Indonesian Journal of Combinatorics **8** (2) (2024), 109–126

INDONESIAN JOURNAL OF COMBINATORICS

Strong 3-Rainbow Indexes of Closed Helm Graphs

Calista Suci Nugrahani[a], Christian Constantine[a], A.N.M. Salman[a]

[a]*Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung, Indonesia*

calistanugrahani@gmail.com, christianconstantine2512@gmail.com, msalman@itb.ac.id

Abstract

Let G be a nontrivial, edge-colored, and connected graph of order $m \geq 3$, where adjacent edges may have the same color. A tree T in the graph G is called a rainbow tree if all the edges in T have different colors. For $S \subseteq V(G)$, the Steiner distance $sd(S)$ of S is the minimum size of a tree in G containing S. Let k be an integer with $2 \leq k \leq m$. An edge-coloring in G is a strong k-rainbow coloring if, for every set S of k vertices in G, there exists a rainbow tree of size $sd(S)$ in G containing S. In this paper, we study the strong 3-rainbow index srx_3 of closed helm graphs. We determine the srx_3 of closed helm graphs.

Keywords: closed helm graph, rainbow coloring, rainbow tree, strong k-rainbow index
Mathematics Subject Classification : 05C05, 05C15, 05C40

1. Introduction

All graphs considered in this paper are simple, finite, and connected. We follow the terminology and notation of Diestel [7]. For simplifying, we define $[a, b]$ as a set of all integers x with $a \leq x \leq b$.

Let G be an edge-colored graph of order $m \geq 3$, where adjacent edges may be colored the same. For $S \subseteq V(G)$, a rainbow S-tree is a rainbow tree that contains the vertices of S. Let k be an integer with $k \in [2, m]$. An edge-coloring of G is called a k-rainbow coloring if, for every set S of k vertices of G, there exists a rainbow S-tree in G [1].

The k-rainbow index $rx_k(G)$ of G, introduced by Chartrand *et al*. [6], is the minimum number of colors needed in the k-rainbow coloring of G. Thus, if $k = 2$, then $rx_2(G)$ is the rainbow

Received: 18 December 2023, Revised: 18 April 2024, Accepted: 12 December 2024.

connection number $rc(G)$ of G, which was introduced by Chartrand *et al.* in 2008 [5]. For every nontrivial connected graph G of order n, it is easy to see that $rx_2(G) \leq rx_3(G) \leq \ldots \leq rx_n(G)$.

The Steiner distance of a set S of vertices in G, denoted by, $sd(S)$, is the minimum size of a tree in G containing S. Such a tree is called a Steiner S-tree. The maximum Steiner distance of S among all sets S of k vertices of G is called the k-Steiner diameter $sdiam_k(G)$ of G. In [6], for every connected graph G of order $n \geq 3$ and each integer k with $k \in [3, m]$,

$$k-1 \leq sdiam_k(G) \leq rx_k(G) \leq n-1.$$

Awanis *et al.* [1] introduced a generalization of the k-rainbow coloring of G called the strong k-rainbow index of G, denoted by $srx_k(G)$. A rainbow Steiner S-tree is a rainbow S-tree of size $sd(S)$. An edge-coloring of G is called a strong k-rainbow coloring of G, if, for every set S of k vertices of G, there exists a rainbow Steiner S-tree in G. The minimum number of colors needed in a strong k-rainbow coloring of G is the strong k-rainbow index $srx_k(G)$ of G. Thus, we have $rx_k(G) \leq srx_k(G)$ for every connected graph G.

Furthermore, if G is a nontrivial connected graph of size $|E(G)|$ whose k-Steiner diameter is $sdiam_k(G)$, then it is easy to check that

$$sdiam_k(G) \leq rx_k(G) \leq srx_k(G) \leq |E(G)|. \tag{1}$$

To find the strong 3-rainbow coloring of the closed helm graph, the following results are needed.

Lemma 1.1. *([1]) For $n \geq 3$, let c be a strong 3-rainbow coloring of fan graph, F_n. Then at most two spokes of F_n may be colored the same. Moreover, if $c(vv_i) = c(vv_j)$ for distinct i, $j \in [1, n]$, then v_i and v_j are adjacent.*

Observation 1.1. ([3]) Awanis and Salman (2022) define a strong 3-rainbow coloring of wheel graph as below:

$$c\left(vv^j\right) = \left\lceil \frac{j}{2} \right\rceil \ for\ j \in [1, m];$$

for odd m,

$$c\left(v^j v^{j+1}\right) = \begin{cases} \left\lceil \frac{j}{2} \right\rceil + 1 & \text{for odd } j \in [1, m-2], \\ 1 & \text{for } j = m, \\ \frac{j}{2} & \text{for even } j \in [1, m-1]; \end{cases}$$

for even m,

$$c\left(v^j v^{j+1}\right) = \begin{cases} \left\lceil \frac{j}{2} \right\rceil + 1 & \text{for odd } j \in [1, m-3], \\ 1 & \text{for } j = m-1, \\ \frac{j}{2} & \text{for even } j \in [1, m]. \end{cases}$$

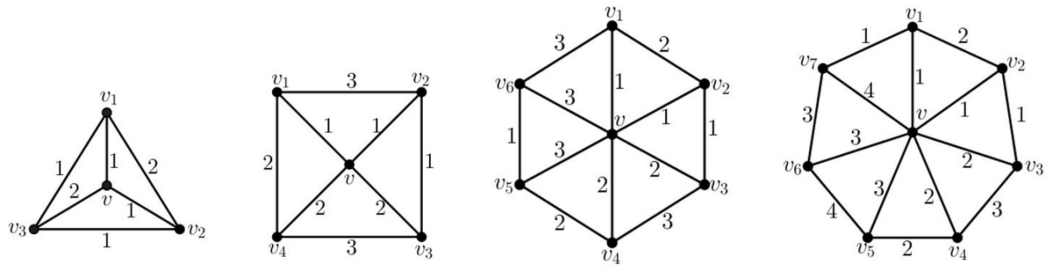


Figure 1. Strong 3-rainbow colorings of W_3, W_4, W_6, and W_7

2. Main Result

Before working on the strong 3-rainbow coloring of closed helm graphs, we must first define the vertices and edges of closed helm graphs. Let CH_m be a closed helm graph with $m \geq 3$. Define the vertices and edges of a closed helm graph, successively, as

$$V\left(CH_m\right) = \{v\} \cup \{v_i^p|\ i \in [1,2]; p \in [1,m]\}$$

and

$$E\left(CH_m\right) = \{vv_1^p|\ p \in [1,m]\} \cup \{v_1^p v_2^p|\ p \in [1,m]\} \cup \left\{v_i^p v_i^{p+1}\middle|\ i \in [1,2]; p \in [1,m]\right\}$$

where $v_i^{m+1} = v_i^1$ with $i \in [1,2]$. A closed helm graph CH_m is shown in Figure 2.

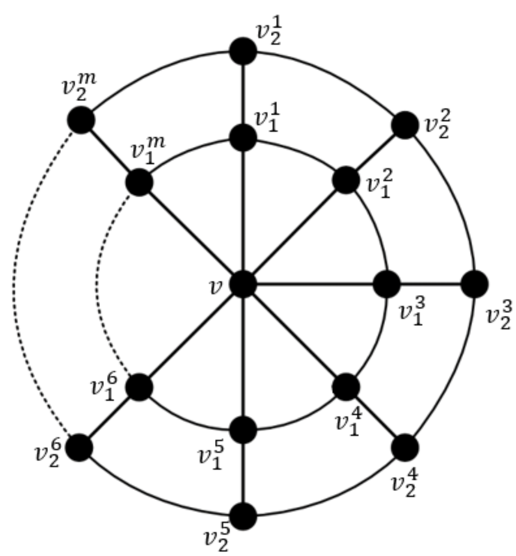


Figure 2. A closed helm graph CH_m

To make our work easier, we group the edges of a closed helm graph into 4 groups. We define the edges $\{vv_1^i| i \in [1,m]\}$ as the *inner spokes* of the graph and the edges $\{v_1^i v_2^i| i \in [1,m]\}$ as the *outer spokes* of the graph. Then, we define the edges $\{v_1^i v_1^{i+1} \mid i \in [1,\ m-1]\} \cup \{v_1^m v_1^i\}$ as the *inner rim* and the edges $\{v_2^i v_2^{i+1} \mid i \in [1, m-1]\} \cup \{v_2^m v_2^i\}$ as the *outer rim*. Lastly, we define *rim distance*, written as $rd\,(v_i^p, v_i^q)$, means the minimum length of the path that connects v_i^p and v_i^q that does not contain v.

The *inner spokes* and the *inner rim* of a closed helm graph can be seen as a wheel graph. As such, we can simply apply the coloring in Observation 1.1 to them. The following results are needed to define the coloring of the *outer spokes* and the *outer rim* of a closed helm graph.

Lemma 2.1. *For $m \geq 6$, let c be a strong 3-rainbow coloring of CH_m. Then at most three outer spokes of CH_m may be colored the same. Moreover, if $c(v_1^p v_2^p) = c(v_1^q v_2^q)$ for distinct p, $q \in [1, m]$, then $rd(v_2^p, v_2^q) \leq 2$.*

Proof. Let c be a strong 3-rainbow coloring of graph CH_m. Suppose that we have four outer spokes with the same color, that is, $c(v_1^{p_1} v_2^{p_1}) = c(v_1^{p_2} v_2^{p_2}) = c(v_1^{p_3} v_2^{p_3}) = c(v_1^{p_4} v_2^{p_4})$. As a result, there exists a pair of vertices in $\{v_2^{p_1}, v_2^{p_2}, v_2^{p_3}, v_2^{p_4}\}$ such that their rim distance is more than 2, we name those two vertices as $v_2^{p_i}$ and $v_2^{p_j}$. By observation, it is clear that the Steiner tree that contains v,$v_2^{p_i}$, and $v_2^{p_j}$ is only $(v_2^{p_i}, v_1^{p_i}, v, v_1^{p_j}, v_2^{p_j})$. Thus, the Steiner tree does not have rainbow coloring. This contradicts our initial assumption that c is a strong 3-rainbow coloring. ☐

Lemma 2.2. *Let c be a strong 3-rainbow coloring of CH_m. For $p, q, r \in [1, m]$, where v_2^r is a vertex adjacent to v_2^q and $rd\,(v_2^p, v_2^r) \leq rd\,(v_2^p, v_2^q)$, if $rd\,(v_2^p, v_2^q) \notin \{3, 4, 5\}$ then $c(v_2^r v_2^q) \neq c(v_1^p v_2^p)$.*

Proof. Let CH_m be a closed helm graph. To prove this lemma, we will look at the following cases:

Case 1, $rd(v_2^p, v_2^q) \in \{1, 2\}$

Suppose $c(v_2^r v_2^q) = c(v_1^p v_2^p)$. For $rd(v_2^p, v_2^q) = 1$, we have $v_2^r = v_2^p$ and the Steiner tree that contains v_1^p, v_2^p, and v_2^q is only (v_1^p, v_2^p, v_2^q). It is obvious that this Steiner tree is not a rainbow tree. For $rd(v_2^p, v_2^q) = 2$, the Steiner tree that contains v_1^p, v_2^p, and v_2^q is only $(v_1^p, v_2^p, v_2^r, v_2^q)$. Thus, this Steiner tree is also not a rainbow tree.

Case 2, $rd(v_2^p, v_2^q) \geq 6$

Suppose $c(v_2^r v_2^q) = c(v_1^p v_2^p)$. The Steiner tree that contains v_2^p,v_2^r,v_2^q is $(v_2^p, v_1^p, v, v_1^q, v_2^q, v_2^r)$ or $(v_2^p, v_1^p, v, v_1^r, v_2^r, v_2^q)$, both trees contain the edges $v_1^p v_2^p$ and $v_2^r v_2^q$. Thus, this Steiner tree is not a rainbow tree.

In conclusion, both cases show that if $rd\,(v_2^p, v_2^q) \notin \{3, 4, 5\}$ and $c(v_2^r v_2^q) = c(v_1^p v_2^p)$ then c is not a strong 3-rainbow coloring of CH_m. ☐

Corollary 2.1. *If $rd(v_2^p, v_2^q) \geq 6$ for p ,$q \in [1, m]$ and v_2^r is a vertex adjacent to v_2^q and $rd\,(v_2^p, v_2^r) \leq rd\,(v_2^p, v_2^q)$, then $c(v_2^r v_2^q) \notin \{c(vv_1^p), c(v_1^p v_2^p)\}$.*

Proof. From Lemma 2.2, we know that $c(v_2^r v_2^q) \neq c(v_1^p v_2^p)$. Furthermore, according to the proof of Lemma 2.2 in Case 2, the Steiner trees that contain v_2^p, v_2^r, v_2^q both have the edges vv_1^p and $v_2^r v_2^q$. Having $c(v_2^r v_2^q) = c(vv_1^p)$, the Steiner tree is not a rainbow tree. ☐

Lemma 2.3. *For $m \geq 10$, let c be a strong 3-rainbow coloring of CH_m. If $2 \leq rd(v_2^p, v_2^q) \leq 5$, then $c(v_2^p v_2^s) \neq c(v_2^r v_2^q)$ for distinct $p, q, r, s \in [1, m]$ where v_2^r is a vertex adjacent to v_2^q with $rd\,(v_2^p, v_2^r) < rd\,(v_2^p, v_2^q)$ and v_2^s is a vertex adjacent to v_2^p with $rd\,(v_2^q, v_2^s) < rd\,(v_2^q, v_2^p)$.*

Proof. Suppose $c(v_2^p v_2^s) = c(v_2^r v_2^q)$. For some k such that the Steiner tree containing v_2^p, v_2^k, v_2^q is only $(v_2^p, v_2^s, v_2^k, v_2^r, v_2^q)$, the tree is not a rainbow tree. □

Lemma 2.4. *Let c be a strong 3-rainbow coloring of CH_m. If $c(vv_1^p) = c(vv_1^{p+1})$, then $c(v_2^p v_2^{p+1}) \neq c(vv_1^p)$ for $p \in [1, m]$.*

Proof. Let $c(v_2^p v_2^{p+1}) = c(vv_1^p)$ and $c(vv_1^p) = c(vv_1^{p+1})$. The Steiner tree that contains v, v_2^p, and v_2^{p+1} is either $(v, v_1^p, v_2^p, v_2^{p+1})$ or $(v, v_1^{p+1}, v_2^{p+1}, v_2^p)$. Suppose $c(v_2^p v_2^{p+1}) = c(vv_1^p)$, so obviously $(v, v_1^p, v_2^p, v_2^{p+1})$ is not a rainbow tree. Since $c(vv_1^p) = c(vv_1^{p+1})$, we find that $(v, v_1^{p+1}, v_2^{p+1}, v_2^p)$ is also not a rainbow tree. Thus, we have no Steiner rainbow tree. □

Lemma 2.5. *Let c be a strong 3-rainbow coloring of CH_m and let $p, q \in [1, m]$ with $p \neq q$, where v_2^r is a vertex adjacent to v_2^q and $rd\,(v_2^p, v_2^r) < rd\,(v_2^p, v_2^q)$. If $rd(v_2^p, v_2^q) > 3$, then $c(v_2^q v_2^r) \neq c(vv_1^p)$.*

Proof. Suppose $c(v_2^q v_2^r) = c(vv_1^p)$ and $rd(v_2^p,\ v_2^q) > 3$. The Steiner tree that contains v_2^q, v_2^r, and v_1^p is either $(v_2^r, v_2^q, v_1^q, v, v_1^p)$ or $(v_2^q, v_2^r, v_1^r, v, v_1^p)$. Both contain the edges $v_2^q v_2^r$ and vv_1^p. Thus, the Steiner tree is not a rainbow tree. □

By the inequality provided in (1), it is important to determine the 3-Steiner diameters of the closed helm graphs before we determine the strong 3-rainbow indexes of the graphs. Theorem 2.1 below defines the $sdiam_3$ of a closed helm graph CH_m.

Theorem 2.1. *For $m \geq 3$, then*

$$sdiam_3(CH_m) = \begin{cases} 3 & \text{for } m = 3; \\ 4 & \text{for } m \in [4, 5]; \\ 5 & \text{for } m \in [6, 8]; \\ 6 & \text{for } m \geq 9. \end{cases}$$

Proof. Note that for every m, there are 7 cases of triple vertices denoted by S_k, for $k \in [1, 7]$ such that $\bigcup_{k=1}^{7} S_k = S$. Thus, S is the set for all possible combinations of triple vertices in CH_m. We have:

1. $S_1 = \{\{v, v_1^p, v_1^q\} | \text{for } p, q \in [1, m], p \neq q\}$,
2. $S_2 = \{\{v, v_2^p, v_2^q\} | \text{for } p, q \in [1, m], p \neq q\}$,
3. $S_3 = \{\{v, v_1^p, v_2^p\} | \text{for } p \in [1, m]\}$,
4. $S_4 = \{\{v, v_1^p, v_2^q\} | \text{for } p, q \in [1, m], p \neq q\}$,
5. $S_5 = \{\{v_i^p, v_i^q, v_j^r\} | \text{for } p, q, r \in [1, m], p \neq q \neq r \neq p \text{ and } i, j \in [1, 2], i \neq j\}$,
6. $S_6 = \{\{v_1^p, v_2^p, v_i^q\} | \text{for } p, q \in [1, m], p \neq q \text{ and } i \in [1, 2]\}$,
7. $S_7 = \{\{v_i^p, v_i^q, v_i^r\} | \text{for } p, q, r \in [1, m], p \neq q \neq r \neq p \text{ and } i \in [1, 2]\}$,

By observation, we know that for every m:

1. $sd(s \in S_1) = 2$,
2. $sd(s \in S_3) = 2$,
3. $sd(s \in S_4) = 3$.

Now, we observe for each case of m.

Table 1. Steiner distance for every triple vertices s in each case and each m

m	max $\{sd(s \in S_2)\}$	max $\{sd(s \in S_5)\}$	max $\{sd(s \in S_6)\}$	max $\{sd(s \in S_7)\}$	$sdiam_3(CH_m)$ $=max\{sd(s \in S)\}$
3	3	3	2	2	3
4	4	3	3	2	4
5	4	4	3	3	4
6	4	5	4	4	5
7	4	5	4	4	5
8	4	5	4	5	5
≥ 9	4	5	4	6	6

For each m, we find the Steiner distance of each cases of the triple vertices s. The table above shows the maximum Steiner distance for cases $s \in \{S_2, S_5, S_6, S_7\}$ and the maximum Steiner distance for all cases. In other words, the last column shows the Steiner diameters of closed helm graphs. Note that it gives a similar result to the theorem. □

Finally, Theorem 2.2 below gives the srx_3 of closed helm graph CH_m.

Theorem 2.2. *For $m \geq 3$, then*

$$srx_3(CH_m) = f(x) = \begin{cases} 3 & \textit{for } m = 3; \\ 4 & \textit{for } m \in \{4,5\}; \\ m & \textit{for } m \in \{6,7,9\}; \\ \lceil \frac{m}{2} \rceil + \lceil \frac{m}{3} \rceil & \textit{for } m = 8 \textit{ or } m \geq 10 \textit{ with } m(mod\ 6) \neq 1; \\ \lceil \frac{m}{2} \rceil + \lceil \frac{m}{3} \rceil - 1 & \textit{for } m \geq 13 \textit{ with } m(mod\ 6) = 1. \end{cases}$$

Proof. **Case 1**, $m = 3$

Since we have $sdiam_3(CH_3) = 3$, we have $srx_3(CH_3) \geq 3$ by (1). Next, we show that $srx_3(CH_3) \leq 3$ by defining a strong 3-rainbow coloring $c : E(CH_3) \to [1,3]$ as follows.

$$c(vv_1^1) = c(vv_1^2) = c(v_i^2 v_i^3) = c(v_i^3 v_i^1) = 1 \text{ for } i \in [1,2],$$

$$c(vv_1^3) = c(v_i^1 v_i^2) = 2 \text{ for } i \in [1,2],$$

$$c(v_1^j v_2^j) = 3 \text{ for } j \in [1,3].$$

Case 2, $m \in \{4,5\}$

Since we have $sdiam_3(CH_m) = 4$, we have $srx_3(CH_m) \geq 4$ by (1). Next, we show that $srx_3(CH_m) \leq 4$ by defining a strong 3-rainbow coloring $c : E(CH_m) \to [1,4]$ as follows.

$$c(vv_1^j) = \lceil \frac{j}{2} \rceil \text{ for } j \in [1,m],$$

$$c(v_1^j v_2^j) = 4 \text{ for } j \in [1,m];$$

for $m = 4$,

$$c(v_i^1 v_i^2) = 3 \text{ for } i \in [1, 2],$$
$$c(v_i^2 v_i^3) = c(v_i^4 v_i^1) = 2 \text{ for } i \in [1, 2],$$
$$c(v_i^3 v_i^4) = 1 \text{ for } i \in [1, 2];$$

for $m = 5$,

$$c(v_i^2 v_i^3) = 3 \text{ for } i \in [1, 2],$$
$$c(v_i^1 v_i^2) = c(v_i^4 v_i^5) = 2 \text{ for } i \in [1, 2],$$
$$c(v_i^3 v_i^4) = c(v_i^5 v_i^1) = 1 \text{ for } i \in [1, 2].$$

The result of the coloring is shown in Figure 3.

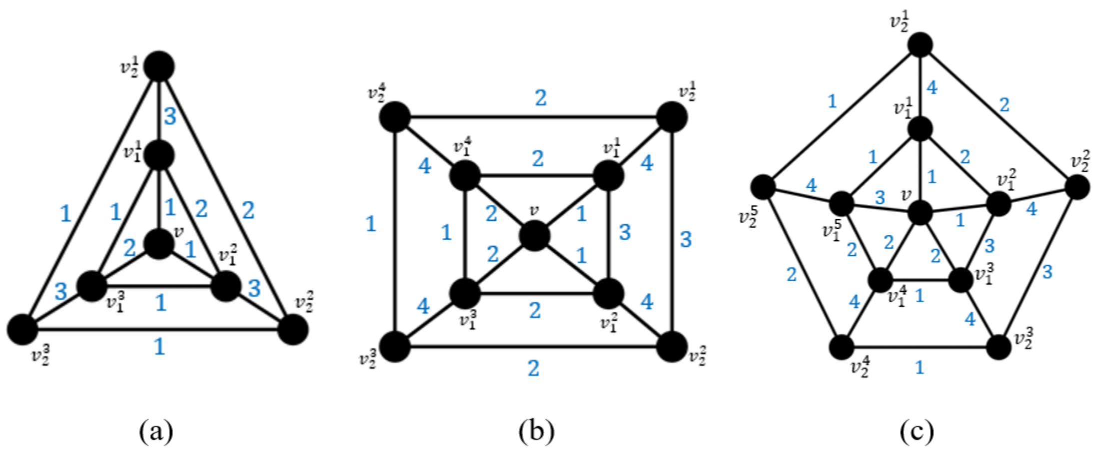
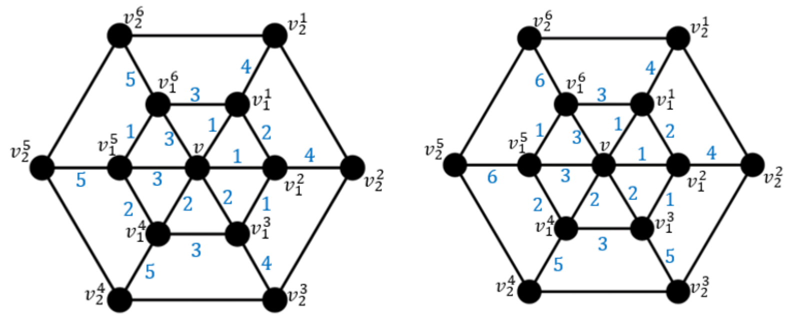


Figure 3. Strong 3-rainbow colorings of closed helm graph (a) CH_3, (b) CH_4, and (c) CH_5

Case 3, $m \in \{6, 7, 9\}$
Subcase 3.1, $m = 6$

Note that $sdiam_3(CH_6) = 5$, thus we have $srx_3(CH_6) \geq 5$ by (1). By Lemma 1.1, we use $\lceil \frac{m}{2} \rceil$ colors and by Lemma 2.1, we use additional $\lceil \frac{m}{3} \rceil$ colors. As a result, for CH_6, we use at least $\lceil \frac{6}{2} \rceil + \lceil \frac{6}{3} \rceil = 5$ colors.

Suppose that $srx_3(CH_6) = 5$, then by Lemma 1.1 and Observation 1.1, we color the inner spokes and inner rims with 3 colors, and by Lemma 2.1, we color the outer spokes with an additional 2 colors. The colors of the spokes and inner rims are shown in Figure 4(a).



(a) (b)

Figure 4. Spokes and Inner rims' color for (a) CH_6 with 5 colors and (b) CH_6 with 6 colors.

By Lemma 2.2, Lemma 2.4, and Lemma 2.5, the colors which we can color the outer rim with are shown in Table 2.

Table 2. The colors of outer rims for CH_6 if $srx_3(CH_6) = 5$

Outer Rim	Lemma 2.2	Lemma 2.4	Lemma 2.5	Available
$c(v_2^1 v_2^2)$	$\neq 4, 5$	$\neq 1$	-	{2,3}
$c(v_2^2 v_2^3)$	$\neq 4, 5$	-	-	{1,2,3}
$c(v_2^3 v_2^4)$	$\neq 4, 5$	$\neq 2$	-	{1,3}
$c(v_2^4 v_2^5)$	$\neq 4, 5$	-	-	{1,2,3}
$c(v_2^5 v_2^6)$	$\neq 4, 5$	$\neq 3$	-	{1,2}
$c(v_2^6 v_2^1)$	$\neq 4, 5$	-	-	{1,2,3}

We have six outer rims to color, but we only have three colors available. Suppose that we have triple vertices v_2^1, v_2^3, v_2^5, then the Steiner tree consists only of the outer rim edges and has a length of 4, which means we need at least one additional color for the outer rim. Thus, it's clear that $srx_3(CH_6) > 5$.

Let us propose a new coloring. We color the spokes and inner rim as shown in Figure 4(b). By Lemma 2.2, Lemma 2.4, and Lemma 2.5, below are the colors which we can color the outer rim with, as shown in Table 3.

Table 3. The colors of outer rims for CH_6 if $srx_3(CH_6) = 6$

Outer Rim	Lemma 2.2	Lemma 2.4	Lemma 2.5	Available
$c(v_2^1 v_2^2)$	$\neq 4, 5, 6$	$\neq 1$	-	{2,3}
$c(v_2^2 v_2^3)$	$\neq 4, 5$	-	-	{1,2,3,6}
$c(v_2^3 v_2^4)$	$\neq 4, 5, 6$	$\neq 2$	-	{1,3}
$c(v_2^4 v_2^5)$	$\neq 5, 6$	-	-	{1,2,3,4}
$c(v_2^5 v_2^6)$	$\neq 4, 5, 6$	$\neq 3$	-	{1,2}
$c(v_2^6 v_2^1)$	$\neq 4, 6$	-	-	{1,2,3,5}

We now have 6 edges to color and 6 colors to use. Then, we are able to define a strong 3-rainbow coloring of CH_6 as shown in Table 4.

Table 4. The colors of edges for CH_6

Edges	Color
$vv_1^1, vv_1^2, v_1^2 v_1^3, v_1^5 v_1^6, v_2^5 v_2^6$	1
$vv_1^3, vv_1^4, v_1^1 v_1^2, v_1^4 v_1^5, v_2^1 v_2^2$	2
$vv_1^5, vv_1^6, v_1^3 v_1^4, v_1^6 v_1^1, v_2^3 v_2^4$	3
$v_1^1 v_2^1, v_1^2 v_2^2, v_2^4 v_2^5$	4
$v_1^3 v_2^3, v_1^4 v_2^4, v_2^6 v_2^1$	5
$v_1^5 v_2^5, v_1^6 v_2^6, v_2^2 v_2^3$	6

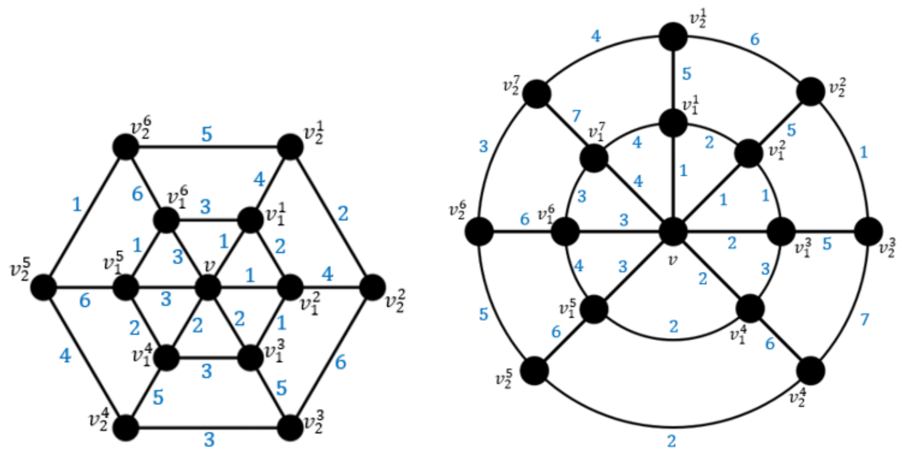
A strong 3-rainbow coloring of CH_6 is shown in Figure 5(a). We conclude that $srx_3(CH_6) = 6 = m$.

Subcase 3.2, $m = 7$

Note that $sdiam_3(CH_7) = 5$, thus we have $srx_3(CH_7) \geq 5$ by (1). By Lemma 1.1 and Lemma 2.1, we use at least $\lceil \frac{7}{2} \rceil + \lceil \frac{7}{3} \rceil = 7$ colors to color the spokes and inner rims of CH_7. The only edges that are yet to be colored are the outer rim edges. It is clear that we would be able to color the last 7 edges with 7 colors. We define a strong 3-rainbow coloring of CH_7 as shown in Table 5.

Table 5. The colors of edges for CH_7

Edges	Color
$vv_1^1, vv_1^2, v_1^2v_1^3, v_1^5v_1^6, v_2^2v_2^3$	1
$vv_1^3, vv_1^4, v_1^1v_1^2, v_1^4v_1^5, v_2^4v_2^5$	2
$vv_1^5, vv_1^6, v_1^3v_1^4, v_1^6v_1^7, v_2^6v_2^7$	3
$vv_1^7, v_1^7v_1^1, v_2^7v_2^1$	4
$v_1^1v_2^1, v_1^2v_2^2, v_1^3v_2^3, v_2^5v_2^6$	5
$v_1^4v_2^4, v_1^5v_2^5, v_1^6v_2^6, v_2^1v_2^2$	6
$v_1^7v_2^7, v_2^3v_2^4$	7



(a) (b)

Figure 5. Strong 3-rainbow colorings of closed helm graph (a) CH_6 and (b) CH_7

A strong 3-rainbow coloring of CH_7 is shown in Figure 5(b). We conclude that $srx_3(CH_7) = 7 = m$.

Subcase 3.3, $m = 9$

Note that $sdiam_3(CH_9) = 6$, so we have $srx_3(CH_9) \geq 6$ by (1). By Lemma 1.1 and Lemma 2.1, we use at least $\lceil \frac{9}{2} \rceil + \lceil \frac{9}{3} \rceil = 8$ colors for CH_9.

Suppose that $srx_3(CH_9) = 8$, then by Lemma 1.1 and Observation 1.1, we color the inner spokes and inner rims, and by Lemma 2.1, we color the outer spokes. The colors of the spokes and inner rims is shown in Figure 6.

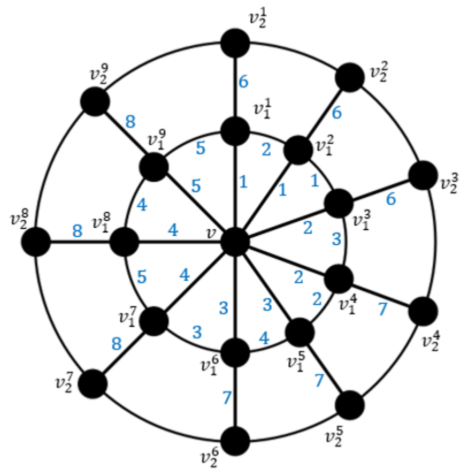


Figure 6. Spokes and Inner rims' color for CH_9 with 8 colors

By Lemma 2.2, Lemma 2.4, and Lemma 2.5, we can color the outer rim as shown in Table 6.

Table 6. The colors of outer rims for CH_9 if $srx_3(CH_9) = 8$

Outer Rim	Lemma 2.2	Lemma 2.4	Lemma 2.5	Available
$c(v_2^1 v_2^2)$	$\neq 6, 8$	$\neq 1$	$\neq 3, 4$	{2,5,7}
$c(v_2^2 v_2^3)$	$\neq 6, 7$	-	$\neq 3, 4$	{1,2,5,8}
$c(v_2^3 v_2^4)$	$\neq 6, 7$	$\neq 2$	$\neq 4, 5$	{1,3,8}
$c(v_2^4 v_2^5)$	$\neq 6, 7$	-	$\neq 1, 4, 5$	{2,3,8}
$c(v_2^5 v_2^6)$	$\neq 7, 8$	$\neq 3$	$\neq 1, 5$	{2,4,6}
$c(v_2^6 v_2^7)$	$\neq 7, 8$	-	$\neq 1, 2$	{3,4,5,6}
$c(v_2^7 v_2^8)$	$\neq 7, 8$	$\neq 4$	$\neq 1, 2$	{3,5,6}
$c(v_2^8 v_2^9)$	$\neq 6, 8$	-	$\neq 2, 3$	{1,4,5,7}
$c(v_2^9 v_2^1)$	$\neq 6, 8$	-	$\neq 2, 3$	{1,4,5,7}

We have 9 edges to color and only 8 colors to use, that means there is at least two edges with the same color. Suppose that those two edges are colored with one of the inner spokes' colors (1,2,3,4,5), we take the outer rim edges with the largest rim distance. Let $c(v_2^2 v_2^3) = c(v_2^6 v_2^7) = 5$. Suppose that we take the triple vertices as $\{v_2^2, v_2^4, v_2^7\}$, then the Steiner tree is $(v_2^2, v_2^3, v_2^4, v_2^5, v_2^6, v_2^7)$. Note that this tree is not a rainbow tree. Notice that we will also run into the same problem if we use the other colors of the inner spoke.

Our next choice is to use the colors from the outer spokes (6,7,8), but all the possible pairs have a rim distance of 3. Thus, it is clear that it will not result in a strong 3-rainbow coloring. Then $srx_3(CH_9)$ must be ≥ 9.

We define a strong 3-rainbow coloring of $srx_3(CH_9)$ as shown in Table 7.

Table 7. The colors of edges for CH_9

Edges	Color
$vv_1^1, vv_1^2, v_1^2v_1^3, v_2^9v_2^1$	1
$vv_1^3, vv_1^4, v_1^1v_1^2, v_1^4v_1^5, v_2^1v_2^2$	2
$vv_1^5, vv_1^6, v_1^3v_1^4, v_1^6v_1^7, v_2^3v_2^4$	3
$vv_1^7, vv_1^8, v_1^5v_1^6, v_1^8v_1^9, v_2^6v_2^7$	4
$vv_1^9, v_1^7v_1^8, v_1^9v_1^1, v_2^7v_2^8$	5
$v_1^1v_2^1, v_1^2v_2^2, v_1^3v_2^3, v_2^5v_2^6$	6
$v_1^4v_2^4, v_1^5v_2^5, v_1^6v_2^6, v_2^8v_2^9$	7
$v_1^7v_2^7, v_1^8v_2^8, v_1^9v_2^9, v_2^2v_2^3$	8
$v_2^4v_2^5$	9

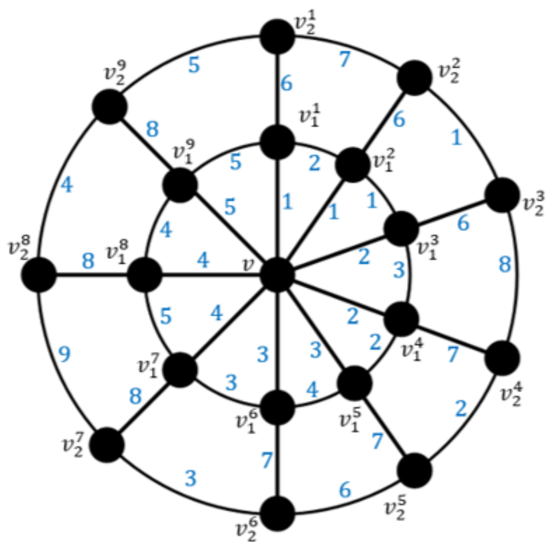


Figure 7. A strong 3-rainbow coloring of a closed helm graph CH_9

A strong 3-rainbow coloring of CH_9 is shown in Figure 7. We conclude that $srx_3(CH_9) = 9 = m$.

Case 4, $m = 8$ or $m \geq 10$ with $m \ (mod\ 6) \neq 1$

Subcase 4.1, $m = 8$

Note that $sdiam_3(CH_8) = 5$, so we have $srx_3(CH_8) \geq 5$ by (1). By Lemma 1.1 and Lemma 2.1, we use at least $\lceil \frac{8}{2} \rceil + \lceil \frac{8}{3} \rceil = 7$ colors for CH_9. Next, we show that $srx_3(CH_8) \leq 7$ by defining a strong 3-rainbow coloring for CH_8, as shown in Table 8.

Table 8. The colors of edges for CH_8

Edges	Color
$vv_1^1, vv_1^2, v_1^2v_1^3, v_1^7v_1^8, v_2^3v_2^4, v_2^7v_2^8$	1
$vv_1^3, vv_1^4, v_1^1v_1^2, v_1^4v_1^5, v_2^4v_2^5$	2
$vv_1^5, vv_1^6, v_1^3v_1^4, v_1^6v_1^7, v_2^6v_2^7$	3
$vv_1^7, vv_1^8, v_1^5v_1^6, v_1^8v_1^1, v_2^1v_2^2$	4
$v_1^1v_2^1, v_1^2v_2^2, v_1^3v_2^3, v_2^5v_2^6$	5
$v_1^4v_2^4, v_1^5v_2^5, v_1^6v_2^6, v_2^8v_2^1$	6
$v_1^7v_2^7, v_1^8v_2^8, v_2^2v_2^3$	7

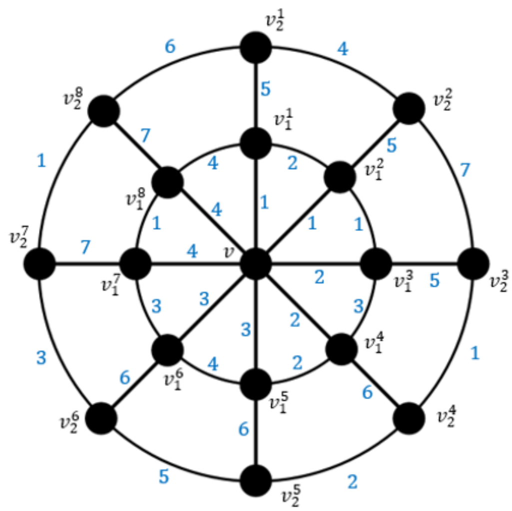


Figure 8. A strong 3-rainbow coloring of a closed helm graph CH_8

We conclude that $srx_3(CH_8) = \lceil \frac{8}{2} \rceil + \lceil \frac{8}{3} \rceil = 7$.

Subcase 4.2, $m \geq 10$ with $m \ (mod\ 6) \neq 1$

Note that $sdiam_3(CH_m) = 6$, so we have $srx_3(CH_m) \geq 6$ by (1). By Lemma 1.1 and Lemma 2.1, we use at least $\lceil \frac{m}{2} \rceil + \lceil \frac{m}{3} \rceil$ colors for CH_m. We define the coloring of the spokes and inner rim as follows.

$$c(vv_1^i) = \left\lceil \frac{i}{2} \right\rceil \text{ for } i \in [1, m],$$

$$c(v_1^iv_2^i) = \left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{i}{3} \right\rceil \text{ for } i \in [1, m];$$

for odd m,

$$c(v_1^iv_1^{i+1}) = \begin{cases} \lceil \frac{i}{2} \rceil + 1 & \text{for odd } i \in [1, m-2], \\ \lceil \frac{m}{2} \rceil & \text{for } i = m, \\ \frac{i}{2} & \text{for even } i \in [1, m-1]; \end{cases}$$

for even m,

$$c(v_1^i v_1^{i+1}) = \begin{cases} \lceil \frac{i}{2} \rceil + 1 & \text{for odd } i \in [1, m-3], \\ 1 & \text{for } i = m-1, \\ \frac{i}{2} & \text{for even } i \in [1, m]. \end{cases}$$

Because $\lceil \frac{m}{2} \rceil + \lceil \frac{m}{3} \rceil < m$ for $m \geq 10$, we have at least two edges in the outer rim with the same color. Following the coloring from Lemma 1.1, to satisfy Lemma 2.5, it is not possible to have more than one color α in the outer rim if α is a color from the inner spokes. We have shown the reasoning from the proof for CH_9. As such, the only possible colors are the colors from the outer spokes, that is, at most $\lceil \frac{m}{3} \rceil$ colors.

Subcase 4.2.1, $m \ (mod\ 3) = 0$

We define a strong 3-rainbow coloring of the outer rim edges as follows.

$$c(v_2^i v_2^{i+1}) = \begin{cases} \lceil \frac{m}{2} \rceil + \frac{i+5}{3} & \text{if } i (mod\ 3) = 1 \text{ for } i \in [1, m-5], \\ \lceil \frac{m}{2} \rceil + \frac{i-2}{3} & \text{if } i (mod\ 3) = 2 \text{ for } i \in [1, m-1], \\ \lceil \frac{m}{2} \rceil + 1 & \text{for } i = m-2, \\ \lceil \frac{m}{2} \rceil + \lceil \frac{m}{3} \rceil & \text{for } i = 2, \\ \lceil \frac{m}{2} \rceil & \text{for } i = m, \\ c(v_1^i v_1^{i+1}) & \text{if } c(vv_1^i) \neq c(vv_1^{i+1}), \\ c(vv_1^i) - 1 & \text{if } c(vv_1^i) = c(vv_1^{i+1}). \end{cases}$$

We attach one of the results of this subcase in Figure 9.

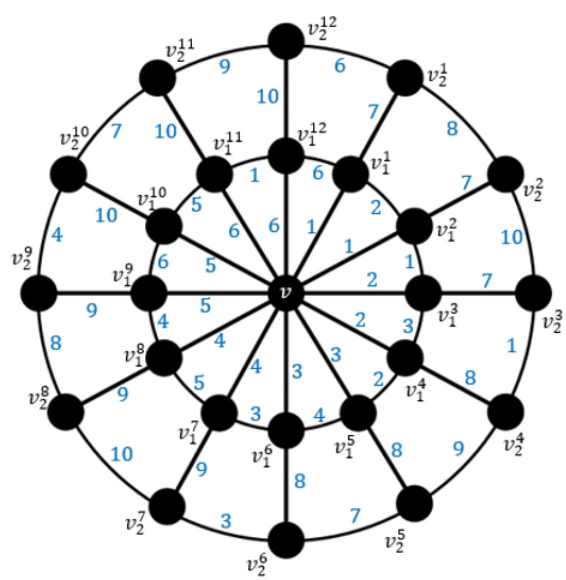


Figure 9. A strong 3-rainbow coloring of a closed helm graph CH_{12}

Subcase 4.2.2, $m \ (mod\ 3) = 1$

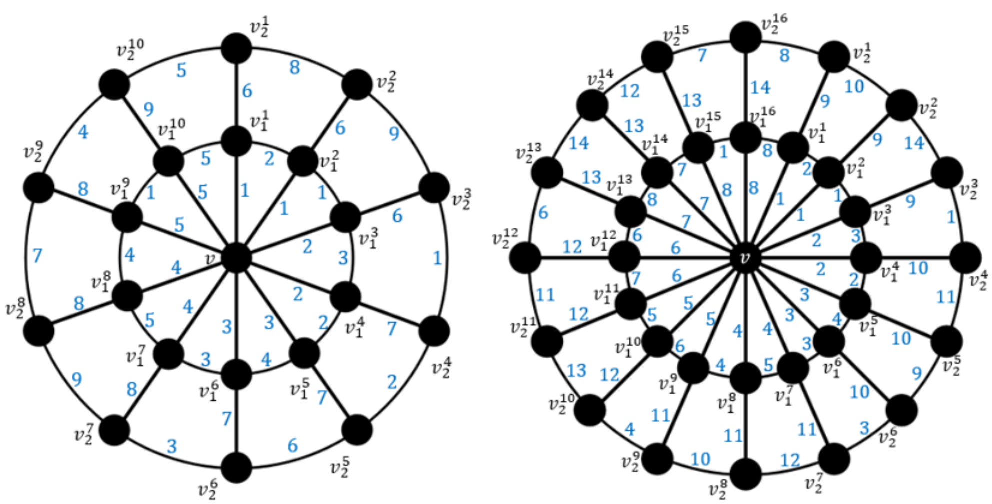
We define a strong 3-rainbow coloring of the outer rim edges for $m = 10$ as follows.

$$c(v_2^i v_2^{i+1}) = \begin{cases} 6 & \text{for } i = 5, \\ 7 & \text{for } i = 8, \\ 8 & \text{for } i = 1, \\ 9 & \text{for } i = 2 \text{ and } i = 7, \\ c(v_1^i v_1^{i+1}) & \text{if } c(vv_1^i) \neq c(vv_1^{i+1}), \\ c(vv_1^i) - 1 & \text{if } c(vv_1^i) = c(vv_1^{i+1}). \end{cases}$$

We define a strong 3-rainbow coloring of the outer rim edges for $m > 13$ as follows.

$$c(v_2^i v_2^{i+1}) = \begin{cases} \left\lceil \frac{m}{2} \right\rceil + \frac{i+5}{3} & \text{if } i(mod\ 3) = 1 \text{ for } i \in [1, m-3], \\ \left\lceil \frac{m}{2} \right\rceil + \frac{i-2}{3} & \text{if } i(mod\ 3) = 2 \text{ for } i \in [1, m-2], \\ \left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{m}{3} \right\rceil & \text{for } i = 2, \\ c(v_1^i v_1^{i+1}) & \text{if } c(vv_1^i) \neq c(vv_1^{i+1}), \\ c(vv_1^i) - 1 & \text{if } c(vv_1^i) = c(vv_1^{i+1}). \end{cases}$$

We attach some results of this subcase in Figure 10.



(a) (b)

Figure 10. Strong 3-rainbow colorings of closed helm graph (a) CH_{10} and (b) CH_{16}

Subcase 4.2.3, $m\ (mod\ 3) = 2$

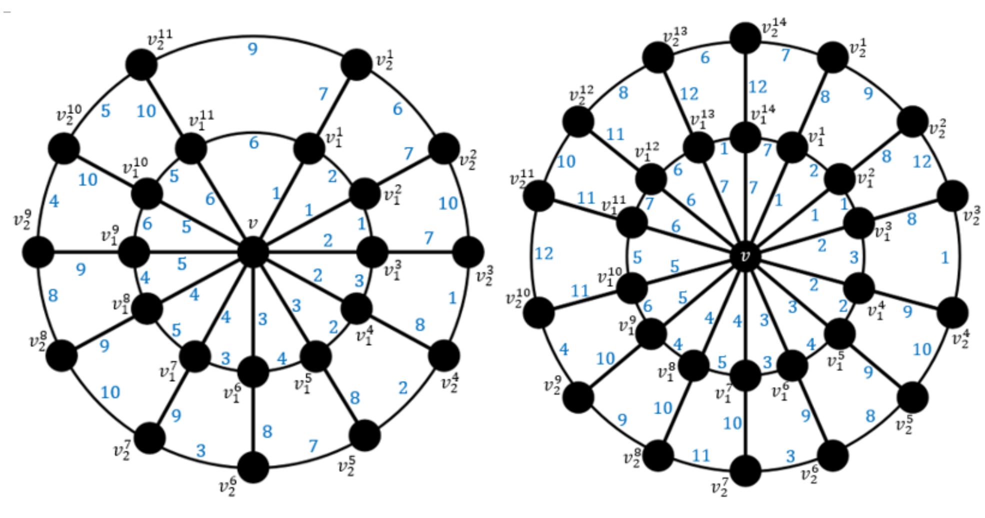
We define a strong 3-rainbow coloring of the outer rim edges for $m = 11$ as follows.

$$c(v_2^i v_2^{i+1}) = \begin{cases} 6 & \text{for } i = 1, \\ 7 & \text{for } i = 5, \\ 8 & \text{for } i = 8, \\ 9 & \text{for } i = m, \\ 10 & \text{for } i = 2 \text{ and } i = 7, \\ c(v_1^i v_1^{i+1}) & \text{if } c(vv_1^i) \neq c(vv_1^{i+1}), \\ c(vv_1^i) - 1 & \text{if } c(vv_1^i) = c(vv_1^{i+1}). \end{cases}$$

We define a strong 3-rainbow coloring of the outer rim edges for $m \geq 14$ as follows.

$$c(v_2^i v_2^{i+1}) = \begin{cases} \left\lceil \frac{m}{2} \right\rceil + \frac{i+5}{3} & \text{if } i (mod\, 3) = 1 \text{ for } i \in [1, m-4], \\ \left\lceil \frac{m}{2} \right\rceil + \frac{i-2}{3} & \text{if } i (mod\, 3) = 2 \text{ for } i \in [5, m], \\ \left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{m}{3} \right\rceil & \text{for } i = 2, \\ \left\lceil \frac{m}{2} \right\rceil + 1 & \text{for } i = m-2, \\ c(v_1^i v_1^{i+1}) & \text{if } c(vv_1^i) \neq c(vv_1^{i+1}), \\ c(vv_1^i) - 1 & \text{if } c(vv_1^i) = c(vv_1^{i+1}). \end{cases}$$

We attach some results of this subcase in Figure 11.



(a) (b)

Figure 11. Strong 3-rainbow colorings of closed helm graph (a) CH_{11} and (b) CH_{14}

By observation, it is clear that $srx_3(CH_m) = \left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{m}{3} \right\rceil$ for $m = 8$ or $m \geq 10$ with m $(mod\, 6) \neq 1$.

Case 5, $m \geq 13$ with $m\ (mod\ 6) = 1$

In this case, we have one outer spoke with the same color as one of the inner spokes' color namely color $\lceil \frac{m}{2} \rceil$. Thus, we use at least $\lceil \frac{m}{2} \rceil + \lceil \frac{m}{3} \rceil - 1$ colors to color the spokes and inner rims. We define a coloring of the spokes and inner rims as follows.

$$c(vv_1^i) = \left\lceil \frac{i}{2} \right\rceil \text{ for } i \in [1, m],$$

$$c(v_1^i v_2^i) = \left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{i-1}{3} \right\rceil \text{ for } i \in [1, m],$$

$$c(v_1^i v_1^{i+1}) = \begin{cases} \lceil \frac{i}{2} \rceil + 1 & \text{for odd } i \in [1, m-2], \\ 1 & \text{for } i = m, \\ \frac{i}{2} & \text{for even } i \in [1, m-1]. \end{cases}$$

We define a strong 3-rainbow coloring for the outer rims as follows.

$$c(v_2^i v_2^{i+1}) = \begin{cases} \lceil \frac{m}{2} \rceil + \frac{i+4}{3} & \text{if } i(mod\ 3) = 2 \text{ for } i \in [2, m-5], \\ \lceil \frac{m}{2} \rceil + \frac{i-3}{3} & \text{if } i(mod\ 3) = 0 \text{ for } i \in [3, m-1], \\ \lceil \frac{m}{2} \rceil & \text{for } i = m-2, \\ 1 & \text{for } i = m, \\ c(v_1^i v_1^{i+1}) + 1 & \text{if } c(vv_1^i) \neq c(vv_1^{i+1}), \\ c(v_1^i v_1^{i+1}) & \text{if } c(vv_1^i) = c(vv_1^{i+1}). \end{cases}$$

We attach one of the results of this subcase in Figure 12.

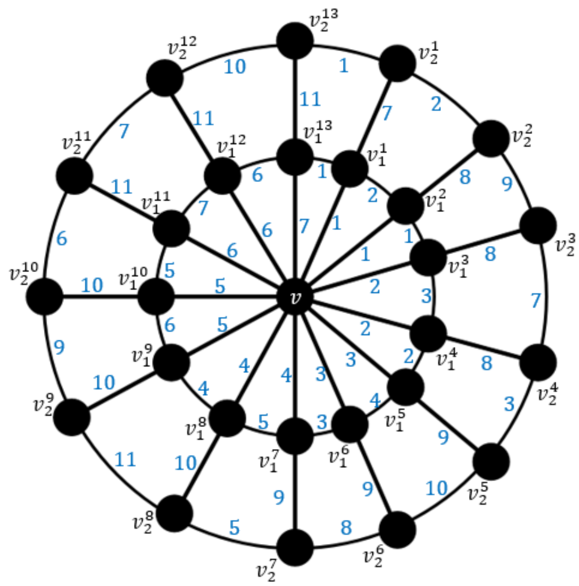


Figure 12. A strong 3-rainbow coloring of a closed helm graph CH_{13}

By observation, it is clear that $srx_3(CH_m) = \lceil \frac{m}{2} \rceil + \lceil \frac{m}{3} \rceil - 1$ for $m \geq 13$ with $m \ (mod\ 6) = 1$. In conclusion, the theorem is proven to be true. □

References

[1] Z.Y. Awanis, et al., The strong 3-rainbow index of edge-amalgamation of some graphs, *Turkish Journal of Mathematics*, **44** (2020), 446–462. Doi: 10.3906/mat-1911-49.

[2] Z.Y. Awanis, A.N.M. Salman, and S.W. Saputro, The strong 3-rainbow index of comb product of a tree and a connected graph, *Journal of Information Processing*, **28** (2020), 865–875. https://www.jstage.jst.go.jp/article/ipsjjip/28/0/28_865/_pdf

[3] Z.Y. Awanis and A.N.M. Salman, The strong 3-rainbow index of some certain graphs and its amalgamation, *Opuscula Math*, **42(4)** (2022), 527–547. Doi: 10.7494/OpMath.2022.42.4.527

[4] Z. Y. Awanis, A.N.M. Salman, and S.W. Saputro, The strong 3-rainbow index of edge-comb product of a path and a connected graph, *Electronic Journal of Graph Theory and Applications*, 10 (1) (2022), 33–50. http://dx.doi.org/10.5614/ejgta.2022.10.1.3

[5] G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang, Rainbow connection in graphs, *Mathematica Bohemica*, **133** (2008), 85–98. http://eudml.org/doc/32579

[6] G. Chartrand, F. Okamoto, and P. Zhang, Rainbow trees in graphs and generalized connectivity, *Networks*, **55** (2010), 360–367. Doi: 10.1002/net.20339.

[7] R. Diestel. Graph Theory, 4th Edition. Heidelberg, Germany: Springer, 2010.