Four new operations related to composition and their reformulated Zagreb Index

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Abstract

The first reformulated Zagreb index $EM_1(G)$ of a simple graph $G$ is defined as the sum of the terms $(d_u + d_v - 2)^2$ over all edges $uv$ of $G$. In 2017, Sarala et al. [3] introduced four new operations ($F$-product) of graphs. In this paper, we study the first reformulated Zagreb index for the $F$-product of some special well-known graphs such as subdivision and total graph.

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1. Introduction

For vertex $u \in V(G)$, the degree of the vertex $u$ in $G$, denoted by $d_G(u)$, is the number of edges incident with $u$ in $G$. A topological index of a graph is a parameter related to the graph; it does not depend on labeling or pictorial representation of the graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [5]. Several types of such indices exist, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index. Two of these topological indices are known under various names, the most commonly used ones are the first and second Zagreb indices.

The Zagreb indices have been introduced more than thirty years ago by Gutman and Trinajstić [6]. They are defined as $M_1(G) = \sum_{u \in V(G)} d_G(u)^2$ and $M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v)$. Note that the
first Zagreb index may also written as \( M_1(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v)) \). The Zagreb indices
are found to have applications in QSPR and QSAR studies as well, see [4]. For the survey on
theory and application of Zagreb indices see [9]. Feng et al. [7] have given a sharp bounds for the
Zagreb indices of graphs with a given matching number. Khalifeh et al. [8] have obtained the
Zagreb indices of the Cartesian product, composition, join, disjunction, and symmetric difference
of graphs.

Furtula and Gutman in [16] recently investigated this index and named this index as forgotten
topological index or \( F \)-index and showed that the predictive ability of this index is almost similar
to that of first Zagreb index and for the entropy and acetic factor, both of them yield correlation
coefficients greater than 0.95. The \( F \)-index of a graph \( G \) is defined as \( F(G) = \sum_{u \in V(G)} d^3_G(u) = \sum_{uv \in E(G)} (d^2_G(u) + d^2_G(v)). \)

Recently, Shirdel et al. [15] introduced a variant of the first Zagreb index called hyper-Zagreb
index. The hyper-Zagreb index of \( G \) is denoted by \( HM(G) \) and defined as \( HM(G) = \sum_{uv \in E(G)} (d(u) + d(v))^2 \). In [15], the hyper-Zagreb indices of the Cartesian product, composition, join and disjunc-
tion of graphs are obtained. The hyper Zagreb indices of some classes of chemical graphs are
obtained in [11, 13, 14]. Pattabiraman and Vijayaragavan have obtained the hyper-Zagreb indices
of some special classes of graphs [20]. Some upper and lower bounds on hyper-Zagreb index for a
connected graph are obtained by Falahati-Nezhad and Azari [19].

Milicević et al. [23] in 2004 reformulated the Zagreb indices in terms of edge-degrees instead of
vertex-degrees \( EM_1(G) = \sum_{e \in E(G)} d(e)^2 \), where \( d(e) \) denotes the degree of the edge \( e \) in \( G \), which
is defined by \( d(e) = d(u) + d(v) - 2 \) with \( e = uv \). The use of these descriptors in QSPR study
was also discussed in their report [23]. Reformulated Zagreb index, particularly its upper/lower
bounds has attracted recently theatention of many mathematicians and computer scientists, see
[21, 22, 23, 24, 25]. In this paper, we study the first reformulated Zagreb index for the \( F \)-product
of some special well-known graphs such as subdivision and total graph.

2. Main Results

The Cartesian product, \( G \square H \), of the graphs \( G \) and \( H \) has the vertex set \( V(G \square H) = V(G) \times V(H) \) and
\( (u, x)(v, y) \) is an edge of \( G \square H \) if \( u = v \) and \( xy \in E(H) \) or \( uv \in E(G) \) and \( x = y \). To
each vertex \( u \in V(G) \), there is an isomorphic copy of \( H \) in \( G \square H \) and to each vertex \( v \in V(H) \),
there is an isomorphic copy of \( G \) in \( G \square H \). The composition of two graphs \( G \) and \( H \) is denoted
by \( G[H] \). The vertex set of \( G[H] \) is \( V(G) \times V(H) \) and any two vertices \( (u_i, v_r) \) and \( (u_k, v_s) \) are
adjacent if and only if \( u_i u_k \in E(G) \) or \( [u_i = u_k \text{ and } v_r v_s \in E(H)] \).

For a connected graph \( G \), there are four related graphs as follows:

(i) The subdivision graph \( S(G) \) is the graph obtained from \( G \) by replacing each edge of \( G \) by a
path of length two.

(ii) \( R(G) \) is obtained from \( G \) by adding a new vertex corresponding to each edge of \( G \), then
joining each new vertex to the end vertices of the corresponding edge.
(iii) \( Q(G) \) is obtained from \( G \) by inserting a new vertex into each edge of \( G \), then joining with edges those pairs of new vertices on adjacent edges of \( G \).

(iv) The total graph \( T(G) \) has as its vertices the edges and vertices of \( G \). Adjacency in \( T(G) \) is defined as adjacency or incidence for the corresponding elements of \( G \), see Figure 1.

![Graphs](image)

Eliasi and Taeri [2] introduced the following four operations of the graphs \( G_1 \) and \( G_2 \) based on the Cartesian product of these graphs.

Let \( F \) be one of the symbols \( S, R, Q, \) or \( T \). The \( F \)-sum \( G_1 +_F H \) is a graph with the set of vertices \( V(G_1 +_F H) = (V(G) \cup E(G)) \times V(H) \) and two vertices \( (g_1, h_1) \) and \( (g_2, h_2) \) of \( G_1 +_F H \) are adjacent if and only if \( g_1 = g_2 \) and \( h_1h_2 \in E(H) \) or \( h_1 = h_2 \) and \( g_1g_2 \in F(G) \), see Figure 2. The Zagreb indices of the \( F \)-sum of graphs are obtained by Deng et al. [17]. The \( F \)-index of four operations on some special graphs are computed by Ghobadi and Ghorbaninejad [18]. Eliasi and Taeri[2] have obtained the Wiener index of four new sums of graphs.

In this sequence, Sarala et al. [3] introduced the following four operations of the graphs \( G_1 \) and \( G_2 \) based on the composition of these graphs.

Let \( F \) be one of the symbols \( S, R, Q \) or \( T \). The \( F \)-product \( G_1 \square F G_2 \), denoted by \( G_1(G_2)_F \) is defined by \( F(G_1)[G_2] - \mathcal{E}^* \), where \( \mathcal{E}^* = \{(x, y_1)(x, y_2) \in E(F(G_1)[G_2]) | x \in V(F(G_1)) - V(G_1), y_1y_2 \in E(G_2) \} \), that is, \( G_1[G_2]_F \) is a graph with the set of vertices either \( x_1 = x_2 \in V(G_1) \) and \( y_1y_2 \in V(G_2) \) or \( x_1x_2 \in V(G_1) \) and \( y_1y_2 \in V(G_2) \), see Figure 3. Sarala et al. [3] have obtained the Zagreb indices of \( F \)-product of graphs.
First we compute the first reformulated Zagreb index of the graph $G_1[G_2]_S$.

**Theorem 2.1.** Let $G_i$, be a graph with $n_i$ vertices and $m_i$ edges, $i = 1, 2$. Then

$$EM_1(G_1[G_2]_S) = n_1HM(G_2) + n_2^4F(G_1) + 2n_2^2(4m_2 + n_2(2n_2 - 2))M_1(G_1) + (10m_1n_2 - 4n_1)M_1(G_2) + 2n_2^2m_1(2n_2 - 2)^2 + 4n_1m_2 + 16(n_2 - 2)m_1m_2n_2.$$  

**Proof.** Let $\{x_1, x_2, \ldots, x_{n_1}\}$ and $\{y_1, y_2, \ldots, y_{n_2}\}$ be the vertex sets of $G_1$ and $G_2$, respectively. From the definition of first reformulated Zagreb index and the structure of the graph $G_1[G_2]_S$, we have

$$EM_1(G_1[G_2]_S) = \sum_{(x_1, y_1)(x_2, y_2) \in E(G_1[G_2]_S)} \left( d_{G_1[G_2]_S}((x_1, y_1)) + d_{G_1[G_2]_S}((x_2, y_2)) - 2 \right)^2$$

$$= \sum_{x_1=x_2 \in V(G_1)} \sum_{y_1y_2 \in E(G_2)} \left( d_{G_1[G_2]_S}((x_1, y_1)) + d_{G_1[G_2]_S}((x_2, y_2)) - 2 \right)^2$$

$$+ \sum_{x_1x_2 \in E(S(G_1))} \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \left( d_{G_1[G_2]_S}((x_1, y_1)) + d_{G_1[G_2]_S}((x_2, y_2)) - 2 \right)^2$$

$$= A_1 + A_2,$$

(1)

where $A_1$ and $A_2$ are the sums of the terms, in order.

We shall calculate $A_1$ and $A_2$ of (1) separately. First we calculate the sum

$$A_1 = \sum_{x_1=x_2 \in V(G_1)} \sum_{y_1y_2 \in E(G_2)} \left( d_{G_1[G_2]_S}((x_1, y_1)) + d_{G_1[G_2]_S}((x_2, y_2)) - 2 \right)^2.$$
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For each vertex \((x_i, y_j)\) in \(G_1[G_2]_S\), the degree of \((x_i, y_j)\) is \(n_2d_{G_1}(x_i) + d_{G_2}(y_j)\). Thus

\[
A_1 = \sum_{x_1 \in V(G_1)} \sum_{y_1, y_2 \in E(G_2)} \left( n_2d_{G_1}(x_1) + d_{G_2}(y_1) + n_2d_{G_1}(x_1) + d_{G_2}(y_2) - 2 \right)^2
\]

\[
= \sum_{x_1 \in V(G_1)} \sum_{y_1, y_2 \in E(G_2)} \left( 2n_2d_{G_1}(x_1) + (d_{G_2}(y_1) + d_{G_2}(y_2)) - 2 \right)^2
\]

\[
= \sum_{x_1 \in V(G_1)} \sum_{y_1, y_2 \in E(G_2)} \left( 4n_2^2d_{G_1}(x_1)^2 + (d_{G_2}(y_1) + d_{G_2}(y_2))^2 + 4 + 4n_2d_{G_1}(x_1)ight. \\
\left. (d_{G_2}(y_1) + d_{G_2}(y_2)) - 8n_2d_{G_1}(x_1) - 4(d_{G_2}(y_1) + d_{G_2}(y_2)) \right).
\]

From the definitions of first and hyper-Zagreb indices, we obtain:

\[
A_1 = 4n_2^2m_2M_1(G_1) + n_1HM(G_2) + (8n_2m_1 - 4n_1)M_1(G_2) + 4n_1m_2 - 16n_2m_1m_2.
\]
Next we find the value of the sum $A_2$.

\[
A_2 = \sum_{x_1,x_2 \in E(S(G_1))} \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \left( d_{G_1[G_2]}((x_1,y_1)) + d_{G_1[G_2]}((x_2,y_2)) - 2 \right)^2
\]

\[
= \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1 \in V(G_1)} \sum_{e \in E(G_1)} \left( d((x_1,y_1)) + d((e,y_2)) - 2 \right)^2
\]

\[
= \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x \in V(G_1)} \left( n_2 d_{G_1}(x) + d_{G_2}(y_1) + 2n_2 - 2 \right)^2
\]

\[
+ 2n_2 d_{G_1}(x_1) d_{G_2}(y_1) + 2n_2(2n_2 - 2) d_{G_1}(x_1) + 2(2n_2 - 2) d_{G_2}(y_1)
\]

By the definitions of $F$-index and first Zagreb index, we get

\[
A_2 = n_2^4 F(G_1) + (4n_2^2 m_2 + 2n_2^3(2n_2 - 2)) M_1(G_1) + 2n_2 m_1 M_1(G_2) + 2m_1 n_2^2 (2n_2 - 2)^2 + 8(2n_2 - 2)n_2 m_1 m_2.
\]

Adding $A_1$ and $A_2$, we obtain the required result. \square

Next we obtain the first reformulated Zagreb index of the graph $G_1[G_2]_R$.

**Theorem 2.2.** Let $G_i$ be a graph with $n_i$ vertices and $m_i$ edges, $i = 1, 2$. Then $EM_1(G_1[G_2]_R) = 4n_2^2 HM(G_1) + n_1 HM(G_2) + 4n_2^2 F(G_1) + 8n_2^2(5m_2 - n_2) M_1(G_1) + (20n_2 m_1 - 4n_1) M_1(G_2) + 4m_1 n_2^2 - 8m_1 m_2 (m_2 + 8n_2 - 2n_2^2)$.

**Proof.** By the definition of first reformulated Zagreb index and the structure of $G_1[G_2]_R$,

\[
EM_1(G_1[G_2]_R) = \sum_{(x_1,y_1)(x_2,y_2) \in E(G_1[G_2]_R)} \left( d_{G_1[G_2]}((x_1,y_1)) + d_{G_1[G_2]}((x_2,y_2)) - 2 \right)^2
\]

\[
= \sum_{x_1=x_2 \in V(G_1)} \sum_{y_1 y_2 \in E(G_2)} \left( d_{G_1[G_2]}((x_1,y_1)) + d_{G_1[G_2]}((x_2,y_2)) - 2 \right)^2
\]

\[
+ \sum_{x_1 x_2 \in E(R(G_1))} \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \left( d_{G_1[G_2]}((x_1,y_1)) + d_{G_1[G_2]}((x_2,y_2)) - 2 \right)^2
\]

\[
= A_1 + A_2,
\]

where $A_1$ and $A_2$ are the sums of the terms, in order.

We shall obtain the value of $A_1$ and $A_2$ of (2) separately.
\[ A_1 = \sum_{x_1 = x_2 \in V(G_1)} \sum_{y_1 y_2 \in E(G_2)} \left( d_{G_1[G_2]}((x_1, y_1)) + d_{G_1[G_2]}((x_2, y_2)) - 2 \right)^2 \]

\[ = \sum_{x_1 \in V(G_1)} \sum_{y_1 y_2 \in E(G_2)} \left( n_2 d_{R(G_1)}(x_1) + d_{G_2}(y_1) + n_2 d_{R(G_1)}(x_1) + d_{G_2}(y_2) - 2 \right)^2 \]

\[ = \sum_{x_1 \in V(G_1)} \sum_{y_1 y_2 \in E(G_2)} \left( 4n_2 d_{G_1}(x_1) + (d_{G_2}(y_1) + d_{G_2}(y_2)) - 2 \right)^2 \]

\[ = \sum_{x_1 \in V(G_1)} \sum_{y_1 y_2 \in E(G_2)} \left( 16n_2^2 d_{G_1}(x_1)^2 + (d_{G_2}(y_1) + d_{G_2}(y_2))^2 + 4 \right. \]

\[ + 8n_2 d_{G_1}(x_1) (d_{G_2}(y_1) + d_{G_2}(y_2)) - 16n_2 d_{G_1}(x_1) - 4(d_{G_2}(y_1) + d_{G_2}(y_2)) \]

\[ = 16n_2^2 m_2 M_1(G_1) + n_1 H M(G_2) + (16n_2 m_1 - 4n_1) M_1(G_2) + 4n_1 m_2 - 32n_2 m_1 m_2. \]

\[ A_2 = \sum_{x_1 x_2 \in E(R(G_1))} \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \left( d_{G_1[G_2]}((x_1, y_1)) + d_{G_1[G_2]}((x_2, y_2)) - 2 \right)^2 \]

\[ = \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1 x_2 \in V(G_1)} \left( d\left((x_1, y_1)\right) + d\left((x_2, y_2)\right) - 2 \right)^2 \]

\[ + \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1 x_2 \in E(R(G_1))} \left( d\left(x_1, y_1\right) + d\left(x_2, y_2\right) - 2 \right)^2 \]

\[ = A'_2 + A''_2, \quad (3) \]

where

\[ A'_2 = \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1 x_2 \in V(G_1)} \left( d\left(x_1, y_1\right) + d\left(x_2, y_2\right) - 2 \right)^2 \]

\[ = \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1 x_2 \in V(G_1)} \left( n_2 d_{R(G_1)}(x_1) + d_{G_2}(y_1) + n_2 d_{R(G_1)}(x_2) + d_{G_2}(y_2) - 2 \right)^2 \]

\[ = \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1 x_2 \in V(G_1)} \left( 2n_2 d_{G_1}(x_1) + d_{G_2}(y_1) + 2n_2 d_{G_1}(x_2) + d_{G_2}(y_2) - 2 \right)^2 \]

\[ = \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1 x_2 \in V(G_1)} \left( 4n_2^2 d_{G_1}(x_1)^2 + d_{G_2}(y_1)^2 + d_{G_2}(y_2)^2 + 4 \right. \]

\[ + 2d_{G_2}(y_1) d_{G_2}(y_2) - 4d_{G_2}(y_1) - 4d_{G_2}(y_2) + \left(4n_2 d_{G_2}(y_1) + 4n_2 d_{G_2}(y_2) - 8n_2 \right) \]

\[ d_{G_1}(x_1) + d_{G_1}(x_2) \right) \]

\[ = 4n_2^4 H M(G_1) + 2m_1 n_2 M_1(G_2) + 4n_2 (4m_2 n_2 - 2n_2^2) M_1(G_1) + m_1 (4n_2^2 - 8m_2^2 - 16n_2 m_2). \]
and

\[ A_2'' = \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1, x_2 \in E(R(G_1))} \left( d(x_1, y_1) + d(x_2, y_2) - 2 \right)^2 \]

\[ = \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1, x_2 \in E(R(G_1))} \left( n_2 d_{R(G_1)}(x_1) + d_{G_2}(y_1) + n_2 d_{R(G_1)}(x_2) - 2 \right)^2 \]

\[ = \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1, x_2 \in E(R(G_1))} \left( 2n_2 d_{G_1}(x_1) + d_{G_2}(y_1) + 2n_2 - 2 \right)^2 \]

\[ = \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1, y_1 \in V(G_1)} \left( 4n_2^2 d_{G_1}(x_1)^2 + d_{G_2}(y_1)^2 + (2n_2 - 2)^2 \right) + 4n_2 d_{G_1}(x_1) d_{G_2}(y_1) + 4n_2 (2n_2 - 2) d_{G_1}(x_1) + 2(2n_2 - 2) d_{G_2}(y_1) \]

\[ = 4n_2^2 F(G_1) + 2m_1 n_2 M_1(G_2) + 8(2n_2 - 2)n_2 m_1 m_2 + (8n_2^2 m_2 + 4n_2^3 (2n_2 - 2))M_1(G_1). \]

From \( A_2' \) and \( A_2'' \), we have \( A_2 = 4n_2^3 HM(G_1) + 4n_2^2 F(G_1) + 4m_1 n_2 M_1(G_2) + 8n_2^2 (3m_2 - n_2) M_1(G_1) + 4n_2^3 m_1 - 8m_1 m_2 (m_2 + 4n_2 - 2n_3). \)

Using (2) and the sums \( A_1, A_2 \) we obtain the desired result. \qed

Next we find the first reformulated Zagreb index of \( G_1[G_2]_Q \).

**Theorem 2.3.** Let \( G_i \) be a graph with \( n_i \) vertices and \( m_i \) edges, \( i = 1, 2 \). Then \( EM_1(G_1[G_2]_Q) = 2n_1^2 HM(G_1) + n_1 LM(G_2) + 3n_1^2 F(G_1) + n_2^2(n_1^2 LM(L(G_1))) + 2n_2 (4n_2 - 2) M_1(L(G_1))) + n_2^2 (16m_2 - 8m_2 - 2) M_1(G_1) + (10n_2 m_1 - 4n_1) M_1(G_2) + 4n_2^4 M_2(G_1) + 4m_1 m_2 + 8m_2 n_1 (n_2 - 4m_2) - n_2^3 m_1 (n_2 - 2)^2. \)

**Proof.** By the definition of first reformulated Zagreb index,

\[ EM_1(G_1[G_2]_Q) = \sum_{(x_1, y_1)(x_2, y_2) \in E(G_1[G_2]_Q)} \left( d_{G_1[G_2]_Q}((x_1, y_1)) + d_{G_1[G_2]_Q}((x_2, y_2)) - 2 \right)^2 \]

\[ = \sum_{x_1 = x_2 \in V(G_1)} \sum_{y_1, y_2 \in V(G_2)} \left( d_{G_1[G_2]_Q}((x_1, y_1)) + d_{G_1[G_2]_Q}((x_2, y_2)) - 2 \right)^2 \]

\[ + \sum_{x_1, x_2 \in E(Q(G_1))} \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \left( d_{G_1[G_2]_Q}((x_1, y_1)) + d_{G_1[G_2]_Q}((x_2, y_2)) - 2 \right)^2 \]

\[ = A_1 + A_2, \quad (4) \]

where \( A_1 \) and \( A_2 \) are the sums of the terms, in order.
We shall calculate $A_1$ and $A_2$ of 4 separately.

\[
A_1 = \sum_{x_1=x_2 \in V(G_1)} \sum_{y_1,y_2 \in E(G_2)} \left( d_{G_1|G_2|Q}((x_1, y_1)) + d_{G_1|G_2|Q}((x_2, y_2)) - 2 \right)^2
\]

\[
= \sum_{x_1 \in V(G_1)} \sum_{y_1,y_2 \in E(G_2)} \left( n_2 d_Q(G_1)(x_1) + d_{G_2}(y_1) + n_2 d_Q(G_1)(x_1) + d_{G_2}(y_2) - 2 \right)^2
\]

\[
= \sum_{x_1 \in V(G_1)} \sum_{y_1,y_2 \in E(G_2)} \left( 2n_2 d_{G_1}(x_1) + (d_{G_2}(y_1) + d_{G_2}(y_2)) - 2 \right)^2
\]

\[
= \sum_{x_1 \in V(G_1)} \sum_{y_1,y_2 \in E(G_2)} \left( 4n_2^2 d_{G_1}(x_1)^2 + (d_{G_2}(y_1) + d_{G_2}(y_2))^2 + 4
\]

\[
+ 4n_2 d_{G_1}(x_1)(d_{G_2}(y_1) + d_{G_2}(y_2)) - 8n_2 d_{G_1}(x_1) - 4(d_{G_2}(y_1) + d_{G_2}(y_2)) \right)
\]

\[
= 4n_2^2 m_2 M_1(G_1) + n_1 H M(G_2) + (8n_2 m_1 - 4n_1) M_1(G_2) + 4n_1 m_2 - 16n_2 m_1 m_2.
\]

\[
A_2 = \sum_{x_1 x_2 \in E(Q(G_1))} \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \left( d_{G_1|G_2|Q}((x_1, y_1)) + d_{G_1|G_2|Q}((x_2, y_2)) - 2 \right)^2
\]

\[
= \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1 x_2 \in E(Q(G_1))} \left( d(x_1, y_1) + d(x_2, y_2) - 2 \right)^2
\]

\[
+ \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1 x_2 \in E(Q(G_1))} \left( d(x_1, y_1) + d(x_2, y_2) - 2 \right)^2
\]

\[
= A_2' + A_2'',
\]
where

\[
A'_2 = \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1, x_2 \in E(Q(G_1))} \left( d((x_1, y_1)) + d((x_2, y_2)) - 2 \right)^2 \\
= \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1, x_2 \in E(Q(G_1))} \left( n_2 d_{Q(G_1)}(x_1) + d_{G_1}(y_1) + n_2 d_{Q(G_1)}(x_2) - 2 \right)^2 \\
= \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1, x_2 \in E(Q(G_1))} \left( n_2 d_{G_1}(x_1) + d_{G_1}(y_1) + n_2 d_{Q(G_1)}(x_2) - 2 \right)^2 \\
+ 2n_2 d_{G_1}(x_1)(d_{G_2}(y_1) - 2) + 2n_2 d_{G_1}(x_1)d_{Q(G_1)}(x_2) + 2n_2(d_{G_2}(y_1) - 2)d_{Q(G_1)}(x_2) \\
= n_1^2 F(G_1) + 2n_2^2 (2m_2 - 2n_2) M_1(G_1) + 2n_2 m_1 (M_1(G_2) + 4n_2 - 8m_2) \\
+ \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1, x_2 \in E(Q(G_1))} \left( n_2^2 d_{Q(G_1)}(x_2)^2 + (2n_2^2 d_{G_1}(x_1) \\
+ 2n_2(d_{G_2}(y_1) - 2))d_{Q(G_1)}(x_2) \right)
\]

One can see that for a vertex \( x_2 \in V(Q(G_1)) - V(G_1) \), \( d_{Q(G_1)}(x_2) = d_{G_1}(x) + d_{G_1}(w) \), where \( x_2 = xw \in E(G_1) \). Thus

\[
A'_2 = n_1^2 F(G_1) + 2n_2^2 (2m_2 - 2n_2) M_1(G_1) + 2n_2 m_1 (M_1(G_2) + 4n_2 - 8m_2) \\
+ \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1, x_2 \in E(Q(G_1))} \left( n_2^2 (d_{G_1}(x) + d_{G_1}(w))^2 \\
+ 2n_2^2 d_{G_1}(x_1)(d_{G_1}(x) + d_{G_1}(w)) + 2n_2(d_{G_2}(y_1) - 2)(d_{G_1}(x) + d_{G_1}(w)) \right) \\
= n_1^2 F(G_1) + 2n_2^2 (2m_2 - 2n_2) M_1(G_1) + 2n_2 m_1 (M_1(G_2) + 4n_2 - 8m_2) \\
+ 2n_2^4 H M(G_1) + 2n_1^2 F(G_1) + 2M_1(G_2) + 4n_2^2 (2m_2 - 2n_2) M_1(G_1) \\
+ 2n_1^2 H M(G_1) + 3n_2^4 F(G_1) + 12n_2^2 (m_2 - n_2) M_1(G_1) + 2n_2 m_1 M_1(G_2) \\
+ 4n_2^4 M_2(G_1) + 2m_1 n_2 (4n_2 - 8m_2).
Finally, we obtain the first formulated Zagreb index of $G$ from the sums
\[
A''_2 = \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1 x_2 \in E(Q(G_1))} (d(x_1, y_1) + d(x_2, y_2) - 2)^2
\]

\[
= \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1 x_2 \in E(Q(G_1))} (n_2 d_{Q(G_1)}(x_1) + n_2 d_{Q(G_1)}(x_2) - 2)^2
\]

\[
= n_2^2 \sum_{w_1, w_2 \in V(G_1)} \left( n_2 (d_G(w_i) + d_G(w_j)) + n_2 (d_G(w_i) + d_G(w_k)) - 2 \right)^2
\]

\[
= n_2^2 \sum_{X_i, X_j \in E(L(G_1))} \left( n_2 (d_{L(G_1)}(X_i) + d_{L(G_1)}(X_j)) + 4n_2 - 2 \right)^2 + 2n_2(4n_2 - 2)(d_{L(G_1)}(X_i) + d_{L(G_1)}(X_j)) + (4n_2 - 2)^2 (\frac{M_1(G_1)}{2} - m_1)
\]

From the sums $A'_2$ and $A''_2$, we have $A_2 = 2n_2^2 HM(G_1) + 3n_2^2 F(G_1) + n_2^2 (12m_2 - 8n_2 - 2) M_1(G_1) + 2n_2 m_1 M_1(G_2) + 4n_2^2 M_2(G_1) + n_2^2 (n_2^2 HM(L(G_1)) + 2n_2(4n_2 - 2) M_1(L(G_1)) + m_1 n_2 (4n_2 - 8m_2) - n_2^2 m_1 (4n_2 - 2)^2$.

Adding $A_1$ and $A_2$, we get the desired result.

Finally, we obtain the first formulated Zagreb index of $G_1 \square G_2_T$.

**Theorem 2.4.** Let $G_i$ be a graph with $n_i$ vertices and $m_i$ edges, $i = 1, 2$. Then $EM_1(G_1 \square G_2_T) = 10n_2^2 HM(G_1) + n_1 HM(G_2) + n_2^2 F(G_1) + n_2^2 HM(L(G_1)) + 2n_2^2 (4n_2 - 2) M_1(L(G_1)) + (32n_2^2 m_2 + 16n_2 m_2 - 16n_2^2 - 8n_2^2 + 2(2n_2 - 1)^2) M_1(G_1) + (20m_1 n_2 - 4n_1) M_1(G_2) + 4n_1 m_2 - 8m_1 m_2 (8n_1 + m_2) - n_2^2 m_1 ((4n_2 - 2)^2 - 12)$. 


Proof. By the definition of first formulated Zagreb index,

\[ EM_1(G_1[G_2]_T) = \sum_{(x_1,y_1)(x_2,y_2)\in E(G_1[G_2]_T)} \left( d_{G_1[G_2]_T}((x_1,y_1)) + d_{G_1[G_2]_T}((x_2,y_2)) - 2 \right)^2 \]

\[ = \sum_{x_1=x_2\in V(G_1)} \sum_{y_1,y_2\in E(G_2)} \left( d_{G_1[G_2]_T}((x_1,y_1)) + d_{G_1[G_2]_T}((x_2,y_2)) - 2 \right)^2 \]

\[ + \sum_{x_1x_2\in E(T(G_1))} \sum_{y_1\in V(G_2)} \sum_{y_2\in V(G_2)} \left( d_{G_1[G_2]_T}((x_1,y_1)) + d_{G_1[G_2]_T}((x_2,y_2)) - 2 \right)^2 \]

\[ = A_1 + A_2 + A_3 + A_4, \tag{5} \]

where \( A_1 \) to \( A_4 \) are the sums of the terms, in order.

We shall calculate \( A_1 \) to \( A_4 \) of 5 separately. A similar arguments of \( A_1 \) and \( A_2' \) in Theorem 2.2, we have

\[ A_1 = \sum_{x_1=x_2\in V(G_1)} \sum_{y_1,y_2\in E(G_2)} \left( d_{G_1[G_2]_T}((x_1,y_1)) + d_{G_1[G_2]_T}((x_2,y_2)) - 2 \right)^2 \]

\[ = 16n_2^2m_2M_1(G_1) + n_1HM(G_2) + (16n_2m_1 - 4m_1)M_1(G_2) + 4n_1m_2 - 32n_2m_1m_2. \]

and

\[ A_2 = \sum_{y_1\in V(G_2)} \sum_{y_2\in E(G_2)} \sum_{x_1x_2\in E(G_1)} \left( d_{G_1[G_2]_T}((x_1,y_1)) + d_{G_1[G_2]_T}((x_2,y_2)) - 2 \right)^2 \]

\[ = 4n_2^2HM(G_1) + 2m_1n_2M_1(G_2) + 4n_2(4m_2n_2 - 2n_2^2)M_1(G_1) + m_1(4n_2^2 - 8m_2^2 - 16n_2m_2). \]
A_3 = \sum_{y_1 \in V(G_2)} \sum_{y_2 \in E(G_2)} \sum_{x_1, x_2 \in E(T(G_1))} \left( d_{G_1[G_2]^T}((x_1, y_1)) + d_{G_1[G_2]^T}((x_2, y_2)) - 2 \right)^2
= \sum_{y_1 \in V(G_2)} \sum_{y_2 \in E(G_2)} \sum_{x_1, x_2 \in E(T(G_1))} \left( d_{T(G_1)}(x_1) + d_{G_2}(y_2) + d_{T(G_1)}(x_2) - 2 \right)^2
= \sum_{y_1 \in V(G_2)} \sum_{y_2 \in E(G_2)} \sum_{x_1, x_2 \in E(T(G_1))} \left( d_{R(G_1)}(x_1) + d_{G_2}(y_2) - 1 \right)^2 + d_{Q(G_1)}(x_2)^2
+ 2d_{R(G_1)}(x_1)(d_{G_2}(y_2) - 2) + 2(d_{G_2}(y_2) - 2)d_{Q(G_1)}(x_2) + 2d_{R(G_1)}(x_1)d_{Q(G_1)}(x_2)^2
= \sum_{y_1 \in V(G_2)} \sum_{y_2 \in E(G_2)} \sum_{x_1, x_2 \in E(T(G_1))} \left( 4d_{G_1}(x_1) + (d_{G_2}(y_2) - 1)^2 + d_{Q(G_1)}(x_2)^2
+ 4d_{G_1}(x_1)(d_{G_2}(y_2) - 1) + 2(d_{G_2}(y_2) - 1)d_{Q(G_1)}(x_2) + 2d_{R(G_1)}(x_1)d_{Q(G_1)}(x_2) \right)^2
= 4n_2F(G_1) + 2n_2m_1M_1(G_2) + 6n_2^2HM(G_1) + (16n_2m_2 - 16n_2^2)M_1(G_1) + 8m_1n_2(n_2 - 2m_2).

A similar argument of \(A_4''\) in Theorem 2.3, we obtain

\[ A_4 = \sum_{y_1 \in V(G_2)} \sum_{y_2 \in V(G_2)} \sum_{x_1, x_2 \in E(Q(G_1))} \left( d_{G_1[G_2]^T}((x_1, y_1)) + d_{G_1[G_2]^T}((x_2, y_2)) - 2 \right)^2 \]
\[ = n_2^2 \left( n_2^2HM(L(G_1)) + 2n_2(4n_2 - 2)M_1(L(G_1)) + (4n_2 - 2)^2 \left( \frac{M_1(G_1)}{2} - m_1 \right) \right). \]

Adding the sums \(A_1\) to \(A_4\), we get the desired result. \(\Box\)

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**References**


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