals have weight 14. (b) No convex quadrilateral has an odd weight.

Figure 2. Two examples.

and

$$f_4(n) \leq 2n - 9 \text{ for } n \geq 7. \tag{3}$$

As for a lower bound of $f_4(n)$, it is shown in [10] that $f_4(n) \geq n - 5$ for any n. In this paper, we show that:

Theorem 1.1. $f_4(n) \geq \dfrac{4}{3}n - 7$ *for any* n.

To prove Theorem 1.1, we first show the following theorem:

Theorem 1.2. $|W_4(P) - \{10\}| \geq 4$ *for any* $P \in \mathcal{P}_7$.

We conclude this section with a theorem due to Guy which we need in our proof of Theorem 1.2.

Theorem 1.3 (Guy [7])**.** *Any* $P \in \mathcal{P}_7$ *contains at least nine convex quadrilaterals.*

2. Proof of Theorem 1.2

We prove Theorem 1.2 by dividing it into two parts:

(I) $|W_4(P)| \geq 4$ for any $P \in \mathcal{P}_7$; and

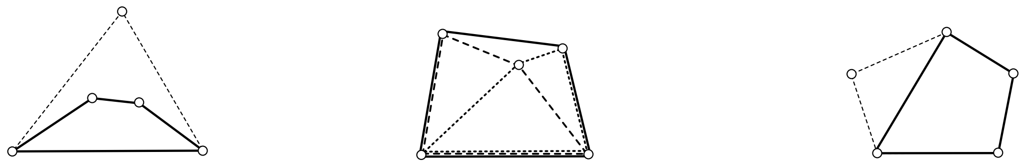
(II) if $(1 + 2 + 3 + 4 =)10 \in W_4(P)$, then $|W_4(P)| \geq 5$.

For a point set S in the plane, we denote by $\mathrm{Conv}(S)$ the convex hull of S.

2.1. Proof of (I)

As implied in the proof of (1) shown in [5], the number of convex quadrilaterals contained in $P \in \mathcal{P}_5$ is 1, 3 or 5 according to whether $\mathrm{Conv}(P)$ is a triangle, a quadrilateral or a pentagon, respectively (Figure 3). Hence

$$\text{any } P \in \mathcal{P}_5 \text{ contains an odd number of convex quadrilaterals.} \tag{4}$$



(a) One convex quadrilateral. (b) Three convex quadrilaterals. (c) One of five convex quadrilaterals.

Figure 3. The number of convex quadrilaterals contained in $P \in \mathcal{P}_5$ is odd.

Table 5 summarizes the 5-subsets of 7-set $\{1,2,3,4,5,6,7\}$, the 4-subsets of each 5-subset, and the sum of the elements of each 4-subset. For simplicity, each 5-subset is represented in the left-most column as a 5-digit number of its elements arranged in increasing order, and its 4-subsets are represented as 4-digit numbers in the same row. The sum of the elements of each 4-subset is shown in the top cell of the same column.

For each $P \in \mathcal{P}_7$, let $\mathcal{Q}_P = \left\{Q_1, Q_2, \ldots, Q_{\binom{7}{4}}\right\}$ be the family of 4-subsets of P, where 4-subsets are indexed in the lexicographic order of the 4-digit numbers. More specifically, since $1234 < 1235 < \cdots < 4567$, we have

$$Q_1 = \{1,2,3,4\},\ Q_2 = \{1,2,3,5\},\ \ldots,\ Q_{\binom{7}{4}} = Q_{35} = \{4,5,6,7\}.$$

For each i with $1 \leq i \leq 35$, let

$$x_i = \begin{cases} 1 & (\ \mathrm{Conv}(Q_i) \text{ is a quadrilateral}), \\ 0 & (\text{otherwise}). \end{cases}$$

It follows from (4) that the number of 4-subsets that are the vertex sets of convex quadrilaterals is odd in each row of Table 1. Thus

$$\text{the following } \binom{7}{5} = 21 \text{ values are all odd integers:} \tag{5}$$

$$\begin{array}{lll}
x_1 + x_2 + x_5 + x_{11} + x_{21}, & x_1 + x_3 + x_6 + x_{12} + x_{22}, & x_1 + x_4 + x_7 + x_{13} + x_{23}, \\
x_2 + x_3 + x_8 + x_{14} + x_{24}, & x_2 + x_4 + x_9 + x_{15} + x_{25}, & x_3 + x_4 + x_{10} + x_{16} + x_{26}, \\
x_5 + x_6 + x_8 + x_{17} + x_{27}, & x_5 + x_7 + x_9 + x_{18} + x_{28}, & x_6 + x_7 + x_{10} + x_{19} + x_{29}, \\
x_8 + x_9 + x_{10} + x_{20} + x_{30}, & x_{11} + x_{12} + x_{14} + x_{17} + x_{31}, & x_{11} + x_{13} + x_{15} + x_{18} + x_{32}, \\
x_{12} + x_{13} + x_{16} + x_{19} + x_{33}, & x_{14} + x_{15} + x_{16} + x_{20} + x_{34}, & x_{17} + x_{18} + x_{19} + x_{20} + x_{35}, \\
x_{21} + x_{22} + x_{24} + x_{27} + x_{31}, & x_{21} + x_{23} + x_{25} + x_{28} + x_{32}, & x_{22} + x_{23} + x_{26} + x_{29} + x_{33}, \\
x_{24} + x_{25} + x_{26} + x_{30} + x_{34}, & x_{27} + x_{28} + x_{29} + x_{30} + x_{35} \ \text{and} & x_{31} + x_{32} + x_{33} + x_{34} + x_{35}.
\end{array}$$

Table 1. 5-subsets of $\{1, 2, \ldots, 7\}$, their 4-subsets, and the sums of the elements of the 4-subsets.

	10	11	12	13	14	15	16	17	18	19	20	21	22
12345	1234 (Q_1)	1235 (Q_2)	1245 (Q_5)	1345 (Q_{11})	2345 (Q_{21})								
12346	1234 (Q_1)		1236 (Q_3)	1246 (Q_6)	1346 (Q_{12})	2346 (Q_{22})							
12347	1234 (Q_1)			1237 (Q_4)	1247 (Q_7)	1347 (Q_{13})	2347 (Q_{23})						
12356		1235 (Q_2)	1236 (Q_3)		1256 (Q_8)	1356 (Q_{14})	2356 (Q_{24})						
12357		1235 (Q_2)		1237 (Q_4)		1257 (Q_9)	1357 (Q_{15})	2357 (Q_{25})					
12367			1236 (Q_3)	1237 (Q_4)			1267 (Q_{10})	1367 (Q_{16})	2367 (Q_{26})				
12456			1245 (Q_5)	1246 (Q_6)	1256 (Q_8)		1456 (Q_{17})	2456 (Q_{27})					
12457			1245 (Q_5)		1247 (Q_7)	1257 (Q_9)		1457 (Q_{18})	2457 (Q_{28})				
12467				1246 (Q_6)	1247 (Q_7)		1267 (Q_{10})		1467 (Q_{19})	2467 (Q_{29})			
12567					1256 (Q_8)	1257 (Q_9)	1267 (Q_{10})			1567 (Q_{20})	2567 (Q_{30})		
13456				1345 (Q_{11})	1346 (Q_{12})	1356 (Q_{14})	1456 (Q_{17})		3456 (Q_{31})				
13457				1345 (Q_{11})		1347 (Q_{13})	1357 (Q_{15})	1457 (Q_{18})		3457 (Q_{32})			
13467					1346 (Q_{12})	1347 (Q_{13})		1367 (Q_{16})	1467 (Q_{19})		3467 (Q_{33})		
13567						1356 (Q_{14})	1357 (Q_{15})	1367 (Q_{16})		1567 (Q_{20})		3567 (Q_{34})	
14567							1456 (Q_{17})	1457 (Q_{18})	1467 (Q_{19})	1567 (Q_{20})			4567 (Q_{35})
23456					2345 (Q_{21})	2346 (Q_{22})	2356 (Q_{24})	2456 (Q_{27})	3456 (Q_{31})				
23457					2345 (Q_{21})		2347 (Q_{23})	2357 (Q_{25})	2457 (Q_{28})	3457 (Q_{32})			
23467						2346 (Q_{22})	2347 (Q_{23})		2367 (Q_{26})	2467 (Q_{29})	3467 (Q_{33})		
23567							2356 (Q_{24})	2357 (Q_{25})	2367 (Q_{26})		2567 (Q_{30})	3567 (Q_{34})	
24567								2456 (Q_{27})	2457 (Q_{28})	2467 (Q_{29})	2567 (Q_{30})		4567 (Q_{35})
34567									3456 (Q_{31})	3457 (Q_{32})	3467 (Q_{33})	3567 (Q_{34})	4567 (Q_{35})

To prove (I), we show that there are at least four columns (except for the left-most one) containing 4-subsets that are the vertex sets of convex quadrilaterals, i.e.,

at least four of the following thirteen values are 1: (6)

$x_1, x_2, \max\{x_3, x_5\}, \max\{x_4, x_6, x_{11}\}, \max\{x_7, x_8, x_{12}, x_{21}\}, \max\{x_9, x_{13}, x_{14}, x_{22}\},$
$\max\{x_{10}, x_{15}, x_{17}, x_{23}, x_{24}\}, \max\{x_{16}, x_{18}, x_{25}, x_{27}\}, \max\{x_{19}, x_{26}, x_{28}, x_{31}\},$
$\max\{x_{20}, x_{29}, x_{32}\}, \max\{x_{30}, x_{33}\}, x_{34}$ and x_{35}.

Under condition (5), we search for the x_i, $1 \leq i \leq 35$, for which (6) does *not* hold by running a Java program on a computer. Then we obtain only one solution:

$$x_i = \begin{cases} 1 & (i = 10, 12, 15, 17, 21, 23, 24, 28, 31), \\ 0 & (\text{otherwise}), \end{cases}$$

which implies that P contains exactly nine convex quadrilaterals with vertex sets

$Q_{10} = \{1, 2, 6, 7\}$, $Q_{12} = \{1, 3, 4, 6\}$, $Q_{15} = \{1, 3, 5, 7\}$, $Q_{17} = \{1, 4, 5, 6\}$, $Q_{21} = \{2, 3, 4, 5\}$, $Q_{23} = \{2, 3, 4, 7\}$, $Q_{24} = \{2, 3, 5, 6\}$, $Q_{28} = \{2, 4, 5, 7\}$ and $Q_{31} = \{3, 4, 5, 6\}$.

To complete the proof of (I), we show that this solution cannot be realized as a 7-set belonging to $\mathcal{P}_7$. By way of contradiction, suppose that there exists $P \in \mathcal{P}_7$ that contains exactly nine convex quadrilaterals $Q_{10}, Q_{12}, \ldots, Q_{31}$ shown above.

Lemma 2.1. *P does not contain a convex pentagon.*

Proof. Suppose that P contains a convex pentagon $S = \{i_1, i_2, i_3, i_4, i_5\}$. Then its 4-subsets $S - \{i_1\}, S-\{i_2\}, \cdots, S-\{i_5\}$ have all different weights, and hence $|W(P)| \geq 5$, a contradiction. □

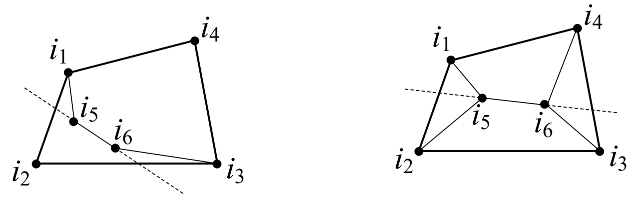
For each $i \in P$, let $m(i)$ denote the number of subsets among the nine 4-subsets $Q_{10}, Q_{12}, \cdots, Q_{31}$ that contain i. We have

$$m(1) = 4,\ m(2) = 5,\ m(3) = 6,\ m(4) = 6,\ m(5) = 6,\ m(6) = 5 \text{ and } m(7) = 4. \qquad (7)$$

Lemma 2.2. *P does not contain a convex quadrilateral with two interior points.*

Proof. Suppose that P contains a quadrilateral with two interior points. Denote by i_1, i_2, i_3 and i_4 the four vertices of the quadrilateral in counterclockwise order, and by i_5 and i_6 the two interior points. If the straight line i_5i_6 intersects two adjacent sides of the quadrilateral, say the sides i_1i_2 and i_2i_3 (Figure 4 (a)), then the five points i_1, i_3, i_4, i_5 and i_6 are five vertices of a convex pentagon, which contradicts Lemma 2.1.

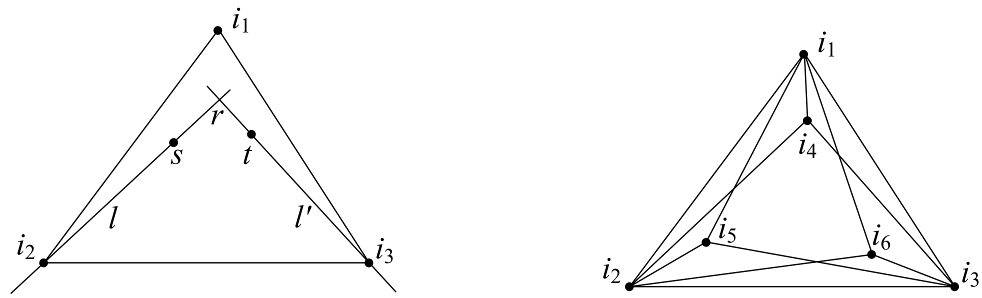
Thus assume that the straight line i_5i_6 intersects two non-adjacent sides, say the sides i_1i_2 and i_3i_4 (Figure 4 (b)). Then there are three quadrilaterals with vertex sets $\{i_1, i_2, i_3, i_4\}, \{i_1, i_5, i_6, i_4\}$ and $\{i_2, i_3, i_6, i_5\}$, respectively, and four quadrilaterals with vertex set consisting of one element of $\{i_5, i_6\}$ and three elements of $\{i_1, i_2, i_3, i_4\}$ (see the dotted quadrilaterals shown in Figure 3 (b)), for a total of seven quadrilaterals. This implies that the remaining point i_7 is a common vertex of exactly two remaining quadrilaterals, i.e., $m(i_7) = 2$, which contradicts (7). □



(a) (b)

Figure 4. A convex quadrilateral with two interior points.

By Lemmas 2.1 and 2.2, $\mathrm{Conv}(P)$ must be a triangle. Denote by i_1, i_2 and i_3 the three vertices of $\mathrm{Conv}(P)$ in counterclockwise order. Let l be the line through i_1 and i_2. Rotate l clockwise around i_2, and let s be the point of P that l first meets, see Figure 5 (a). Similarly, let l' be the line through i_1 and i_3, rotate l' counterclockwise around i_3, and let t be the point of P that l' first meets.



(a) The case where $s \neq t$. (b) Labeling of i_4, i_5 and i_6.

Figure 5. Labeling of points.

Suppose that $s \neq t$, and let r be the intersection point of lines i_2s and i_3t. If triangle $\{r, s, t\}$ contains a point, say q, of P in its interior, then P contains a convex pentagon $\{i_2, i_3, t, q, s\}$, which contradicts Lemma 2.1. Thus the two points of $P - \{i_1, i_2, i_3, s, t\}$ must be in the interior of convex quadrilateral $\{i_2, i_3, t, s\}$, which contradicts Lemma 2.2. Consequently, we must have $s = t$, and denote this point by i_4 (Figure 5 (b)). By the choice of i_4,

triangles $\{i_2, i_4, i_1\}$ and $\{i_3, i_1, i_4\}$ contain no point of P in their interiors.

Arguing similarly as above, we see that there exists two points i_5 and i_6 of P such that

triangles $\{i_3, i_5, i_2\}, \{i_1, i_2, i_5\}, \{i_1, i_6, i_3\}$ and $\{i_2, i_3, i_6\}$ contain no point of P in their interiors.

(note that i_4, i_5 and i_6 are all different since otherwise triangle $\{i_1, i_2, i_3\}$ can contain only one point of P in its interior, a contradiction). Let i_7 denote the remaining point of P. We further define $l_{j,k}, 1 \leq j < k \leq 6$, as the straight lines passing through i_j and i_k. Now let p_1, p_2 and

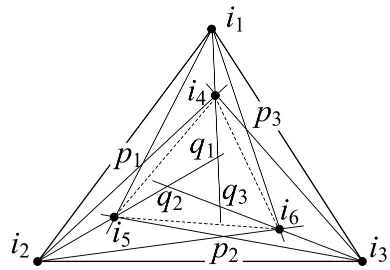


Figure 6. The intersection points p_i and q_i, $1 \leq i \leq 3$.

p_3 be the intersection points of $l_{1,5}$ and $l_{2,4}$, $l_{2,6}$ and $l_{3,5}$, and $l_{3,4}$ and $l_{1,6}$, respectively (Figure 6). Furthermore, let q_1, q_2 and q_3 be the intersection points of $l_{1,4}$ and $l_{2,5}$, $l_{2,5}$ and $l_{3,6}$, and $l_{3,6}$ and $l_{1,4}$, respectively (Figure 6; the case where i_4, q_1 and q_3 appear on $l_{1,4}$ in this order).

If i_7 is contained in the interior of triangle $\{q_1, i_4, i_5\}$, then $\{i_1, i_2, i_5, i_7, i_4\}$ is the vertex set of a convex pentagon, a contradiction. Combined with similar arguments, we see that i_7 is not contained in the interior of any of triangles $\{q_1, i_4, i_5\}$, $\{q_2, i_5, i_6\}$ or $\{q_3, i_6, i_4\}$. Thus i_7 is contained in the interior of one of triangles $\{q_1, q_2, q_3\}$, $\{p_1, i_5, i_4\}$, $\{p_2, i_6, i_5\}$ or $\{p_3, i_4, i_6\}$. By symmetry, we may assume that i_7 is contained in the interior of either triangle $\{q_1, q_2, q_3\}$ or $\{p_1, i_5, i_4\}$.

Case 1. i_7 is contained in the interior of triangle $\{q_1, q_2, q_3\}$.

By symmetry, we may assume that i_4, q_1 and q_3 appear on $l_{1,4}$ in this order as shown in Figures 6 and 7 (a). Then there are exactly nine convex quadrilaterals:

$$\{i_1, i_2, i_5, i_4\},\ \{i_2, i_3, i_6, i_5\},\ \{i_3, i_1, i_4, i_6\},\ \{i_1, i_2, i_7, i_4\},\ \{i_1, i_5, i_7, i_4\},$$
$$\{i_2, i_3, i_7, i_5\},\ \{i_2, i_6, i_7, i_5\},\ \{i_3, i_1, i_7, i_6\}\ \text{and}\ \{i_3, i_4, i_7, i_6\}.$$

Let $m_1(i_j)$, $1 \leq j \leq 7$, denote the number of convex quadrilaterals that have i_j as their vertices. Then

$$m_1(i_1)=5,\ m_1(i_2)=5,\ m_1(i_3)=5,\ m_1(i_4)=5,\ m_1(i_5)=5,\ m_1(i_6)=5\ \text{and}\ m_1(i_7)=6. \tag{8}$$

We have $m_1(i_j) \neq 4$ for any j with $1 \leq j \leq 7$, which contradicts (7).

Case 2. i_7 is contained in the interior of triangle $\{p_1, i_5, i_4\}$.

By symmetry, we may assume that i_7 lies on the same side as i_1 with respect to $l_{3,6}$ (Figure 7 (b)). Then we have following nine convex quadrilaterals:

$$\{i_1, i_2, i_5, i_4\},\ \{i_2, i_3, i_6, i_5\},\ \{i_3, i_1, i_4, i_6\},\ \{i_1, i_2, i_7, i_4\},\ \{i_1, i_2, i_5, i_7\},$$
$$\{i_1, i_3, i_6, i_7\},\ \{i_3, i_4, i_7, i_5\},\ \{i_3, i_4, i_7, i_6\}\ \text{and}\ \{i_4, i_7, i_5, i_6\},$$

and if we let $m_2(i_j)$, $1 \leq j \leq 7$, denote the number of convex quadrilaterals that have i_j as their vertices,

$$m_2(i_1)=5,\ m_2(i_2)=4,\ m_2(i_3)=5,\ m_2(i_4)=6,\ m_2(i_5)=5,\ m_2(i_6)=5\ \text{and}\ m_2(i_7)=6. \tag{9}$$

We have $m_1(i_j) = 4$ for only $j = 2$, which contradicts (7).

This is the end of the proof of (I).

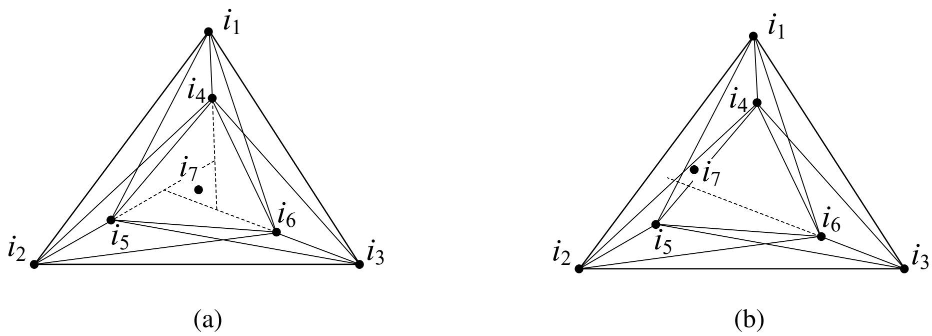


Figure 7. Two cases according to the position of i_7

2.2. *Proof of (II)*

The proof of (II) follows essentially the same line of argument as the proof of (I). First we determine the sets of x_i's that satisfy the condition (5) with $x_1 = 1$, but does *not* satisfy the following condition (10) in place of (6):

$$\text{at least five of the thirteen values shown just below (6) are 1.} \tag{10}$$

By running a Java program on a computer, we obtain the following solutions:

$$x_i = \begin{cases} 1 & (i = 1, 10, 16, 18, 24, 25, 26, 27, 31), \\ 0 & (\text{otherwise}); \end{cases} \tag{11}$$

$$x_i = \begin{cases} 1 & (i = 1, 8, 15, 19, 26, 28, 31), \\ 0 & (\text{otherwise}); \end{cases} \tag{12}$$

$$x_i = \begin{cases} 1 & (i = 1, 15, 16, 17, 18, 20, 24, 29, 32), \\ 0 & (\text{otherwise}); \end{cases} \tag{13}$$

and

$$x_i = \begin{cases} 1 & (i = 1, 10, 15, 17, 24, 28, 33), \\ 0 & (\text{otherwise}). \end{cases} \tag{14}$$

Each of solutions (12) and (14) corresponds to the case where P contains only seven convex quadrilaterals, which contradicts Theorem 1.3. Therefore, we consider solutions (11) and (13).

The convex quadrilaterals corresponding to $x_i = 1$ in (11) are

$$\left.\begin{array}{lll} Q_1 = \{1,2,3,4\}, & Q_{10} = \{1,2,6,7\}, & Q_{16} = \{1,3,6,7\}, \\ Q_{18} = \{1,4,5,7\}, & Q_{24} = \{2,3,5,6\}, & Q_{25} = \{2,3,5,7\}, \\ Q_{26} = \{2,3,6,7\}, & Q_{27} = \{2,4,5,6\} \text{ and} & Q_{31} = \{3,4,5,6\}. \end{array}\right\} \tag{15}$$

By the condition that the x_i's does not satisfy (10), Lemma 2.1 holds again in this case. Furthermore, since no element $i \in P$ belongs to exactly two Q_j's, Lemma 2.2 also holds in this case (recall the last sentence of the proof of Lemma 2.2). Thus, $\mathrm{Conv}(P)$ is a triangle, and with appropriate relabeling, (8) or (9) must be satisfied. However, in (15), each of 1 and 4 belongs to exactly four Q_j's, a contradiction.

Next consider convex quadrilaterals corresponding to $x_i = 1$ in (13):

$$\left.\begin{array}{lll} Q_1 = \{1,2,3,4\}, & Q_{15} = \{1,3,5,7\}, & Q_{16} = \{1,3,6,7\}, \\ Q_{17} = \{1,4,5,6\}, & Q_{18} = \{1,4,5,7\}, & Q_{20} = \{1,5,6,7\}, \\ Q_{24} = \{2,3,5,6\}, & Q_{29} = \{2,4,6,7\} \text{ and } & Q_{32} = \{3,4,5,7\}. \end{array}\right\} \tag{16}$$

We can argue in a similar way in this case as well. Eventually, we obtain a contradiction since 2 belongs to exactly three Q_j's in (16).

This is the end of the proof of (II). □

3. Proof of Theorem 1.2

Let n be a positive integer. For two integers l and m with $1 \le l \le m \le n$ and for $P \in \mathcal{P}_n$, let

$$P[l,m] = \{p \in P : l \le w(p) \le m\}$$

(note that $W_4(P[l,m])$ stands for the set of weights of convex quadrilaterals with vertices in $P[l,m]$). By subtracting $l-1$ from each weight of the points of $P[l,l+6]$, we see from Theorem 1.2 that

$$|W_4(P[l,l+6]) - \{4l+6\}| = |W_4(P[1,7]) - \{10\}| \ge 4. \tag{17}$$

Using (17), we show Theorem 1.1 by induction on n. From (2), we can verify that

$$f_4(n) \ge \frac{4}{3}n - 7 \tag{18}$$

holds for $n \le 6$. Assume now that $n \ge 7$ and let $P \in \mathcal{P}_n$. Since

$$\begin{aligned} W_4(P[1,n-3]) \cap W_4(P[n-6,n]) &\subseteq \{(n-6)+(n-5)+(n-4)+(n-3)\} \\ &= \{4n-18\}, \end{aligned}$$

it follows from the induction hypothesis and (17) with $l = n-6$ that

$$\begin{aligned} |W_4(P)| &\ge |W_4(P[1,n-3])| + |(W_4(P[n-6,n]) - \{4n-18\}| \\ &\ge \frac{4}{3}(n-3) - 7 + 4 \\ &= \frac{4}{3}n - 7, \end{aligned}$$

as desired. □

4. Conclusion

In this study, with the aid of a computer, we obtained $4n/3-7$ as a new lower bound for $f_4(n)$. However, there is still a large gap between this lower bound and the upper bound shown in (3), i.e., $f_4(n) \leq 2n-9$ for $n \geq 7$. It remains a future work to narrow the gap between the upper and lower bounds.

Related to this study, a similar problem can be considered by restricting the convex quadrilaterals to *empty* convex quadrilaterals, i.e., convex quadrilaterals that do not contain any points of P in their interior. We can also consider a similar problem even for empty quadrilaterals by excluding the condition of convexity. However, we have not obtained satisfactory results even on the problem of the number of weights of empty triangles: while an upper bound $2n-5$ for $n \geq 3$ is obtained for the number of weights of the empty triangles [10], a linear lower bound is not obtained so far.

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