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Γ-supermagic labeling of products of two cycles with dihedral groups

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Abstract

A Γ-supermagic labeling of a graph $G = (V, E)$ is a bijection from E to a group Γ of order $|E|$ such that for every vertex $x \in V$ a product of labels of all edges incident with x is equal to the same element $\mu \in \Gamma$. A Γ-supermagic labeling of the Cartesian product of two cycles, $C_m \square C_n$ for every $m, n \geq 3$ of the same parity was found recently [5, 6] for all Abelian groups of order $2mn$. In this paper we present a D_{2k}-supermagic labeling of the Cartesian, direct, and strong product of cycles C_m and C_n by dihedral group D_{2k} of order $2k$ for any integers $m, n \geq 3$, where $2k$ is equal to the number of edges of the respective product.

Keywords: Magic-type labeling, supermagic labeling, vertex-magic edge labeling, group supermagic labeling, product of cycles
Mathematics Subject Classification: 05C78

1. Motivation

The Cartesian product of two cycles $C_m \square C_n$ can be viewed as the Cayley graph of the group $Z_m \oplus Z_n$ generated by the group elements $(1, 0)$ and $(0, 1)$. The following question was asked and partially answered by several sets of authors. Given the Cartesian product $C_m \square C_n$ with vertex set V and edge set E and an Abelian group $\Gamma = Z_{k_1} \oplus Z_{k_2} \oplus \cdots \oplus Z_{k_t}$ of order $|E| = 2mn$, can the edges be labeled bijectively with elements of Γ so that the weight of every vertex is the same element of Γ? The weight of a vertex x is the sum of the labels of edges incident with x. Similar questions can be (and have been) asked for labelings of vertices and also vertices and edges. Exact definitions of the above notions are given in Section 2.

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The question was originally studied for labelings of Cartesian products of two cycles by positive integers in [7] and later extended to labelings by Abelian groups (see, e.g., [3, 4, 5, 6]). A more detailed overview of the above mentioned results is given in Section 3.

The first attempt to find Γ-supermagic labelings of any graphs by non-Abelian groups was made by the author in [2] where D_{2k}-supermagic labelings of 4-regular Archimedean graphs by the dihedral group D_{2k} were found.

We continue this effort to find Γ-supermagic labelings of some infinite families of vertex-transitive graphs for non-Abelian groups Γ. We again choose dihedral groups. In Section 4 we present a construction for D_{2mn}-supermagic labeling of the Cartesian product $C_m \square C_n$ by the dihedral group D_{2mn}. In Section 5 we do the same for D_{2mn}-supermagic labeling of the direct product $C_m \times C_n$, and finally in Section 6 for D_{4mn}-supermagic labeling of the strong product $C_m \boxtimes C_n$.

Disclaimer. As noted above, the topic of this paper is very similar to the topic of [5] and [6]. Most of the known results cited in this paper have been also cited in these two papers and the statements of the cited theorems here are therefore identical. Also, some text in Sections 2 and 3 is taken directly from [5] or [6].

2. Definitions

For the sake of completeness, we start with the definitions of various products of two graphs. We start with the Cartesian product.

Definition 2.1. The *Cartesian product* $G = G_1 \square G_2$ of graphs G_1 and G_2 with disjoint vertex and edge sets V_1, V_2, and E_1, E_2, respectively, is the graph with vertex set $V = V_1 \times V_2$ where any two vertices $u = (u_1, u_2) \in G$ and $v = (v_1, v_2) \in G$ are adjacent in G if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1.

Another well known product is the direct product, sometimes also called the tensor or Kronecker product.

Definition 2.2. The *direct product* $G = G_1 \times G_2$ of graphs G_1 and G_2 with disjoint vertex and edge sets V_1, V_2, and E_1, E_2, respectively, is the graph with vertex set $V = V_1 \times V_2$ where any two vertices $u = (u_1, u_2) \in G$ and $v = (v_1, v_2) \in G$ are adjacent in G if and only if u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2.

The strong product is just the union of the above two.

Definition 2.3. The *strong product* $G = G_1 \boxtimes G_2$ of graphs G_1 and G_2 with disjoint vertex and edge sets V_1, V_2, and E_1, E_2, respectively, is the graph with vertex set $V = V_1 \times V_2$ where any two vertices $u = (u_1, u_2) \in G$ and $v = (v_1, v_2) \in G$ are adjacent in G if and only if u is adjacent to v in $G_1 \square G_2$ or u is adjacent to v in $G_1 \times G_2$.

The notion of supermagic labeling was also studied under the name of vertex-magic edge labeling.

Definition 2.4. A *supermagic labeling* of a graph $G(V, E)$ with $|E| = q$ is a bijection f from E to the set $\{1, 2, \ldots, q\}$ such that the sum of labels of all incident edges of every vertex $x \in V$, called the *weight* of x and denoted $w(x)$, is equal to the same positive constant c, called the *magic constant*. That is,

$$w(x) = \sum_{xy \in E} f(xy) = c$$

for every vertex $x \in V$. A graph that admits a supermagic labeling is called a *supermagic graph*.

There were also some more general forms of edge labelings studied by Sedláček [9] and Stanley [11, 12]. Stewart [13] introduced the notion of supermagic labeling, where the set of labels consisted of $|E|$ consecutive integers. When a supermagic graph is regular, then the edge labels can start with any positive integer, and therefore are always considered to be $1, 2, \ldots, |E|$.

Definition 2.5. A Γ-*supermagic labeling* of a graph $G(V, E)$ with $|E| = q$ is a bijection f from E to a group Γ of order q such that for every vertex $x \in V$ and its incident edges $e_1, e_2, \ldots, e_r$ there exists an ordering $e_{i_1}, e_{i_2}, \ldots, e_{i_r}$ for which the *weight* of x, denoted $w(x)$ and defined as

$$w(x) = f(e_{i_r})f(e_{i_{r-1}}) \ldots f(e_{i_1})$$

is equal to the same element $\mu \in \Gamma$, called the *magic constant*.

A graph that admits a Γ-supermagic labeling is called a Γ-*supermagic graph*.

While for Abelian groups the order in which the edge labels are considered is irrelevant, for non-Abelian groups different orders may produce different weights. It is indeed desirable that the order for every vertex is in some way predictable or uniform. Although for general graphs it may be hard to achieve, for many classes of graphs it can be done in a simple way. For instance, when the graph is drawn in the plane or on the torus. We will provide such definitions later in subsequent sections.

The dihedral group D_{2k} of order $2k$ (sometimes also denoted by D_k) is the group consisting of k rotations r_i and k reflections s_i, where the rotations form a cyclic group of order k and and each reflection generates a subgroup of order 2. A more formal definition is below.

Definition 2.6. The *dihedral group* of order $2k$ where $k \geq 3$, denoted by D_{2k}, is defined on the set of elements $\{r_0, r_1, \ldots, r_{k-1}, s_0, s_1, \ldots, s_{k-1}\}$ where $r_0 = e, r_i = r^i$, $s_0 = s$, $s_i = r^i s$, $s_i^2 = e$ and $r^i s = sr^{-i}$ for $i = 0, 1, \ldots, k-1$. The elements r_i are called *rotations*, and s_i are called *reflections*.

An important property of D_{2k} will be used in our constructions. If follows directly from the definition.

Proposition 2.1. *In any dihedral group D_{2k}, we have $sr^i s = r^{-i}$ for every $i = 0, 1, \ldots, k-1$.*

3. Known results

Research in this area was initiated by Ivančo [7] who investigated labelings with positive integers. He proved two results.

Theorem 3.1 ([7]). *Let $n \geq 3$. Then the Cartesian product $C_n \square C_n$ has a supermagic labeling.*

Theorem 3.2 ([7]). *Let $m, n \geq 4$ be even integers. Then $C_m \square C_n$ has a supermagic labeling.*

Ivančo also conjectured that there exists a supermagic labeling for all Cartesian products $C_m \square C_n$.

Conjecture 1 ([7]). *The Cartesian product $C_m \square C_n$ allows a supermagic labeling for any integer $m, n \geq 3$.*

Froncek in an unpublished manuscript [1] verified that the conjecture is true also when m, n are both odd and not relatively prime.

Theorem 3.3 ([1]). *Let $m, n \geq 3$ be odd integers and $\gcd(m, n) > 1$. Then $C_m \square C_n$ has a supermagic labeling.*

Froncek, McKeown, McKeown, and McKeown [3] proved a result analogical to Theorems 3.2 and 3.3 for the cyclic group Z_{2mn} where at least one of m, n is odd.

Theorem 3.4 ([3]). *The Cartesian product $C_m \square C_n$ admits a Z_{2mn}-supermagic labeling for all odd $m \geq 3$ and any $n \geq 3$.*

Notice that Theorem 3.2 implies the existence of Z_{2mn}-supermagic labeling for m, n both even. Therefore, a complete characterization was established.

Later, Froncek and McKeown [4] used a different construction to prove the complete result and showed that the labeling is different from the one obtained in the proof of the previous theorem.

Theorem 3.5 ([4]). *The Cartesian product $C_m \square C_n$ admits a Z_{2mn}-supermagic labeling for all $m, n \geq 3$.*

The construction method from [4] was then used by Sorensen [10] and Paananen [8].[1] Notice that when mn is even, the group used in the theorem is not cyclic, hence their result is more general.

Theorem 3.6 ([8, 10]). *For any $m, n \geq 3$, the Cartesian product $C_m \square C_n$ admits a Γ-supermagic labeling for $\Gamma = Z_{mn} \oplus Z_2$.*

Paananen [8] and Sorensen [10] also proved some more partial results that were later generalized by Froncek, Paananen, and Sorensen [5, 6].

Theorem 3.7 ([5, 6]). *Let $m, n \geq 3$ have the same parity. Then the Cartesian product $C_m \square C_n$ admits a Γ-supermagic labeling by any Abelian group Γ of order $2mn$.*

The case of m and n having different parity remains open except for the groups Z_{2mn} and $Z_{mn} \oplus Z_2$.

[1] Paananen [8] (2021) and Sorensen [10] (2020) worked on a joint project for their MS theses. While all results cited here are their joint work, their theses were written and defended independently. Both theses contain Theorem 3.6.

4. Cartesian products

Now we present a D_{2mn}-supermagic labeling of the Cartesian product $C_m \square C_n$.

We use the following notation. We always assume that $m \le n$ and the vertices of $C_m \square C_n$ will be denoted by $x_{i,j}$ with $0 \le i \le m-1, 0 \le j \le n-1$. We define *vertical edges* $v_{i,j} = x_{i,j}x_{i+1,j}$ and *horizontal edges* $h_{i,j} = x_{i,j}x_{i,j+1}$ for all admissible i, j.

A vertex $x_{i,j}$ is then incident with vertical edges $v_{i,j} = x_{i,j}x_{i+1,j}$ and $v_{i-1,j} = x_{i-1,j}x_{i,j}$, horizontal edges $h_{i,j} = x_{i,j}x_{i,j+1}$ and $h_{i,j-1} = x_{i,j-1}x_{i,j}$ and has neighbors $x_{i-1,j}, x_{i,j+1}, x_{i+1,j}, x_{i,j-1}$.

We label horizontal edges in each horizontal cycle C_n consecutively with elements of one coset of the subgroup of rotations $\langle r^m \rangle$ of order n and every vertical cycle C_m with elements of a coset of reflections of order m.

First we define some types of labelings. To make our notation descriptive, we use compass directions and call the edge $v_{i-1,j}$ NORTHBOUND with respect to the vertex $x_{i,j}$ (N for short). Similarly, $h_{i,j}$ is called EASTBOUND (E), $v_{i,j}$ is SOUTHBOUND (S), and $h_{i,j-1}$ is WESTBOUND (W). Hence, when we define the weight $w(x_{i,j})$ of a vertex $x_{i,j}$ as

$$w(x_{i,j}) = f(h_{i,j-1})f(v_{i,j})f(h_{i,j})f(v_{i-1,j}),$$

we speak of type WSEN labeling, because we first use the northbound facing edge, then eastbound, then southbound, and last the westbound.

Definition 4.1. A labeling f of $C_m \square C_n$ is called *uniform* if for every vertex $x_{i,j}$ the weight $w(x_{i,j})$ is calculated in the same way, that is, the sequence of labels of edges $v_{i-1,j}, v_{i,j}, h_{i,j-1}, h_{i,j}$ in their product is the same.

A labeling f is called *revolving of type* ESWN when

$$w(x_{i,j}) = f(h_{i,j})f(v_{i,j})f(h_{i,j-1})f(v_{i-1,j}).$$

Now we present an ESWN-type labeling.

Construction 4.2. *Revolving D_{2mn}-supermagic labeling of type* ESWN.
Let $m, n \ge 3$. We label the i-th horizontal cycle consecutively with elements of the coset $\langle r^m \rangle r^i$. That is, with elements $r^i, r^{m+i}, r^{2m+i}, \ldots, r^{(n-1)m+i}$ such that $f(h_{i,j}) = r^{jm+i}$.

Similarly, we label the j-th vertical cycle consecutively with elements of the coset $\langle r^n \rangle r^j s$. Namely, with elements $r^j s, r^{n+j}s, r^{2n+j}s, \ldots, r^{(m-1)n+j}s$ such that $f(v_{i,j}) = r^{in+j}s$.

This way, the weight of a vertex $x_{i,j}$ is given by

$$\begin{aligned} w(x_{i,j}) &= f(h_{i,j})f(v_{i,j})f(h_{i,j-1})f(v_{i-1,j}) \\ &= (r^{jm+i})(r^{in+j}s)(r^{(j-1)m+i})(r^{(i-1)n+j}s) \\ &= (r^{jm+i+in+j})(sr^{jm-m+i+in-n+j}s) \\ &= (r^{jm+i+in+j})(r^{-(jm-m+i+in-n+j)}) \\ &= r^{m+n}, \end{aligned}$$

because by Proposition 2.1, we have $sr^{jm-m+i+in-n+j}s = r^{-(jm-m+i+in-n+j)}$.

We illustrate the labeling of edges incident with a vertex $x_{i,j}$ in Figure 1. The vertex and the incident edges are printed in blue.

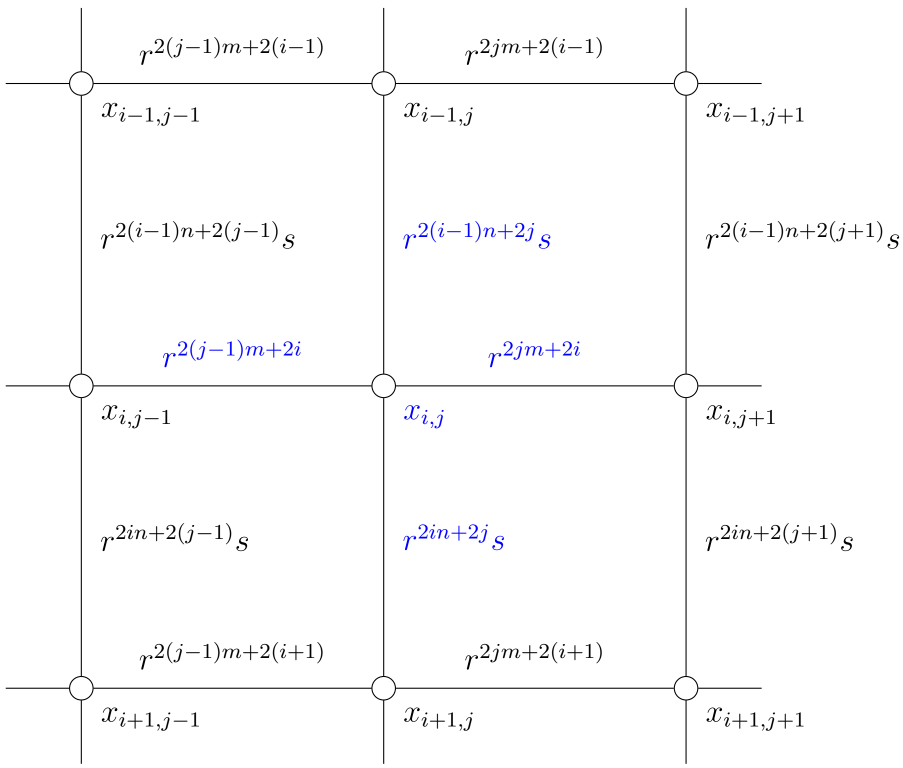

Figure 1. Cartesian product

By constructing the labeling above, we proved the following.

Theorem 4.3. *Let $m, n \geq 3$ and D_{2mn} be the dihedral group of order $2mn$. Then there exists a revolving D_{2mn}-supermagic labeling of the Cartesian product $C_m \square C_n$.*

5. Direct products

We use the same vertex notation as in Section 4. This time we define *forward diagonal edges* $d_{i,j} = x_{i,j}x_{i+1,j+1}$ and *backward diagonal edges* $b_{i,j} = x_{i,j}x_{i-1,j-1}$ for all admissible i, j.

A vertex $x_{i,j}$ is then incident with forward diagonal edges $d_{i,j} = x_{i,j}x_{i+1,j+1}$ and $d_{i-1,j-1} = x_{i-1,j-1}x_{i,j}$, backward diagonal edges $b_{i,j} = x_{i,j}x_{i+1,j-1}$ and $b_{i-1,j+1} = x_{i-1,j+1}x_{i,j}$ and has neighbors $x_{i-1,j-1}, x_{i+1,j+1}, x_{i+1,j-1}, x_{i-1,j+1}$.

The cycle containing the edge $d_{0,0}$ is called the *forward diagonal* D^0, and the one containing the edge $b_{0,0}$ is the *backward diagonal* B^0. Notice that the length of each diagonal is $l = \mathrm{lcm}(m, n)$ and the number of diagonals of the same type is $g = \gcd(m, n)$. It follows that when m, n are relatively prime, there is exactly one diagonal of each type. When $g > 1$, we denote the diagonals containing the vertex $x_{0,t}$ by D^t and B^t, respectively.

It is well known (see [14]) that when at least one of m, n is odd, then $C_m \times C_n$ is connected, and when m, n are both even, then it contains two components.

We label the edges in each forward diagonal D^t consecutively with elements of one coset of the subgroup of rotations $\langle r^g \rangle$ of order l and every backward diagonal with elements of a coset of reflections of order l.

We again define a particular type of labeling. We do not have a good "geographical" description this time, but the notation should be clear from the definition. A rather awkward description would be that each X̂ direction is the X direction rotated 45 degrees clockwise. So for instance, N̂ is in fact NORTHEAST.

Definition 5.1. A labeling f of $C_m \times C_n$ is called *revolving of type N̂ÊŜŴ* when

$$w(x_{i,j}) = f(b_{i-1,j+1})f(d_{i,j})f(b_{i,j})f(d_{i-1,j-1})$$

Construction 5.2. *Revolving D_{2mn}-supermagic labeling of type* N̂ÊŜŴ.
We label the t-th forward diagonal D^t consecutively with elements of the coset $\langle r^g \rangle r^t$ of order l. That is, with elements $r^t, r^{g+t}, r^{2g+t}, \ldots, r^{(l-1)g+t}$ such that $f(d_{0,t}) = r^t$.

Similarly, we label the t-th backward diagonal B^t consecutively with elements of the coset $\langle r^g \rangle r^t s$. Namely, with elements $r^t s, r^{g+t} s, r^{2g+t} s, \ldots, r^{(l-1)g+t} s$ such that $f(b_{0,t}) = r^t s$.

Calculating the weight of each vertex based on its coordinates would be clumsy. Therefore, we use the observation that if $f(d_{i-1,j-1}) = r^a$ for some $a \in \{0, 1, \ldots, mn-1\}$, then the next edge in the same forward diagonal is labeled with $f(d_{i,j}) = r^{a+g}$. Similarly, if $f(b_{i-1,j+1}) = r^b s$ for some $b \in \{0, 1, \ldots, mn-1\}$, then the next edge in the same backward diagonal is labeled with $f(b_{i,j}) = r^{b+g} s$.

This way, the weight of any vertex $x_{i,j}$ is given by

$$\begin{aligned} w(x_{i,j}) &= f(b_{i-1,j+1})f(d_{i,j})f(b_{i,j})f(d_{i-1,j-1}) \\ &= (r^b s)(r^{a+g})(r^{b+g} s)(r^a) \\ &= (r^b)(s r^{a+g} r^{b+g} s)(r^a) \\ &= (r^b)(s r^{a+b+2g} s)(r^a) \\ &= (r^b)(r^{-(a+b+2g)})(r^a) \\ &= r^{-2g} \end{aligned}$$

and the weight is independent on the vertex location. Notice that we used Proposition 2.1 that asserts that $s r^{a+b+2g} s = r^{-(a+b+2g)}$.

We illustrate the labeling of edges incident with a vertex $x_{i,j}$ in Figures 2 and 3. The vertex and the incident edges are printed in blue. We present the labels in two different ways. First with explicit formulas in Figure 2, and then in Figure 3 with notation corresponding to the explanation in Construction 5.2.

The theorem below follows immediately for the above construction.

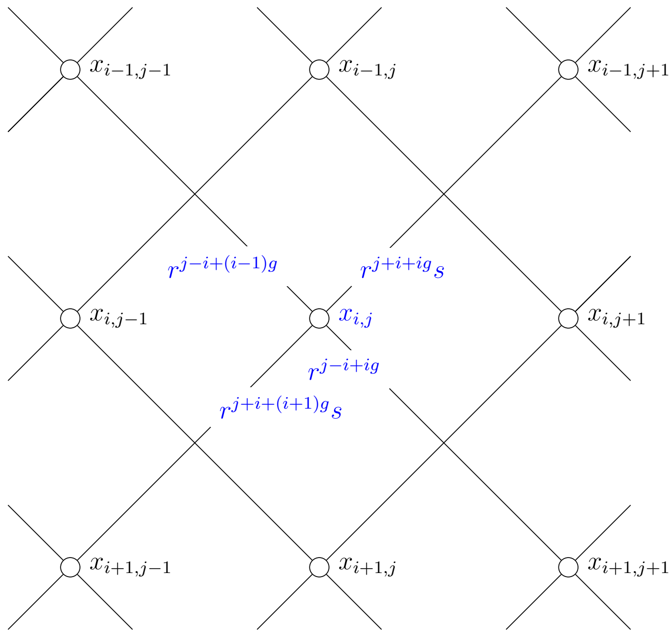

Figure 2. Direct product with explicit labels

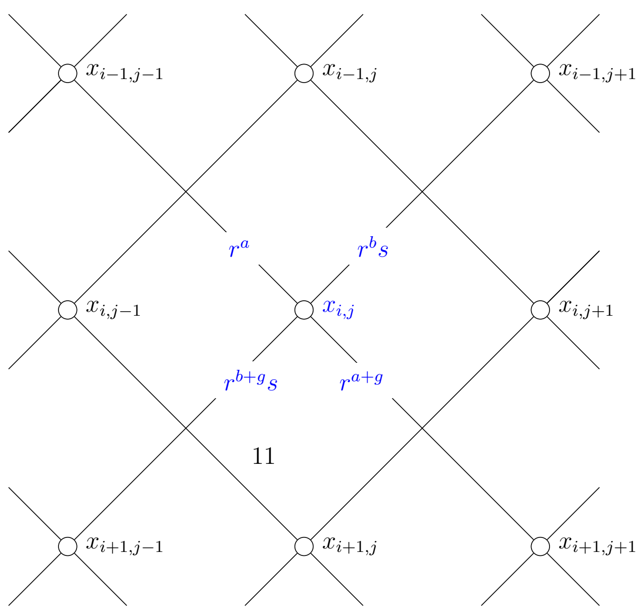

Figure 3. Direct product with simplified labels

Theorem 5.3. *Let $m, n \geq 3$ and D_{2mn} be the dihedral group of order $2mn$. Then there exists a revolving D_{2mn}-supermagic labeling of the direct product $C_m \times C_n$.*

6. Strong products

We use the same notation as in Sections 4 and 5. This time, we construct only a uniform labeling. We illustrate the strong product without labels in Figure 4.

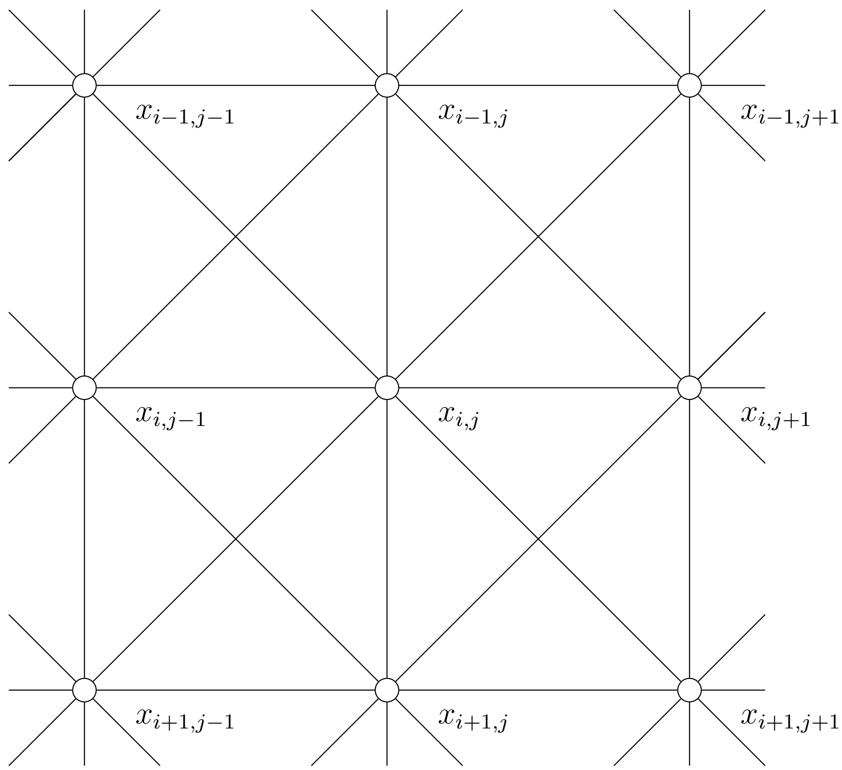

Figure 4. Strong product

Definition 6.1. A labeling f of the strong product $C_m \boxtimes C_n$ is called *uniform of type N̂ÊŜŴESWN* when for every vertex $x_{i,j}$ the weight $w(x_{i,j})$ is calculated as

$$w(x_{i,j}) = f(b_{i-1,j+1})f(d_{i,j})f(b_{i,j})f(d_{i-1,j-1})f(h_{i,j})f(v_{i,j})f(h_{i,j-1})f(v_{i-1,j}).$$

Construction 6.2. *Uniform D_{4mn}-supermagic labeling of type* N̂ÊŜŴESWN.
We use the labelings constructed before for the Cartesian and direct product and combine them into one.

Let $m, n \geq 3$. We label the i-th horizontal cycle consecutively with the elements of the even coset $\langle r^{2m} \rangle r^{2i}$. That is, with elements $r^{2i}, r^{2m+2i}, r^{4m+2i}, \ldots, r^{2(n-1)m+2i}$ such that $f(h_{i,j}) = r^{2jm+2i}$.

Similarly, we label the j-th vertical cycle consecutively with the elements of the even coset $\langle r^{2n}\rangle r^{2j}s$. Namely, with elements $r^{2j}s, r^{2n+2j}s, r^{4n+2j}s, \ldots, r^{2(m-1)n+2j}s$ such that $f(v_{i,j}) = r^{2in+2j}s$.

Then we label the t-th forward diagonal D^t consecutively with the elements of the odd coset $\langle r^{2g}\rangle r^{2t+1}$ of order l. That is, with elements r^{2t+1}, $r^{2g+2t+1}$, $r^{4g+2t+1}$, $\ldots$, $r^{2(l-1)g+2t+1}$ such that $f(d_{0,t}) = r^{2t+1}$.

Also, we label the t-th backward diagonal B^t consecutively with the elements of the odd coset $\langle r^{2g}\rangle r^{2t+1}s$. Namely, with elements $r^{2t+1}s, r^{2g+2t+1}s, r^{4g+2t+1}s, \ldots, r^{2(l-1)g+2t+1}s$ such that $f(d_{0,t}) = r^{2t+1}s$.

We again simplify the construction as in Construction 5.2. So at any $x_{i,j}$, we have

$$\begin{aligned} f(h_{i,j-1}) &= r^{2a}, & f(h_{i,j}) &= r^{2a+2m} \\ f(v_{i-1,j}) &= r^{2b}s, & f(v_{i,j}) &= r^{2b+2n}s \\ f(d_{i-1,j-1}) &= r^{2u+1}, & f(d_{i,j}) &= r^{2u+2g+1} \\ f(b_{i-1,j+1}) &= r^{2v+1}s, & f(b_{i,j}) &= r^{2v+2g+1}s \end{aligned}$$

for some $a, b, u, v \in \{0, 1, \ldots, mn-1\}$ and $g = \gcd(m, n)$.

Now the first four labels on the horizontal and vertical edges contribute to the weight of $x_{i,j}$ by

$$\begin{aligned} w'(x_{i,j}) &= f(h_{i,j})f(v_{i,j})f(h_{i,j-1})f(v_{i-1,j}) \\ &= (r^{2a+2n})(r^{2b+2n}s)(r^{2a})(r^{2b}s) \\ &= (r^{2a+2b+4n})(sr^{2a+2b}s) \\ &= r^{2a+2b+4n}r^{-(2a+2b)} \\ &= r^{4n}. \end{aligned}$$

The next four labels on the diagonal edges contribute to the weight of $x_{i,j}$ by

$$\begin{aligned} w''(x_{i,j}) &= f(b_{i-1,j+1})f(d_{i,j})f(b_{i,j})f(d_{i-1,j-1}) \\ &= (r^{2v+1}s)(r^{2u+2g+1})(r^{2v+2g+1}s)(r^{2u+1}) \\ &= (r^{2v+1})(sr^{2u+2v+4g+2}s)(r^{2u+1}) \\ &= r^{2v+1}r^{-(2u+2v+4g+2)}r^{2u+1} \\ &= r^{-4g}. \end{aligned}$$

But now we have

$$w(x_{i,j}) = w''(x_{i,j})w'(x_{i,j}) = r^{-4g}r^{4n} = r^{4n-4g}$$

for every vertex $x_{i,j}$ and the labeling is D_{4mn}-supermagic.

We illustrate the labeling by showing just a vertex $x_{i,j}$ and the incident edges in 5. We again use the simplified notation corresponding to the explanation in Construction 6.2.

By constructing the labeling above, we proved the following.

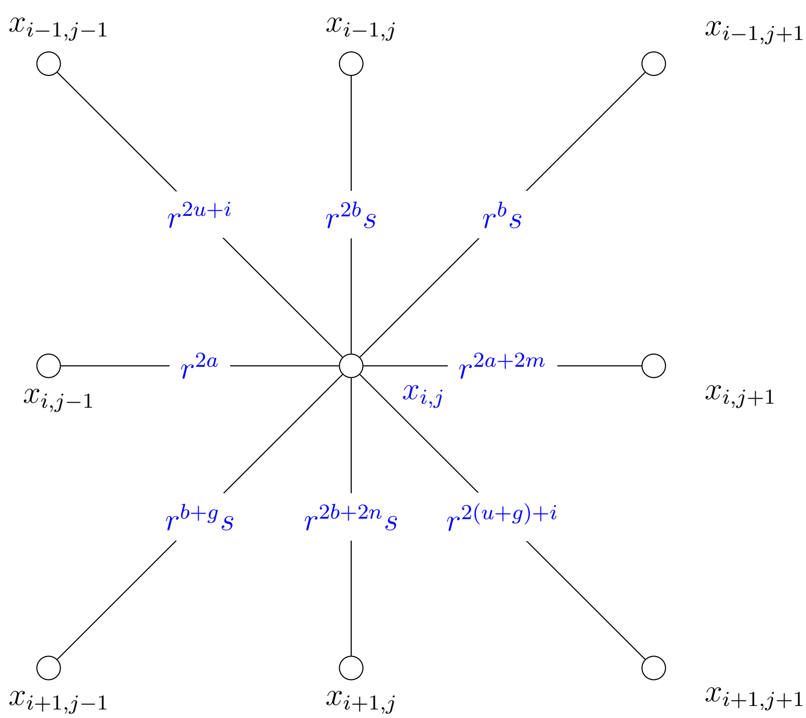

Figure 5. Edge labels in strong product

Theorem 6.3. *Let $m, n \geq 3$ and D_{4mn} be the dihedral group of order $4mn$. Then there exists a uniform D_{4mn}-supermagic labeling of the strong product $C_m \boxtimes C_n$.*

7. Conclusion

We continued research started in [2] by finding uniform revolving D_{2mn}-supermagic labelings of Cartesian and direct products of two cycles, and a uniform D_{4mn}-supermagic of strong product of two cycles. According to our knowledge, there have been so far no other attempts to find Γ-supermagic labelings of any graphs for a non-Abelian group Γ.

Looking at the results in this paper and in [2], we observe that all graphs with a known D_{2k}-supermagic labeling are d-regular of degree four or eight. In other words, $d \equiv 0 \pmod 4$. We do not currently know any d-regular D_{2k}-supermagic graphs for any positive integers k and d such that $d \equiv 2 \pmod 4$. Therefore, we pose the following open problem.

Open Problem. Find a d-regular D_{2k}-supermagic graph G with $2k$ edges for positive integers d and k such that $d \equiv 2 \pmod 4$.

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