Indonesian Journal of Combinatorics **8** (2) (2024), 76–82

INDONESIAN JOURNAL OF COMBINATORICS

Characteristic Polynomial of Antiadjacency Matrix of Several Classes of Graph Join

Anthony Arthur Ataupah[a], Ganesha Lapenangga Putra[a]

[a]*Mathematics Study Program, FST UNDANA, Kupang, Indonesia, 85228*

anthonyataupah0508@gmail.com, ganesha.lapenangga@staf.undana.ac.id

Abstract

Suppose G is a simple and undirected graph. The adjacency matrix of graph G, denoted by $A(G)$ is a square matrix that representing graph G based on the adjacency of vertices on G, denoted by $A(G)$. The antiadjacency matrix of graph G is a matrix $B(G) = J - A(G)$ where J is an $n \times n$ matrix with all the entries equal to 1. This paper deliver the result of study about the characteristic polynomial of antiadjacency matrix of several graph join, such as multipartite graph, windmill graph, and cone graph.

Keywords: antiadjacency matrix, graph join, characteristic polynomial
Mathematics Subject Classification : 05C50

1. Introduction

Throughout this paper, we assume graph G as a simple and undirected graph. Then, we also assume adjacency and antiadjacency matrix of graph G respectively by $A(G)$ and $B(G) = J - A(G)$ where J is a square matrix having all entries equal to 1 [1, 2]. Moreover, we denoted the characteristic polynomial of antiadjacency matrix on graph G by $\chi_B(G; \lambda)$. Diwyacitta et al. [3] gave the determinant of directed cycle with chords. Edwina et al. [4] also give the properties of the determinant of antiadjacency matrix of two operations (join and union) between several classes of graph, such as bipartite, cycle, complete graph, and wheel. Widiastuti et al. [5], Anzana et al. [6], Hasyyati et al. [7] and Aji et al. [8] determined the characteristic polynomial of antiadjacency matrix of directed cyclic wheel graph, friendship graph, unicyclic corona graph, and unicyclic flower vase graph. [7, 8] also added the result related to the eigen values of the graph. Prayitno et al. [9]

Received: 5 January 2024, Revised: 16 December 2024, Accepted: 30 December 2024.

determined the characteristic polynomial of a line digraph. Irawan et al. [10] have given the results of the characteristic polynomial of graph join such as bipartite graph, complete split, and friendship. Lastly, Sugeng et al. [11] have determined several characteristics of antiadjacency matrices for directed graph. Based on these results, we deliver the result of our study about the characteristic polynomial of antiadjacency matrix of simple and undirected graphs which are obtained from join operation graph, such as multipartite graph $K_{n_1,n_2,\ldots,n_k}$, windmill graph $W_{m,n}$ and cone graph $C_{m,n}$. However, let us have a look into the definition and lemmas required for this paper.

Definition 1.1. [2] For $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be two graphs having no intersection between the vertex set, the join graph of G and H is a graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2 \cup A$ where $A = \{uv \mid u \in V_1, v \in V_2\}$.

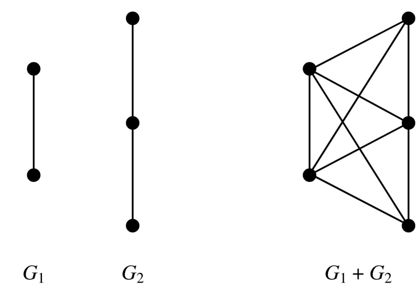

Figure 1. Example of join graph

The $A(G)$ can be written as partitioned matrix as follows.

$$A(G) = \begin{bmatrix} A(G_1) & J \\ J & A(G_2) \end{bmatrix}.$$

Therefore, the $B(G)$ is as follows.

$$B(G) = \begin{bmatrix} B(G_1) & 0 \\ 0 & B(G_2) \end{bmatrix}.$$

Lemma 1.1. *[10] For null graph $\overline{K}_n$ with $n \geq 1$, then*

$$\chi_B(\overline{K}_n; \lambda) = \det(\lambda I - B(\overline{K}_n)) = \lambda^{n-1}(\lambda - n).$$

Lemma 1.2. *[10] For complete graph K_n with $n \geq 1$, then*

$$\chi_B(K_n; \lambda) = \det(\lambda I - B(K_n)) = (\lambda - 1)^n.$$

2. Main Results

Theorem 2.1. *For multipartite graph* $K_{n_1,n_2,\ldots,n_k} = \overline{K}_{n_1} + \overline{K}_{n_2} + \cdots + \overline{K}_{n_k}$ *with* $n_i \geq 1$ *and* $B(K_{n_1,n_2,\ldots,n_k})$ *is an antiadjacency matrix of graf* $K_{n_1,n_2,\ldots,n_k}$ *then characteristic polynomial of the multipartite graph* $K_{n_1,n_2,\ldots,n_k}$,

$$\chi_B(K_{n_1,n_2,\ldots,n_k};\lambda) = \prod_{i=1}^{k}(\lambda^{n_i-1}(\lambda - n_i)).$$

Proof. Let $K_{n_1,n_2,\ldots,n_k} = \overline{K}_{n_1} + \overline{K}_{n_2} + \cdots + \overline{K}_{n_k}$ be a multipartite graph. $A(K_{n_1,n_2,\ldots,n_k})$ can be written as partitioned matrix as follows.

$$A(K_{n_1,n_2,\ldots,n_k}) = \begin{bmatrix} A(\overline{K}_{n_1}) & J_{n_1\times n_2} & \cdots & J_{n_1\times n_{k-1}} & J_{n_1\times n_k} \\ J_{n_2\times n_1} & A(\overline{K}_{n_2}) & \ddots & J_{n_2\times n_{k-1}} & J_{n_2\times n_k} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ J_{n_{k-1}\times n_1} & J_{n_{k-1}\times n_2} & \ddots & A(\overline{K}_{n_{k-1}}) & J_{n_{k-1}\times n_k} \\ J_{n_k\times n_1} & J_{n_k\times n_2} & \cdots & J_{n_k\times n_{k-1}} & A(\overline{K}_{n_k}) \end{bmatrix}.$$

Where $J_{x\times y}$ is an $x \times y$ matrix having all the entries equal to 1. Thus, $B(K_{n_1,n_2,\ldots,n_k})$ can be written as partitioned matrix as follows.

$$B(K_{n_1,n_2,\ldots,n_k}) = \begin{bmatrix} B(\overline{K}_{n_1}) & 0_{n_1\times n_2} & \cdots & 0_{n_1\times n_{k-1}} & 0_{n_1\times n_k} \\ 0_{n_2\times n_1} & B(\overline{K}_{n_2}) & \ddots & 0_{n_2\times n_{k-1}} & 0_{n_2\times n_k} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{n_{k-1}\times n_1} & 0_{n_{k-1}\times n_2} & \ddots & B(\overline{K}_{n_{k-1}}) & 0_{n_{k-1}\times n_k} \\ 0_{n_k\times n_1} & 0_{n_k\times n_2} & \cdots & 0_{n_k\times n_{k-1}} & B(\overline{K}_{n_k}) \end{bmatrix}.$$

Thus,

$$\chi_B(K_{n_1,n_2,\ldots,n_k};\lambda) = \prod_{i=1}^{k}\det(\lambda I_{n_i\times n_i} - B(\overline{K}_{n_i})).$$

$$\chi_B(K_{n_1,n_2,\ldots,n_k};\lambda) = \prod_{i=1}^{k}(\chi_B(\overline{K}_{n_i};\lambda)). \tag{1}$$

According to **Lemma 1.1** and Equation (1), the characteristic polynomial of a multipartite graph $K_{n_1,n_2,\ldots,n_k}$ can be written as follows.

$$\chi_B(K_{n_1,n_2,\ldots,n_k};\lambda) = \prod_{i=1}^{k}(\lambda^{n_i-1}(\lambda - n_i)).$$

□

Theorem 2.2. *For windmill graph $W_{m,n} = mK_{n-1} + K_1$ with $m, n \geq 2$ then*

$$\chi_B(W_n^{(m)}; \lambda) = (\lambda - 1)^{(m-1)(n-2)^2+1}(\lambda - (m-1)(n-1) - 1)^{m-1}(\lambda + n - 2).$$

Proof. Let $W_{m,n} = mK_{n-1} + K_1$ be a windmill graph. $A(W_{m,n})$ can be written as partitioned matrix as follows.

$$B(W_{m,n}) = \begin{bmatrix} J_{1\times 1} - A(K_1) & J_{1\times m(n-1)} - J_{1\times m(n-1)} \\ J_{m(n-1)\times 1} - J_{m(n-1)\times 1} & J_{m(n-1)\times m(n-1)} - A(mK_{n-1}) \end{bmatrix}$$
$$B(W_{m,n}) = \begin{bmatrix} B(K_1) & 0_{1\times m(n-1)} \\ 0_{m(n-1)\times 1} & B(mK_{n-1}) \end{bmatrix}$$

Where $J_{x\times y}$ is an $x \times y$ matrix having all the entries equal to 1. Note that mK_{n-1} is similar to $K_{n-1} \cup K_{n-1} \cup \cdots \cup K_{n-1}$. Thus, $A(mK_{n-1})$ can be written as partitioned matrix as follows.

$$B(mK_{n-1}) = \begin{bmatrix} B(K_{n-1}) & J_{n-1\times n-1} & \cdots & J_{n-1\times n-1} & J_{n-1\times n-1} \\ J_{n-1\times n-1} & B(K_{n-1}) & \ddots & J_{n-1\times n-1} & J_{n-1\times n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ J_{n-1\times n-1} & J_{n-1\times n-1} & \ddots & B(K_{n-1}) & J_{n-1\times n-1} \\ J_{n-1\times n-1} & J_{n-1\times n-1} & \cdots & J_{n-1\times n-1} & B(K_{n_1}) \end{bmatrix}.$$

Thus,

$$\begin{aligned} \chi_B(mK_{n-1}; \lambda) &= (\det(\lambda I_{n-1} - B(K_{n-1}) + J_{n-1}))^{m-1}(\lambda I_{n-1} - B(K_{n-1}) - (m-1)J_{n-1}) \\ &= ((\lambda - 1)^{n-2}(\lambda + n - 2))^{m-1}((\lambda - 1)^{n-2}(\lambda - (n-1)(n-1) - 1)) \\ \chi_B(mK_{n-1}; \lambda) &= (\lambda - 1)^{m(n-2)}(\lambda - (m-1)(n-1) - 1)(\lambda + n - 2)^{m-1}. \end{aligned}$$

This implies that the characteristic polynomial of a windmill graph $W_{m.n}$ can be written as follows.

$$\begin{aligned} \chi_B(W_{m,n}; \lambda) &= \chi_B(mK_{n-1}; \lambda)\chi_B(K_1; \lambda) \\ &= (\lambda - 1)^{m(n-2)}(\lambda - (m-1)(n-1) - 1)(\lambda + n - 2)^{m-1}(\lambda - 1) \\ \chi_B(W_{m,n}; \lambda) &= (\lambda - 1)^{m(n-2)+1}(\lambda - (m-1)(n-1) - 1)(\lambda + n - 2)^{m-1}. \end{aligned}$$

□

Theorem 2.3. *For cone graph $C_{m,n} = C_m + \overline{K}_n$ with $m \geq 3$ and $n \geq 1$. Then*

$$\chi_B(C_{m,n}; \lambda) = \lambda^{n-1}(\lambda - m + 2)(\lambda - n) \prod_{r=1}^{m-1} \left(\lambda + 2\cos\left(\frac{2\pi r}{m}\right)\right).$$

Proof. Let C_m be a cycle graph. Eigenvalues of the adjacency matrix of C_m is as follows.

$$\lambda_{ci} = 2\cos\left(\frac{2\pi i}{m}\right) \text{ for } i \in \{1, 2, \ldots, m\}.$$

Let J is an $m \times m$ matrix having all the entries equal to 1. Eigenvalues of J is m with geometric multiplicity 1 and 0 with geometric multiplicity $m-1$. There exist invertible matrix P such that $A(C_m)$ and J can be simultanously diagonalized by P [12]. Let λ_{ci} be an eigenvalue of $A(C_m)$ and λ_{ji} be an eigenvalue of J for $i \in \{1, 2, \ldots, m\}$.

$$\begin{aligned}
\det(\lambda I - B(C_m)) &= \det(\lambda I - J + A(C_m)) \\
&= \det(P^T)\det(\lambda I - J + A(C_m))\det(P) \\
&= \det(\lambda P^{-1}P - P^{-1}JP + P^{-1}A(C_m)P) \\
\det(\lambda I - B(C_m)) &= \prod_{i=1}^{m}(\lambda - \lambda_{ji} + \lambda_{ci}).
\end{aligned}$$

Let v_k be an eigenvector corresponding to $\lambda_{ck} = 2$ and $\lambda_{jk} = n$ for $1 \leq k \leq m$. Note that $\lambda_{ji} = 0$ for $i \neq k$. Thus,

$$\begin{aligned}
\det(\lambda I - B(C_m)) &= \left(\frac{\lambda - \lambda_{jk} + \lambda_{ck}}{\lambda + \lambda_{ck}}\right)\prod_{r=1}^{m}(\lambda + \lambda_{ci}) \\
\det(\lambda I - B(C_m)) &= \left(\frac{\lambda - n + 2}{\lambda + 2}\right)\prod_{i=1}^{m}\left(\lambda + 2\cos\left(\frac{2\pi i}{m}\right)\right) \\
\det(\lambda I - B(C_m)) &= (\lambda - n + 2)\prod_{i=1}^{m-1}\left(\lambda + 2\cos\left(\frac{2\pi i}{m}\right)\right). \qquad (2)
\end{aligned}$$

Let $C_{m,n} = C_m + \overline{K}_n$ be a cone graph. $B(C_{m,n})$ can be written as partitioned matrix as follows.

$$\begin{aligned}
B(C_{m,n}) &= \begin{bmatrix} J_{m\times m} - A(C_m) & J_{m\times n} - J_{m\times n} \\ J_{n\times m} - J_{n\times m} & J_{n\times n} - A(\overline{K}_n) \end{bmatrix} \\
B(C_{m,n}) &= \begin{bmatrix} B(C_m) & 0_{m\times n} \\ 0_{n\times m} & B(\overline{K}_n) \end{bmatrix}.
\end{aligned}$$

Where $J_{x,y}$ is an $x \times y$ matrix having all the entries equal to 1. Thus, according to **Lemma 1.1** and equation (2), the characteristic polynomial of a cone graph $C_{m.n}$ can be written as follows.

$$\begin{aligned}
\chi_B(C_{m,n};\lambda) &= \det(\lambda I_{n\times n} - B(\overline{K}_n))\det(\lambda I_{m\times m} - B(C_m)) \\
\chi_B(C_{m,n};\lambda) &= \lambda^{n-1}(\lambda - m + 2)(\lambda - n)\prod_{r=1}^{m-1}\left(\lambda + 2\cos\left(\frac{2\pi r}{m}\right)\right).
\end{aligned}$$

□

Conclusion

This paper have derived the characteristic polynomials of the antiadjacency matrices for several graph joins, including multipartite graphs, windmill graphs, and cone graphs. These characteristic polynomials provide a basis for determining key properties such as the determinant, eigenvalues, and spectrum of the antiadjacency matrices of these graphs. We are confident that this study can be extended to explore the characteristics of antiadjacency matrices of other graphs formed through additional graph-theoretical operations, such as composition, Cartesian products, and beyond.

Acknowledgement

We would like to thank UNDANA for the support and reviewers for helping us improving this paper.

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