Indonesian Journal of Combinatorics **2** (1) (2018), 35–49

INDONESIAN JOURNAL OF COMBINATORICS

Four new operations related to composition and their reformulated Zagreb Index

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Abstract

The first reformulated Zagreb index $EM_1(G)$ of a simple graph G is defined as the sum of the terms $(d_u + d_v - 2)^2$ over all edges uv of G. In 2017, Sarala et al. [3] introduced four new operations(F-product) of graphs. In this paper, we study the first reformulated Zagreb index for the F-product of some special well-known graphs such as subdivision and total graph.

Keywords: Reformulated Zagreb index, composition, subdivision, total graph.
Mathematics Subject Classification : 05C12, 05C76
DOI: 10.19184/ijc.2018.2.1.5

1. Introduction

For vertex $u \in V(G)$, the degree of the vertex u in G, denoted by $d_G(u)$, is the number of edges incident with u in G. A *topological index* of a graph is a parameter related to the graph; it does not depend on labeling or pictorial representation of the graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [5]. Several types of such indices exist, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index. Two of these topological indices are known under various names, the most commonly used ones are the first and second Zagreb indices.

The Zagreb indices have been introduced more than thirty years ago by Gutman and Trinajestić [6]. They are defined as $M_1(G) = \sum\limits_{u \in V(G)} d_G(u)^2$ and $M_2(G) = \sum\limits_{uv \in E(G)} d_G(u) d_G(v)$. Note that the

Received: 30 Aug 2017, Revised: 29 Sep 2017, Accepted: 01 May 2018.

first Zagreb index may also written as $M_1(G) = \sum\limits_{uv \in E(G)} (d_G(u) + d_G(v))$. The Zagreb indices are found to have appilications in QSPR and QSAR studies as well, see [4]. For the survey on theory and application of Zagreb indices see [9]. Feng et al.[7] have given a sharp bounds for the Zagreb indices of graphs with a given matching number. Khalifeh et al. [8] have obtained the Zagreb indices of the Cartesian product, composition, join, disjunction, and symmetric difference of graphs.

Furtula and Gutman in [16] recently investigated this index and named this index as *forgotten topological index* or F-index and showed that the predictive ability of this index is almost similar to that of first Zagreb index and for the entropy and acetic factor, both of them yield correlation coefficients greater than 0.95. The F-index of a graph G is defined as $F(G) = \sum\limits_{u \in V(G)} d_G^3(u) = \sum\limits_{uv \in E(G)} (d_G^2(u) + d_G^2(v))$.

Recently, Shirdel et al.[15] introduced a variant of the first Zagreb index called hyper-Zagreb index. The *hyper-Zagreb index* of G is denoted by $HM(G)$ and defined as $HM(G) = \sum\limits_{uv \in E(G)} (d(u)+d(v))^2$. In [15], the hyper-Zagreb indices of the Cartesian product, composition, join and disjunction of graphs are obtained. The hyper Zagreb indices of some classes of chemical graphs are obtained in [11, 13, 14]. Pattabiraman and Vijayaragavan have obtained the hyper-Zagreb indices of some special classes of graphs[20]. Some upper and lower bounds on hyper-Zagreb index for a connected graph are obtained by Falahati-Nezhad and Azari[19].

Milicević et al. [23] in 2004 reformulated the Zagreb indices in terms of edge-degrees instead of vertex-degrees $EM_1(G) = \sum\limits_{e \in E(G)} d(e)^2$, where $d(e)$ denotes the degree of the edge e in G, which is defined by $d(e) = d(u) + d(v) - 2$ with $e = uv$. The use of these descriptors in QSPR study was also discussed in their report [23]. Reformulated Zagreb index, particularly its upper/lower bounds has attracted recently theat tention of many mathematicians and computer scientists, see [21, 22, 23, 24, 25]. In this paper, we study the first reformulated Zagreb index for the F-product of some special well-known graphs such as subdivision and total graph.

2. Main Results

The *Cartesian product,* $G \,\square\, H$, of the graphs G and H has the vertex set $V(G \,\square\, H) = V(G) \times V(H)$ and $(u,x)(v,y)$ is an edge of $G \,\square\, H$ if $u = v$ and $xy \in E(H)$ or $uv \in E(G)$ and $x = y$. To each vertex $u \in V(G)$, there is an isomorphic copy of H in $G \,\square\, H$ and to each vertex $v \in V(H)$, there is an isomorphic copy of G in $G \,\square\, H$. The *composition* of two graphs G and H is denoted by $G[H]$. The vertex set of $G[H]$ is $V(G) \times V(H)$ and any two vertices (u_i, v_r) and (u_k, v_s) are adjacent if and only if $u_i u_k \in E(G)$ or $[u_i = u_k \; and \; v_r v_s \in E(H)]$.

For a connected graph G, there are four related graphs as follows:

(i) The *subdivision graph* $S(G)$ is the graph obtained from G by replacing each edge of G by a path of length two.

(ii) $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge.

(iii) $Q(G)$ is obtained from G by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G.

(iv) The *total graph* $T(G)$ has as its vertices the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G, see Figure 1.

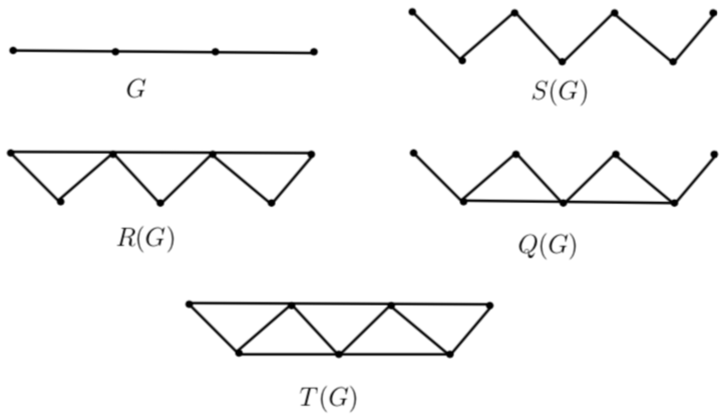

Figure 1. The graph G its $S(G)$,$R(G)$,$Q(G)$, and $S(G)$.

Eliasi and Taeri [2] introduced the following four operations of the graphs G_1 and G_2 based on the Cartesian product of these graphs.

Let F be one of the symbols S, R, Q, or T. The F-*sum* $G +_F H$ is a graph with the set of vertices $V(G +_F H) = (V(G) \cup E(G)) \times V(H)$ and two vertices (g_1, h_1) and (g_2, h_2) of $G +_F H$ are adjacent if and only if $g_1 = g_2$ and $h_1h_2 \in E(H)$ or $h_1 = h_2$ and $g_1g_2 \in F(G)$, see Figure 2. The Zagreb indices of the F-sum of graphs are obtained by Deng et al. [17]. The F-index of four operations on some special graphs are computed by Ghobadi and Ghorbaninejad [18]. Eliasi and Taeri[2] have obtained the Wiener index of four new sums of graphs.

In this sequence, Sarala et al. [3] introduced the following four operations of the graphs G_1 and G_2 based on the composition of these graphs.

Let F be one of the symbols S, R, Q or T. The F-*product* of G_1 and G_2, denoted by $G_1[G_2]_F$ is defined by $F(G_1)[G_2] - E^*$, where $E^* = \{(x, y_1)(x, y_2) \in E(F(G_1)[G_2]) | x \in V(F(G_1)) - V(G_1),\ y_1y_2 \in E(G_2)\}$, that is, $G_1[G_2]_F$ is a graph with the set of vertices either $[x_1 = x_2 \in V(G_1)$ and $y_1y_2 \in E(G_2)]$ or $[x_1x_2 \in E(G_1)$ and $y_1, y_2 \in V(G_2)]$, see Figure 3. Sarala et al. [3] have obtained the Zagreb indices of F-product of graphs.

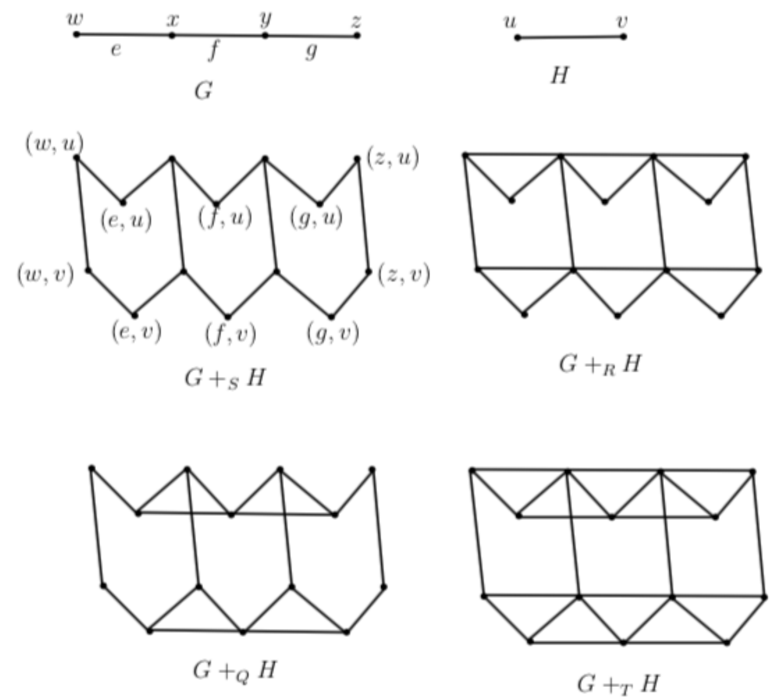

Figure 2. The graph G, H, and its $G +_F H$.

First we compute the first reformulated Zagreb index of the graph $G_1[G_2]_S$.

Theorem 2.1. *Let G_i be a graph with n_i vertices and m_i edges, $i = 1, 2$. Then* $EM_1(G_1[G_2]_S) = n_1HM(G_2) + n_2^4F(G_1) + 2n_2^2(4m_2 + n_2(2n_2 - 2))M_1(G_1) + (10m_1n_2 - 4n_1)M_1(G_2) + 2n_2^2m_1(2n_2 - 2)^2 + 4n_1m_2 + 16(n_2 - 2)m_1m_2n_2.$

Proof. Let $\{x_1, x_2, \ldots, x_{n_1}\}$ and $\{y_1, y_2, \ldots, y_{n_2}\}$ be the vertex sets of G_1 and G_2, respectively. From the definition of first reformulated Zagreb index and the structure of the graph $G_1[G_2]_S$, we have

$$\begin{aligned}
EM_1(G_1[G_2]_S) &= \sum_{(x_1,y_1)(x_2,y_2)\in E(G_1[G_2]_s)} \Big(d_{G_1[G_2]_S}((x_1, y_1)) + d_{G_1[G_2]_S}((x_2, y_2)) - 2\Big)^2 \\
&= \sum_{x_1=x_2\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big(d_{G_1[G_2]_S}((x_1, y_1)) + d_{G_1[G_2]_S}((x_2, y_2)) - 2\Big)^2 \\
&\quad + \sum_{x_1x_2\in E(S(G_1))} \sum_{y_1\in V(G_2)} \sum_{y_2\in V(G_2)} \Big(d_{G_1[G_2]_S}((x_1, y_1)) + d_{G_1[G_2]_S}((x_2, y_2)) - 2\Big)^2 \\
&= A_1 + A_2, \qquad (1)
\end{aligned}$$

where A_1 and A_2 are the sums of the terms, in order.
We shall calculate A_1 and A_2 of (1) separately. First we calculate the sum

$$A_1 = \sum_{x_1=x_2\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big(d_{G_1[G_2]_s}((x_1, y_1)) + d_{G_1[G_2]_s}((x_2, y_2)) - 2\Big)^2.$$

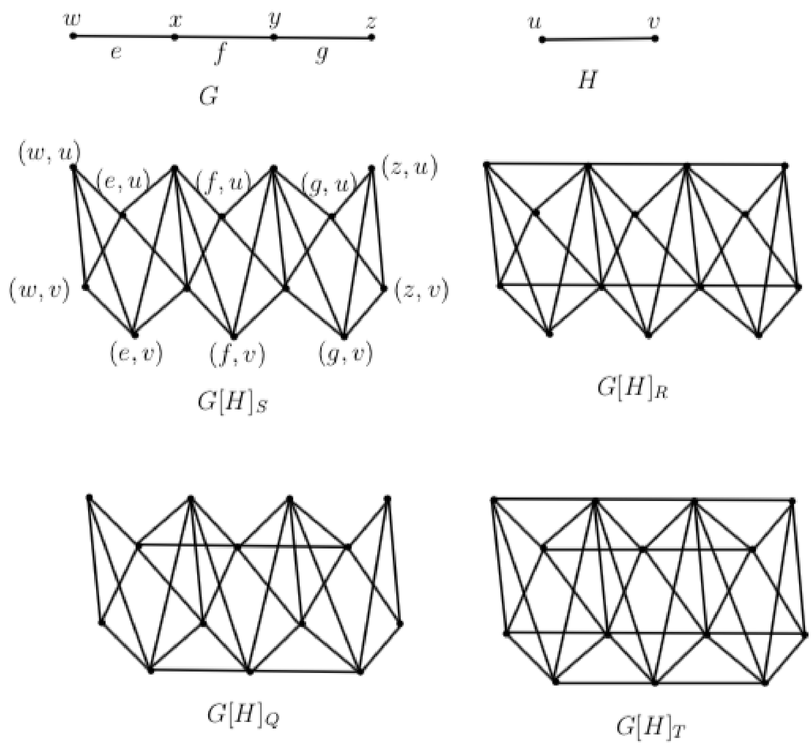

Figure 3. The graph G, H, and its $G[H]_T$.

For each vertex (x_i, y_j) in $G_1[G_2]_S$, the degree of (x_i, y_j) is $n_2 d_{G_1}(x_i) + d_{G_2}(y_j)$. Thus

$$\begin{aligned}
A_1 &= \sum_{x_1 \in V(G_1)} \sum_{y_1y_2 \in E(G_2)} \Big(n_2 d_{G_1}(x_1) + d_{G_2}(y_1) + n_2 d_{G_1}(x_1) + d_{G_2}(y_2) - 2\Big)^2 \\
&= \sum_{x_1 \in V(G_1)} \sum_{y_1y_2 \in E(G_2)} \Big(2n_2 d_{G_1}(x_1) + (d_{G_2}(y_1) + d_{G_2}(y_2)) - 2\Big)^2 \\
&= \sum_{x_1 \in V(G_1)} \sum_{y_1y_2 \in E(G_2)} \Big(4n_2^2 d_{G_1}(x_1)^2 + (d_{G_2}(y_1) + d_{G_2}(y_2))^2 + 4 + 4n_2 d_{G_1}(x_1) \\
&\quad (d_{G_2}(y_1) + d_{G_2}(y_2)) - 8n_2 d_{G_1}(x_1) - 4(d_{G_2}(y_1) + d_{G_2}(y_2))\Big).
\end{aligned}$$

From the definitions of first and hyper-Zagreb indices, we obtain:
$A_1 = 4n_2^2 m_2 M_1(G_1) + n_1 HM(G_2) + (8n_2 m_1 - 4n_1) M_1(G_2) + 4n_1 m_2 - 16 n_2 m_1 m_2.$

Next we find the value of the sum A_2.

$$\begin{aligned}
A_2 &= \sum_{x_1x_2\in E(S(G_1))}\sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\Big(d_{G_1[G_2]_s}((x_1,y_1))+d_{G_1[G_2]_s}((x_2,y_2))-2\Big)^2\\
&= \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{\substack{x_1\in V(G_1),\,e\in E(G_1)\\ x_1 \text{ and } e \text{ are incident in } G_1}}\Big(d((x,y_1))+d((e,y_2))-2\Big)^2\\
&= \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{\substack{x_1\in V(G_1),\,e\in E(G_1)\\ x_1 \text{ and } e \text{ are incident in } G_1}}\Big(n_2d_{G_1}(x_1)+d_{G_2}(y_1)+2n_2-2\Big)^2\\
&= \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{x\in V(G_1)}d_{G_1}(x_1)\Big(n_2^2d_{G_1}(x_1)^2+d_{G_2}(y_1)^2+(2n_2-2)^2\\
&\quad +2n_2d_{G_1}(x_1)d_{G_2}(y_1)+2n_2(2n_2-2)d_{G_1}(x_1)+2(2n_2-2)d_{G_2}(y_1)\Big).
\end{aligned}$$

By the definitions of F-index and first Zagreb index, we get

$$\begin{aligned}
A_2 &= n_2^4F(G_1)+(4n_2^2m_2+2n_2^3(2n_2-2))M_1(G_1)+2n_2m_1M_1(G_2)+2m_1n_2^2(2n_2-2)^2\\
&\quad +8(2n_2-2)n_2m_1m_2.
\end{aligned}$$

Adding A_1 and A_2, we obtain the required result. □

Next we obtain the first reformulated Zagreb index of the graph $G_1[G_2]_R$.

Theorem 2.2. *Let G_i be a graph with n_i vertices and m_i edges, $i = 1, 2$. Then* $EM_1(G_1[G_2]_R) = 4n_2^4HM(G_1)+n_1HM(G_2)+4n_2^4F(G_1)+8n_2^2(5m_2-n_2)M_1(G_1)+(20n_2m_1-4n_1)M_1(G_2)+4m_1n_2^2-8m_1m_2(m_2+8n_2-2n_2^2)$.

Proof. By the definition of first reformulated Zagreb index and the structure of $G_1[G_2]_R$,

$$\begin{aligned}
EM_1(G_1[G_2]_R) &= \sum_{(x_1,y_1)(x_2,y_2)\in E(G_1[G_2]_R)}\Big(d_{G_1[G_2]_R}((x_1,y_1))+d_{G_1[G_2]_R}((x_2,y_2))-2\Big)^2\\
&= \sum_{x_1=x_2\in V(G_1)}\sum_{y_1y_2\in E(G_2)}\Big(d_{G_1[G_2]_R}((x_1,y_1))+d_{G_1[G_2]_R}((x_2,y_2))-2\Big)^2\\
&\quad +\sum_{x_1x_2\in E(R(G_1))}\sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\Big(d_{G_1[G_2]_R}((x_1,y_1))+d_{G_1[G_2]_R}((x_2,y_2))-2\Big)^2\\
&= A_1+A_2, \qquad (2)
\end{aligned}$$

where A_1 and A_2 are the sums of the terms, in order.
We shall obtain the value of A_1 and A_2 of (2) separately.

$$\begin{aligned}
A_1 &= \sum_{x_1=x_2\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big(d_{G_1[G_2]_R}((x_1,y_1)) + d_{G_1[G_2]_R}((x_2,y_2)) - 2\Big)^2 \\
&= \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big(n_2 d_{R(G_1)}(x_1) + d_{G_2}(y_1) + n_2 d_{R(G_1)}(x_1) + d_{G_2}(y_2) - 2\Big)^2 \\
&= \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big(4n_2 d_{G_1}(x_1) + (d_{G_2}(y_1) + d_{G_2}(y_2)) - 2\Big)^2 \\
&= \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big(16n_2^2 d_{G_1}(x_1)^2 + (d_{G_2}(y_1) + d_{G_2}(y_2))^2 + 4 \\
&\quad +8n_2 d_{G_1}(x_1)(d_{G_2}(y_1) + d_{G_2}(y_2)) - 16n_2 d_{G_1}(x_1) - 4(d_{G_2}(y_1) + d_{G_2}(y_2))\Big) \\
&= 16n_2^2 m_2 M_1(G_1) + n_1 HM(G_2) + (16n_2m_1 - 4n_1)M_1(G_2) + 4n_1m_2 - 32n_2m_1m_2.
\end{aligned}$$

$$\begin{aligned}
A_2 &= \sum_{x_1x_2\in E(R(G_1))} \sum_{y_1\in V(G_2)} \sum_{y_2\in V(G_2)} \Big(d_{G_1[G_2]_R}((x_1,y_1)) + d_{G_1[G_2]_R}((x_2,y_2)) - 2\Big)^2 \\
&= \sum_{y_1\in V(G_2)} \sum_{y_2\in V(G_2)} \sum_{x_1x_2\in V(G_1)} \Big(d((x_1,y_1)) + d((x_2,y_2)) - 2\Big)^2 \\
&\quad + \sum_{y_1\in V(G_2)} \sum_{y_2\in V(G_2)} \sum_{\substack{x_1x_2\in E(R(G_1)) \\ x_1\in V(G_1),\, x_2\in V(R(G_1))-V(G_1)}} \Big(d(x_1,y_1) + d(x_2,y_2) - 2\Big)^2 \\
&= A_2' + A_2'',
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
A_2' &= \sum_{y_1\in V(G_2)} \sum_{y_2\in V(G_2)} \sum_{x_1x_2\in V(G_1)} \Big(d(x_1,y_1) + d(x_2,y_2)) - 2\Big)^2 \\
&= \sum_{y_1\in V(G_2)} \sum_{y_2\in V(G_2)} \sum_{x_1x_2\in V(G_1)} \Big(n_2 d_{R(G_1)}(x_1) + d_{G_2}(y_1) + n_2 d_{R(G_1)}(x_2) + d_{G_2}(y_2) - 2\Big)^2 \\
&= \sum_{y_1\in V(G_2)} \sum_{y_2\in V(G_2)} \sum_{x_1x_2\in V(G_1)} \Big(2n_2 d_{G_1}(x_1) + d_{G_2}(y_1) + 2n_2 d_{G_1}(x_2) + d_{G_2}(y_2) - 2\Big)^2 \\
&= \sum_{y_1\in V(G_2)} \sum_{y_2\in V(G_2)} \sum_{x_1x_2\in V(G_1)} \Big(4n_2^2(d_{G_1}(x_1) + d_{G_1}(x_2))^2 + d_{G_2}(y_1)^2 + d_{G_2}(y_2)^2 + 4 \\
&\quad +2d_{G_2}(y_1)d_{G_2}(y_2) - 4d_{G_2}(y_1) - 4d_{G_2}(y_2) + (4n_2 d_{G_2}(y_1) + 4n_2 d_{G_2}(y_2) - 8n_2) \\
&\quad (d_{G_1}(x_1) + d_{G_1}(x_2))\Big) \\
&= 4n_2^4 HM(G_1) + 2m_1n_2M_1(G_2) + 4n_2(4m_2n_2 - 2n_2^2)M_1(G_1) + m_1(4n_2^2 - 8m_2^2 - 16n_2m_2).
\end{aligned}$$

and

$$\begin{aligned}
A_2'' &= \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{\substack{x_1x_2\in E(R(G_1))\\ x_1\in V(G_1),\, x_2\in V(R(G_1))-V(G_1)}}\Big(d(x_1,y_1)+d(x_2,y_2)-2\Big)^2\\
&= \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{\substack{x_1x_2\in E(R(G_1))\\ x_1\in V(G_1),\, x_2\in V(R(G_1))-V(G_1)}}\Big(n_2d_{R(G_1)}(x_1)+d_{G_2}(y_1)+n_2d_{R(G_1)}(x_2)-2\Big)^2\\
&= \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{\substack{x_1x_2\in E(R(G_1))\\ x_1\in V(G_1),\, x_2\in V(R(G_1))-V(G_1)}}\Big(2n_2d_{G_1}(x_1)+d_{G_2}(y_1)+2n_2-2\Big)^2\\
&= \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{x_1\in V(G_1)} d_{G_1}(x_1)\Big(4n_2^2d_{G_1}(x_1)^2+d_{G_2}(y_1)^2+(2n_2-2)^2\\
&\quad +4n_2d_{G_1}(x_1)d_{G_2}(y_1)+4n_2(2n_2-2)d_{G_1}(x_1)+2(2n_2-2)d_{G_2}(y_1)\Big)\\
&= 4n_2^4F(G_1)+2m_1n_2M_1(G_2)+8(2n_2-2)n_2m_1m_2+(8n_2^2m_2+4n_2^3(2n_2-2))M_1(G_1).
\end{aligned}$$

From A_2' and A_2'', we have $A_2 = 4n_2^4HM(G_1) + 4n_2^2F(G_1) + 4m_1n_2M_1(G_2) + 8n_2^2(3m_2 - n_2)M_1(G_1) + 4n_2^2m_1 - 8m_1m_2(m_2 + 4n_2 - 2n_2^2)$.
Using (2) and the sums A_1,A_2 we obtain the desired result. □

Next we find the first reformulated Zagreb index of $G_1[G_2]_Q$.

Theorem 2.3. *Let G_i be a graph with n_i vertices and m_i edges, $i = 1, 2$. Then* $EM_1(G_1[G_2]_Q) = 2n_2^4HM(G_1) + n_1HM(G_2) + 3n_2^4F(G_1) + n_2^2(n_2^2HM(L(G_1)) + 2n_2(4n_2 - 2)M_1(L(G_1))) + n_2^2(16m_2 - 8n_2 - 2)M_1(G_1) + (10n_2m_1 - 4n_1)M_1(G_2) + 4n_2^4M_2(G_1) + 4n_1m_2 + 8n_2m_1(n_2 - 4m_2) - n_2^2m_1(4n_2 - 2)^2$.

Proof. By the definition of first reformulated Zagreb index,

$$\begin{aligned}
EM_1(G_1[G_2]_Q) &= \sum_{(x_1,y_1)(x_2,y_2)\in E(G_1[G_2]_Q)}\Big(d_{G_1[G_2]_R}((x_1,y_1))+d_{G_1[G_2]_Q}((x_2,y_2))-2\Big)^2\\
&= \sum_{x_1=x_2\in V(G_1)}\sum_{y_1y_2\in E(G_2)}\Big(d_{G_1[G_2]_Q}((x_1,y_1))+d_{G_1[G_2]_Q}((x_2,y_2))-2\Big)^2\\
&\quad+\sum_{x_1x_2\in E(Q(G_1))}\sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\Big(d_{G_1[G_2]_Q}((x_1,y_1))+d_{G_1[G_2]_Q}((x_2,y_2))-2\Big)^2\\
&= A_1+A_2, \qquad (4)
\end{aligned}$$

where A_1 and A_2 are the sums of the terms, in order.

We shall calculate A_1 and A_2 of 4 separately.

$$\begin{aligned}
A_1 &= \sum_{x_1=x_2\in V(G_1)}\sum_{y_1y_2\in E(G_2)} \Big(d_{G_1[G_2]_Q}((x_1,y_1)) + d_{G_1[G_2]_Q}((x_2,y_2)) - 2\Big)^2 \\
&= \sum_{x_1\in V(G_1)}\sum_{y_1y_2\in E(G_2)} \Big(n_2 d_{Q(G_1)}(x_1) + d_{G_2}(y_1) + n_2 d_{Q(G_1)}(x_1) + d_{G_2}(y_2) - 2\Big)^2 \\
&= \sum_{x_1\in V(G_1)}\sum_{y_1y_2\in E(G_2)} \Big(2n_2 d_{G_1}(x_1) + (d_{G_2}(y_1) + d_{G_2}(y_2)) - 2\Big)^2 \\
&= \sum_{x_1\in V(G_1)}\sum_{y_1y_2\in E(G_2)} \Big(4n_2^2 d_{G_1}(x_1)^2 + (d_{G_2}(y_1) + d_{G_2}(y_2))^2 + 4 \\
&\quad +4n_2 d_{G_1}(x_1)(d_{G_2}(y_1) + d_{G_2}(y_2)) - 8n_2 d_{G_1}(x_1) - 4(d_{G_2}(y_1) + d_{G_2}(y_2))\Big) \\
&= 4n_2^2 m_2 M_1(G_1) + n_1 HM(G_2) + (8n_2m_1 - 4n_1)M_1(G_2) + 4n_1m_2 - 16n_2m_1m_2.
\end{aligned}$$

$$\begin{aligned}
A_2 &= \sum_{x_1x_2\in E(Q(G_1))}\sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)} \Big(d_{G_1[G_2]_Q}((x_1,y_1)) + d_{G_1[G_2]_Q}((x_2,y_2)) - 2\Big)^2 \\
&= \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)} \sum_{\substack{x_1x_2\in E(Q(G_1)) \\ x_1\in V(G_1),\, x_2\in V(Q(G_1))-V(G_1)}} \Big(d(x_1,y_1) + d(x_2,y_2)) - 2\Big)^2 \\
&\quad + \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)} \sum_{\substack{x_1x_2\in E(Q(G_1)) \\ x_1,\, x_2\in V(Q(G_1))-V(G_1)}} \Big(d(x_1,y_1) + d(x_2,y_2) - 2\Big)^2 \\
&= A_2' + A_2'',
\end{aligned}$$

where

$$\begin{aligned}
A_2' &= \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{\substack{x_1x_2\in E(Q(G_1))\\ x_1\in V(G_1),\, x_2\in V(Q(G_1))-V(G_1)}} \Big(d((x_1,y_1))+d((x_2,y_2))-2\Big)^2\\
&= \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{\substack{x_1x_2\in E(Q(G_1))\\ x_1\in V(G_1),\, x_2\in V(Q(G_1))-V(G_1)}} \Big(n_2d_{Q(G_1)}(x_1)+d_{G_2}(y_1)+n_2d_{Q(G_1)}(x_2)-2\Big)^2\\
&= \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{\substack{x_1x_2\in E(Q(G_1))\\ x_1\in V(G_1),\, x_2\in V(Q(G_1))-V(G_1)}} \Big(n_2d_{G_1}(x_1)+d_{G_2}(y_1)+n_2d_{Q(G_1)}(x_2)-2\Big)^2\\
&= \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{\substack{x_1x_2\in E(Q(G_1))\\ x_1\in V(G_1),\, x_2\in V(Q(G_1))-V(G_1)}} \Big(n_2^2d_{G_1}(x_1)^2+(d_{G_2}(y_1)-2)^2+n_2^2d_{Q(G_1)}(x_2)^2\\
&\quad +2n_2d_{G_1}(x_1)(d_{G_2}(y_1)-2)+2n_2^2d_{G_1}(x_1)d_{Q(G_1)}(x_2)+2n_2(d_{G_2}(y_1)-2)d_{Q(G_1)}(x_2)\Big)\\
&= n_2^4F(G_1)+2n_2^2(2m_2-2n_2)M_1(G_1)+2n_2m_1(M_1(G_2)+4n_2-8m_2)\\
&\quad +\sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{\substack{x_1x_2\in E(Q(G_1))\\ x_1\in V(G_1),\, x_2\in V(Q(G_1))-V(G_1)}} \Big(n_2^2d_{Q(G_1)}(x_2)^2+(2n_2^2d_{G_1}(x_1)\\
&\quad +2n_2(d_{G_2}(y_1)-2))d_{Q(G_1)}(x_2)\Big)
\end{aligned}$$

One can see that for a vertex $x_2\in V(Q(G_1))-V(G_1)$, $d_{Q(G_1)}(x_2)=d_{G_1}(x)+d_{G_1}(w)$, where $x_2=xw\in E(G_1)$. Thus

$$\begin{aligned}
A_2' &= n_2^4F(G_1)+2n_2^2(2m_2-2n_2)M_1(G_1)+2n_2m_1(M_1(G_2)+4n_2-8m_2)\\
&\quad +\sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{\substack{x_1x_2\in E(Q(G_1))\\ x_1\in V(G_1),\, x_2\in V(Q(G_1))-V(G_1)}} \Big(n_2^2(d_{G_1}(x)+d_{G_1}(w))^2\\
&\quad +2n_2^2d_{G_1}(x_1)(d_{G_1}(x)+d_{G_1}(w))+2n_2(d_{G_2}(y_1)-2)(d_{G_1}(x)+d_{G_1}(w))\Big)\\
&= n_2^4F(G_1)+2n_2^2(2m_2-2n_2)M_1(G_1)+2n_2m_1(M_1(G_2)+4n_2-8m_2)\\
&\quad +2n_2^4HM(G_1)+2n_2^4(F(G_1)+2M_2(G_1))+4n_2^2(2m_2-2n_2)M_1(G_1)\\
&= 2n_2^4HM(G_1)+3n_2^4F(G_1)+12n_2^2(m_2-n_2)M_1(G_1)+2n_2m_1M_1(G_2)\\
&\quad +4n_2^4M_2(G_1)+2m_1n_2(4n_2-8m_2).
\end{aligned}$$

$$\begin{aligned}
A_2'' &= \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{\substack{x_1x_2\in E(Q(G_1))\\ x_1,\,x_2\in V(Q(G_1))-V(G_1)}}\Big(d(x_1,y_1)+d(x_2,y_2)-2\Big)^2\\
&= \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{\substack{x_1x_2\in E(Q(G_1))\\ x_1,\,x_2\in V(Q(G_1))-V(G_1)}}\Big(n_2d_{Q(G_1)}(x_1)+n_2d_{Q(G_1)}(x_2)-2\Big)^2\\
&= n_2^2\sum_{w_iw_j,w_jw_k\in E(G_1)}\Big(n_2(d_{G_1}(w_i)+d_{G_1}(w_j))+n_2(d_{G_1}(w_j)+d_{G_1}(w_k))-2\Big)^2\\
&= n_2^2\sum_{X_iX_j\in E(L(G_1))}\Big(n_2d_{L(G_1)}(X_i)+n_2d_{L(G_1)}(X_j)+4n_2-2\Big)^2,\\
&\quad \text{where } X_i \text{ and } X_j \text{ are vertices of } L(G_1)\\
&= n_2^2\sum_{X_iX_j\in E(L(G_1))}\Big(n_2^2(d_{L(G_1)}(X_i)+d_{L(G_1)}(X_j))^2+(4n_2-2)^2\\
&\quad +2n_2(4n_2-2)(d_{L(G_1)}(X_i)+d_{L(G_1)}(X_j))\Big)\\
&= n_2^2\Big(n_2^2HM(L(G_1))+2(4n_2-2)n_2M_1(L(G_1))+(4n_2-2)^2(\frac{M_1(G_1)}{2}-m_1)\Big).
\end{aligned}$$

From the sums A_2' and A_2'', we have $A_2 = 2n_2^4HM(G_1)+3n_2^4F(G_1)+n_2^2(12m_2-8n_2-2)M_1(G_1)+2n_2m_1M_1(G_2)+4n_2^4M_2(G_1)+n_2^2(n_2^2HM(L(G_1))+2n_2(4n_2-2)M_1(L(G_1)))+2m_1n_2(4n_2-8m_2)-n_2^2m_1(4n_2-2)^2$.
Adding A_1 and A_2, we get the desired result. □

Finally, we obtain the first formulated Zagreb index of $G_1[G_2]_T$.

Theorem 2.4. *Let G_i be a graph with n_i vertices and m_i edges, $i=1,2$. Then* $EM_1(G_1[G_2]_T) = 10n_2^2HM(G_1)+n_1HM(G_2)+4n_2^2F(G_1)+n_2^4HM(L(G_1))+2n_2^3(4n_2-2)M_1(L(G_1))+(32n_2^2m_2+16n_2m_2-16n_2^2-8n_2^3+2(2n_2-1)^2)M_1(G_1)+(20m_1n_2-4n_1)M_1(G_2)+4n_1m_2-8m_1m_2(8n_1+m_2)-n_2^2m_1((4n_2-2)^2-12)$.

Proof. By the definition of first formulated Zagreb index,

$$\begin{aligned}
EM_1(G_1[G_2]_T) &= \sum_{(x_1,y_1)(x_2,y_2)\in E(G_1[G_2]_T)} \Big(d_{G_1[G_2]_T}((x_1,y_1))+d_{G_1[G_2]_T}((x_2,y_2))-2\Big)^2\\
&= \sum_{x_1=x_2\in V(G_1)}\sum_{y_1y_2\in E(G_2)} \Big(d_{G_1[G_2]_T}((x_1,y_1))+d_{G_1[G_2]_T}((x_2,y_2))-2\Big)^2\\
&\quad + \sum_{x_1x_2\in E(T(G_1))}\sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)} \Big(d_{G_1[G_2]_T}((x_1,y_1))+d_{G_1[G_2]_T}((x_2,y_2))-2\Big)^2\\
&= \sum_{x_1=x_2\in V(G_1)}\sum_{y_1y_2\in E(G_2)} \Big(d_{G_1[G_2]_T}((x_1,y_1))+d_{G_1[G_2]_T}((x_2,y_2))-2\Big)^2\\
&\quad + \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\Big(\sum_{x_1x_2\in E(G_1)} + \sum_{\substack{x_1x_2\in E(T(G_1))\\ x_1\in V(G_1),\,x_2\in V(T(G_1))-V(G_1)}}\\
&\quad + \sum_{\substack{x_1x_2\in E(T(G_1))\\ x_1,\,x_2\in V(T(G_1))-V(G_1)}}\Big)\Big(d_{G_1[G_2]_T}((x_1,y_1))+d_{G_1[G_2]_T}((x_2,y_2))-2\Big)^2\\
&= A_1+A_2+A_3+A_4, \qquad (5)
\end{aligned}$$

where A_1 to A_4 are the sums of the terms, in order.
We shall calculate A_1 to A_4 of 5 separately. A similar arguments of A_1 and A_2' in Theorem 2.2, we have

$$\begin{aligned}
A_1 &= \sum_{x_1=x_2\in V(G_1)}\sum_{y_1y_2\in E(G_2)} \Big(d_{G_1[G_2]_T}((x_1,y_1))+d_{G_1[G_2]_T}((x_2,y_2))-2\Big)^2\\
&= 16n_2^2m_2M_1(G_1)+n_1HM(G_2)+(16n_2m_1-4n_1)M_1(G_2)+4n_1m_2-32n_2m_1m_2.
\end{aligned}$$

and

$$\begin{aligned}
A_2 &= \sum_{y_1\in V(G_2)}\sum_{y_2\in E(G_2)}\sum_{x_1x_2\in E(G_1)} \Big(d_{G_1[G_2]_T}((x_1,y_1))+d_{G_1[G_2]_T}((x_2,y_2))-2\Big)^2\\
&= 4n_2^4HM(G_1)+2m_1n_2M_1(G_2)+4n_2(4m_2n_2-2n_2^2)M_1(G_1)+m_1(4n_2^2-8m_2^2-16n_2m_2).
\end{aligned}$$

$$
\begin{aligned}
A_3 &= \sum_{y_1\in V(G_2)}\sum_{y_2\in E(G_2)}\sum_{\substack{x_1x_2\in E(T(G_1))\\ x_1\in V(G_1),\,x_2\in V(T(G_1))-V(G_1)}}\Big(d_{G_1[G_2]_T}((x_1,y_1))+d_{G_1[G_2]_T}((x_2,y_2))-2\Big)^2\\
&= \sum_{y_1\in V(G_2)}\sum_{y_2\in E(G_2)}\sum_{\substack{x_1x_2\in E(T(G_1))\\ x_1\in V(G_1),\,x_2\in V(T(G_1))-V(G_1)}}\Big(d_{T(G_1)}(x_1)+d_{G_2}(y_2)+d_{T(G_1)}(x_2)-2\Big)^2\\
&= \sum_{y_1\in V(G_2)}\sum_{y_2\in E(G_2)}\sum_{\substack{x_1x_2\in E(T(G_1))\\ x_1\in V(G_1),\,x_2\in V(T(G_1))-V(G_1)}}\Big(d_{R(G_1)}(x_1)^2+(d_{G_2}(y_2)-1)^2+d_{Q(G_1)}(x_2)^2\\
&\quad +2d_{R(G_1)}(x_1)(d_{G_2}(y_2)-2)+2(d_{G_2}(y_2)-2)d_{Q(G_1)}(x_2)+2d_{R(G_1)}(x_1)d_{Q(G_1)}(x_2)\Big)^2\\
&= \sum_{y_1\in V(G_2)}\sum_{y_2\in E(G_2)}\sum_{\substack{x_1x_2\in E(T(G_1))\\ x_1\in V(G_1),\,x_2\in V(T(G_1))-V(G_1)}}\Big(4d_{G_1}(x_1)^2+(d_{G_2}(y_2)-1)^2+d_{Q(G_1)}(x_2)^2\\
&\quad +4d_{G_1}(x_1)(d_{G_2}(y_2)-1)+2(d_{G_2}(y_2)-1)d_{Q(G_1)}(x_2)+2d_{R(G_1)}(x_1)d_{Q(G_1)}(x_2)\Big)^2\\
&= 4n_2^2F(G_1)+2n_2m_1M_1(G_2)+6n_2^2HM(G_1)+(16n_2m_2-16n_2^2)M_1(G_1)+8m_1n_2(n_2-2m_2).
\end{aligned}
$$

A similar argument of A_2'' in Theorem 2.3 , we obtain

$$
\begin{aligned}
A_4 &= \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{\substack{x_1x_2\in E(Q(G_1))\\ x_1,x_2\in V(Q(G_1))-V(G_1)}}\Big(d_{G_1[G_2]_T}((x_1,y_1))+d_{G_1[G_2]_T}((x_2,y_2))-2\Big)^2\\
&= n_2^2\Big(n_2^2HM(L(G_1))+2n_2(4n_2-2)M_1(L(G_1))+(4n_2-2)^2(\frac{M_1(G_1)}{2}-m_1)\Big).
\end{aligned}
$$

Adding the sums A_1 to A_4, we get the desired result. □

Acknowledgement

The authors wish to thank the anonymous referees for their careful reading and constructive comments on earlier version of this article, which resulted in better presentation of this article.

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