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Reflexive Edge Strength on Slanting Ladder Graph and Corona of Centipede and Null Graph

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Abstract

Assume G is a graph that is simple, undirected, and connected. If every edge label is a positive integer in the range 1 to k_e, and every vertex label is a non-negative even number from 0 to $2k_v$, then a graph G is considered to have an edge irregular reflexive k-labeling, where k is defined as the maximum of k_e and $2k_v$. The edge weight $w_t(ab)$ in the graph G, for the labeling λ, is defined as the function w_t applied to the edge ab. The symbol $res(G)$ denotes the reflexive edge strength, which is the largest label of the smallest k. The results of this research are as follows: $res(SL_m)$ for $m \geq 2$ is $\lceil \frac{3m-3}{3} \rceil$ for $3m-3 \not\equiv 2, 3 \pmod 6$, and $\lceil \frac{3m-3}{3} \rceil + 1$ for $3m-3 \equiv 2, 3 \pmod 6$. $res(Cp_n \odot N_m)$ for $n \geq 2$, $m \geq 1$ is $\lceil \frac{2nm+2n-1}{3} \rceil$ for $2nm+2n-1 \not\equiv 2, 3 \pmod 6$, and $\lceil \frac{2nm+2n-1}{3} \rceil + 1$ for $2nm+2n-1 \equiv 2, 3 \pmod 6$.

Keywords: Reflexive edge strength, slanting ladder graph, corona of centipede and null graph
Mathematics Subject Classification : 05C38, 05C78

1. Introduction

Assume a simple undirected graph G in this article. The given graph G has a set of vertices $V(G) = \{v_1, v_2, \ldots, v_m\}$, where V is a finite non-empty set and a set of edges $E(G) = \{e_1, e_2, \ldots, e_n\}$, which may be empty. A graph labeling is described as a function that maps the components of a graph (vertices and edges) to a set of positive or non-negative integers as the codomain [11].

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A total irregular k-labeling can be classified into two categories: one for the labeling of vertices and the other for the labeling of edges [6]. A new idea of total irregular k-labeling is proposed, specifically referring to the edge irregular k-labeling of both vertices and edges in the graph G [5]. In a graph G, an edge irregular reflexive k-labeling is defined when each vertex is mapped to an even integer ranging from 0 to $2k_v$, while the labeling of each edge involves a positive integer ranging from 1 to k_e. Thus, the weight of each edge is distinct, where k is defined as the maximum of k_e and $2k_v$. The edge weight $w_t(ab)$ in the graph G, for the labeling λ, is defined as the function w_t applied to the edge ab. The function $w_t(ab)$ denotes the total obtained by adding the vertex label of a and b, and the edge label of ab, expressed as $w_t(ab) = \lambda(a) + \lambda(ab) + \lambda(b)$. The notation $res(G)$ represents the reflexive edge strength, defined as the maximum label of the smallest k [4].

This paper examines $res(G)$ of the slanting ladder graph (SL_m) where $m \geq 2$ and corona of centipede and null graph $(Cp_n \odot N_m)$ where $n \geq 2$ and $m \geq 1$. To obtain a lower bound of $res(G)$, we have the inequality given in Lemma 1.1. [4].

Lemma 1.1.

$$res(G) \geq \begin{cases} \left\lceil \frac{|E(G)|}{3} \right\rceil, & \text{for } |E(G)| \not\equiv 2,3 \pmod 6, \\ \left\lceil \frac{|E(G)|}{3} \right\rceil + 1, & \text{for } |E(G)| \equiv 2,3 \pmod 6. \end{cases}$$

Several studies on certain graphs have been conducted, including tadpole graph $T_{m,1}$ and $T_{m,2}$ [8], the corona of a path graph and another graph, such as $P_n \odot K_1$ and $P_n \odot P_2$ [1], umbrella graph $U_{3,n}$ and $U_{4,n}$ [7], and others. In this paper, we will determine $res(SL_m)$ for $m \geq 2$ and $res(Cp_n \odot N_m)$ for $n \geq 2$ and $m \geq 1$.

2. Main Result

2.1. Slanting Ladder Graph

A slanting ladder graph, denoted by SL_m, is the graph obtained by taking two path graphs U_m and V_m, then connecting the vertex u_p to the vertex v_{p+1} with an edge, where $p = 1, 2, 3, \ldots, m$ [9]. The slanting ladder graph is a connected graph in which the set of edges is composed of the pairs u_pu_{p+1}, u_pv_{p+1}, and v_pv_{p+1} for each $1 \leq p \leq m-1$, while the set of vertices consists of the pairs u_p and v_p for each $1 \leq p \leq m$. The slanting ladder graph has an order of $|V(SL_m)| = 2m$ and a size of $|E(SL_m)| = 3m - 3$. The slanting ladder graph SL_m is illustrated in Figure 1.

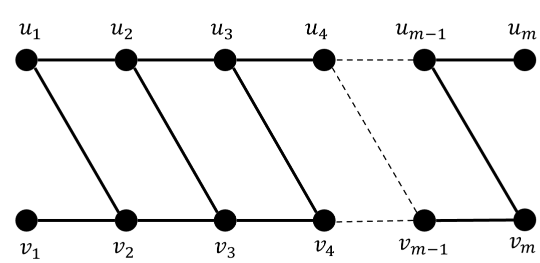

Figure 1. Slanting ladder graph SL_m

Theorem 2.1. *For SL_m with $m \geq 2$,*

$$res(SL_m) = \begin{cases} \left\lceil \frac{3m-3}{3} \right\rceil, & \text{for } 3m-3 \not\equiv 2,3 \pmod 6, \\ \left\lceil \frac{3m-3}{3} \right\rceil + 1, & \text{for } 3m-3 \equiv 2,3 \pmod 6. \end{cases} \quad (1)$$

Proof. Since $|E|$ of SL_m is $3m-3$, then by Lemma 1.1 we obtain

$$res(SL_m) \geq \begin{cases} \left\lceil \frac{3m-3}{3} \right\rceil, & \text{for } 3m-3 \not\equiv 2,3 \pmod 6, \\ \left\lceil \frac{3m-3}{3} \right\rceil + 1, & \text{for } 3m-3 \equiv 2,3 \pmod 6. \end{cases} \quad (2)$$

In the next step, we will establish the upper bound of $res(G)$ for the slanting ladder graph SL_m where $m \geq 2$. We define a λ k-labeling for the slanting ladder graph SL_m where $k = \left\lceil \frac{3m-3}{3} \right\rceil$ when $3m-3 \not\equiv 2,3 \pmod 6$ and $k = \left\lceil \frac{3m-3}{3} \right\rceil + 1$ when $3m-3 \equiv 2,3 \pmod 6$ in the following manner.

For the vertex labeling of v_p and u_p where $1 \leq p \leq m$, we use the following,

$$\lambda(v_p) = \begin{cases} 0, & \text{for } p = 2, \\ p, & \text{for } p \text{ even}, p \neq 2, \\ p-1, & \text{for } p \text{ odd}. \end{cases}$$

$$\lambda(u_p) = \begin{cases} p-1, & \text{for } p \text{ odd}, \\ p, & \text{for } p \text{ even}. \end{cases}$$

The labeling of the edges $v_p v_{p+1}$, $u_p v_{p+1}$, and $u_p u_{p+1}$ for $1 \leq p \leq m-1$, we use the following,

$$\lambda(v_p v_{p+1}) = \begin{cases} p, & \text{for } p = 1 \text{ and } 2, \\ p-2, & \text{for } p \neq 1 \text{ and } 2. \end{cases}$$

$$\lambda(u_p v_{p+1}) = \begin{cases} 2, & \text{for } p = 1, \\ p-1, & \text{for } p \neq 1. \end{cases}$$

$$\lambda(u_p u_{p+1}) = p.$$

Based on labeling obtained, the maximum label, which serves as the upper bound $res(SL_m)$ is found at vertex u_p with $p = m$

$$\lambda(u_m) = \begin{cases} m-1, & \text{for } m \text{ odd}, \\ m, & \text{for } m \text{ even}. \end{cases} \quad (3)$$

It is shown that the upper bound of $res(SL_m)$ in (3) satisfies the right side of (2) as follows: For $\lambda(u_m) = m-1$, when m is odd, we show that $m-1$ satisfies $\left\lceil \frac{3m-3}{3} \right\rceil$ for $3m-3 \not\equiv 2,3$ (mod 6). As m is an odd number, we express it as $m = 2k+1$, where k represents an integer. Substituting this value of m into the expression $3m-3$, we get $3m-3 = 3(2k+1)-3 = 6k \equiv 0 \pmod 6$. Thus, it is concluded that $m-1$ satisfies the equation for odd m. Meanwhile, for $\lambda(u_m) = m$, when m is even. We show that m satisfies $\left\lceil \frac{3m-3}{3} \right\rceil + 1$ for $3m-3 \equiv 2,3 \pmod 6$. Since m is even, let $m = 2k$ be an integer k. Substituting m into $3m-3$, we have $3m-3 = 3(2k)-3 = 6k-3 \equiv 3 \pmod 6$. Therefore, it is concluded that $m-1$ satisfies the equation for even m.

Based on the two cases discussed above, it can be concluded that if the upper bound satisfies the right side of (2), we obtain

$$res(SL_m) \leq \begin{cases} \left\lceil \frac{3m-3}{3} \right\rceil, & \text{for } 3m-3 \not\equiv 2,3 \pmod 6, \\ \left\lceil \frac{3m-3}{3} \right\rceil + 1, & \text{for } 3m-3 \equiv 2,3 \pmod 6. \end{cases}$$

After showing the lower and upper bounds of $res(SL_m)$, we obtain the largest labels for the vertices and edges are k. The edge weights of the slanting ladder graph SL_m are as follows.

$$w_t(v_p v_{p+1}) = 3p-2, \text{ for } 1 \leq p \leq m-1.$$
$$w_t(u_p v_{p+1}) = 3p-1, \text{ for } 1 \leq p \leq m-1.$$
$$w_t(u_p u_{p+1}) = 3p, \text{ for } 1 \leq p \leq m-1.$$

From this, we can deduce that every edge in the slanting ladder graph SL_m has a unique weight. The upper bound of SL_m is equivalent to the lower bound of SL_m, thus Theorem 1 is proven. □

The edge irregular reflexive 4-labeling for SL_m, with $m = 4$, as shown in Figure 2.

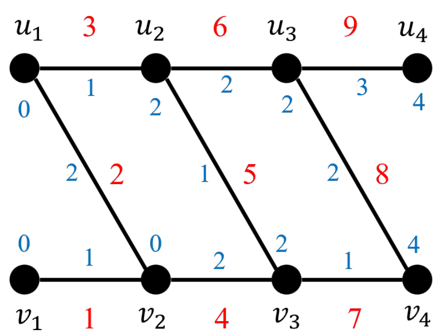

Figure 2. The SL_m graph for $m = 4$ with edge irregular reflexive 4-labeling.

The edge and vertex labels are represented by blue numbers, the edge weights are shown using red numbers, and the vertex names are indicated by black letters in Figure 2. From Theorem 1, we obtain $res(SL_4) = 4$.

2.2. Corona of Centipede and Null Graph

A null graph, represented by N_m, consists of m vertices and no edges, meaning that $|E(N_m)| = 0$ [2]. A centipede graph, denoted by Cp_n, a graph is formed by taking a path graph P_n and a null graph N_n, then an edge connects the p-th vertex of P_n to the p-th vertex of N_m [3]. Based on the definition of the corona [10], a graph denoted as $Cp_n \odot N_m$, called the corona of the centipede graph and the null graph, is obtained by combining Cp_n with $|V(Cp_n)|$ copies of N_m, then connecting each vertex $v_p \in V(Cp_n)$ to each vertex of the p-th copy of N_m for $1 \leq p \leq |V(Cp_n)|$. Figure 3 illustrates the corona product of the centipede and null graph $Cp_n \odot N_m$.

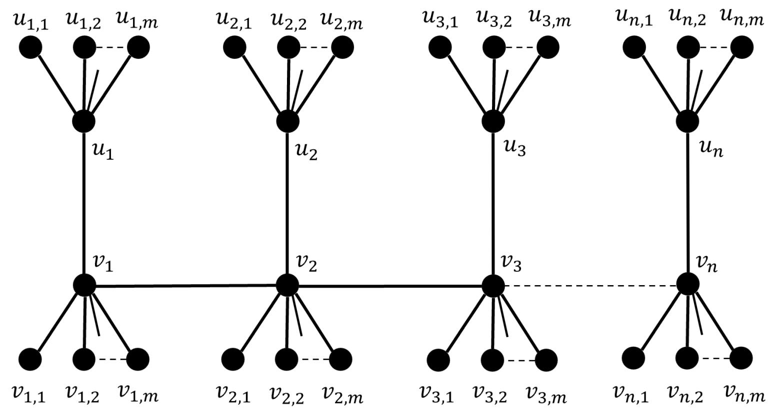

Figure 3. Corona of centipede and null graph $Cp_n \odot N_m$

Theorem 2.2. *For $Cp_n \odot N_m$ where $n \geq 2$ and $m \geq 1$,*

$$res(Cp_n \odot N_m) = \begin{cases} \left\lceil \frac{2nm+2n-1}{3} \right\rceil, & \text{for } 2nm+2n-1 \not\equiv 2,3 \pmod 6, \\ \left\lceil \frac{2nm+2n-1}{3} \right\rceil + 1, & \text{for } 2nm+2n-1 \equiv 2,3 \pmod 6. \end{cases} \quad (4)$$

Proof. Since $|E|$ of $Cp_n \odot N_m$ is $2nm + 2n - 1$, then based on Lemma 1.1 obtained

$$res(Cp_n \odot N_m) \geq \begin{cases} \left\lceil \frac{2nm+2n-1}{3} \right\rceil, & \text{for } 2nm+2n-1 \not\equiv 2,3 \pmod 6, \\ \left\lceil \frac{2nm+2n-1}{3} \right\rceil + 1, & \text{for } 2nm+2n-1 \equiv 2,3 \pmod 6. \end{cases} \quad (5)$$

In the next step, we will prove the upper bound of $res(G)$ for the graph $Cp_n \odot N_m$ where $n \geq 2$ and $m \geq 1$. We construct the labeling λ as the k-labeling for the $Cp_n \odot N_m$, where $k = \left\lceil \frac{2nm+2n-1}{3} \right\rceil$ when $2nm+2n-1 \not\equiv 2,3 \pmod 6$, and $k = \left\lceil \frac{2nm+2n-1}{3} \right\rceil + 1$ when $2nm+2n-1 \equiv 2,3 \pmod 6$, in the following manner.

For the vertex labels v_p and u_p where $1 \leq p \leq n$,

$$\lambda(v_p) = \begin{cases} 0, & \text{for } p = 1, \\ \frac{2pm}{3}, & \text{for } p = 2, m \equiv 0 \pmod 3, \\ \frac{2pm+2p}{3}, & \text{for } p \equiv 0 \pmod 3, m \geq 1 \text{ and} \\ & \text{for } p \equiv 1,2 \pmod 3, m \equiv 2 \pmod 3, p \neq 1, \\ \frac{2pm+2p-2}{3}, & \text{for } p \equiv 1 \pmod 3, m \equiv 0 \pmod 3, p \neq 1 \\ & \text{and } p \equiv 2 \pmod 3, m \equiv 1 \pmod 3, \\ \frac{2pm+2p+2}{3}, & \text{for } p, m \equiv 1 \pmod 3, p \neq 1 \text{ and} \\ & p \equiv 2 \pmod 3, m \equiv 0 \pmod 3, p \neq 2. \end{cases}$$

$$\lambda(u_p) = \begin{cases} m, & \text{for } p = 1, m \text{ even,} \\ m-1, & \text{for } p = 1, m \text{ odd,} \\ \frac{2pm+2p}{3}, & \text{for } p \equiv 0 \pmod 3, m \geq 1 \text{ and} \\ & p \equiv 1,2 \pmod 3, m \equiv 2 \pmod 3, p \neq 1, \\ \frac{2pm+2p-2}{3}, & \text{for } p \equiv 1 \pmod 3, m \equiv 0 \pmod 3, p \neq 1 \\ & \text{and } p \equiv 2 \pmod 3, m \equiv 1 \pmod 3, \\ \frac{2pm+2p+2}{3}, & \text{for } p, m \equiv 1 \pmod 3, p \neq 1 \text{ and} \\ & p \equiv 2 \pmod 3, m \equiv 0 \pmod 3. \end{cases}$$

Vertex label $v_{p,q}$ and $u_{p,q}$ for $1 \leq p \leq n$ and $1 \leq q \leq m$,

$$\lambda(v_{p,q}) = \begin{cases} 0, & \text{for } p = 1, \\ \frac{2m+6}{3}, & \text{for } p = 2, m \equiv 0 \pmod 3, \\ \frac{2m+4}{3}, & \text{for } p = 2, m \equiv 1 \pmod 3, \\ \frac{2m+2}{3}, & \text{for } p = 2, m \equiv 2 \pmod 3, \\ \frac{2pm+2p}{3}, & \text{for } p \equiv 0 \pmod 3, m \geq 1 \text{ and} \\ & p \equiv 1,2 \pmod 3, m \equiv 2 \pmod 3, p \neq 1, 2, \\ \frac{2pm+2p-2}{3}, & \text{for } p \equiv 1 \pmod 3, m \equiv 0 \pmod 3, p \neq 1 \\ & \text{and } p \equiv 2 \pmod 3, m \equiv 1 \pmod 3, p \neq 2, \\ \frac{2pm+2p+2}{3}, & \text{for } p, m \equiv 1 \pmod 3, p \neq 1 \text{ and} \\ & p \equiv 2 \pmod 3, m \equiv 0 \pmod 3, p \neq 2. \end{cases}$$

$$\lambda(u_{p,q}) = \begin{cases} 0, & \text{for } p = 1, m \text{ even,} \\ 2, & \text{for } p = 1, m \text{ odd,} \\ \frac{2pm+2p}{3}, & \text{for } p \equiv 0 \pmod 3, m \geq 1 \text{ and} \\ & p \equiv 1,2 \pmod 3, m \equiv 2 \pmod 3, p \neq 1, \\ \frac{2pm+2p-2}{3}, & \text{for } p \equiv 1 \pmod 3, m \equiv 0 \pmod 3, p \neq 1 \\ & \text{and } p \equiv 2 \pmod 3, m \equiv 1 \pmod 3, \\ \frac{2pm+2p+2}{3}, & \text{for } p, m \equiv 1 \pmod 3, p \neq 1 \text{ and} \\ & p \equiv 2 \pmod 3, m \equiv 0 \pmod 3. \end{cases}$$

Edge label v_pv_{p+1} for $1 \leq p \leq n-1$,

$$\lambda(v_pv_{p+1}) = \begin{cases} \frac{2m+6}{3}, & \text{for } p = 1 \text{ and } 2, m \equiv 0 \pmod 3, \\ \frac{2m+4}{3}, & \text{for } p = 1 \text{ and } 2, m \equiv 1 \pmod 3, \\ \frac{2m+2}{3}, & \text{for } p = 1 \text{ and } 2, m \equiv 2 \pmod 3, \\ \frac{2pm-2m+2p}{3}, & \text{for } p, m \equiv 0 \pmod 3 \text{ and} \\ & p \equiv 2 \pmod 3, m \equiv 1 \pmod 3, p \neq 2, \\ \frac{2pm-2m+2p-4}{3}, & \text{for } p \equiv 0 \pmod 3, m \equiv 1 \pmod 3 \text{ and} \\ & p \equiv 2 \pmod 3, m \equiv 0 \pmod 3, p \neq 2, \\ \frac{2pm-2m+2p-2}{3}, & \text{for } p \equiv 1 \pmod 3, m \geq 1, p \neq 1 \text{ and} \\ & p \equiv 0, 2 \pmod 3, m \equiv 2 \pmod 3, p \neq 2. \end{cases}$$

Edge label v_pu_p for $1 \leq p \leq n$,

$$\lambda(v_pu_p) = \begin{cases} 1, & \text{for } p = 1, m \text{ even}, \\ 2, & \text{for } p = 1, m \text{ odd}, \\ \frac{m+3}{3}, & \text{for } p = 2, m \equiv 0 \pmod 3, \\ \frac{2pm-3m+2p-3}{3}, & \text{for } p \equiv 0 \pmod 3, m \geq 1 \text{ and} \\ & p \equiv 1, 2 \pmod 3, m \equiv 2 \pmod 3, p \neq 1, \\ \frac{2pm-3m+2p+1}{3}, & \text{for } p \equiv 1 \pmod 3, m \equiv 0 \pmod 3, p \neq 1 \\ & \text{and } p \equiv 2 \pmod 3, m \equiv 1 \pmod 3, \\ \frac{2pm-3m+2p-7}{3}, & \text{for } p, m \equiv 1 \pmod 3, p \neq 1 \text{ and} \\ & p \equiv 2 \pmod 3, m \equiv 0 \pmod 3, p \neq 2. \end{cases}$$

Edge label $v_pv_{p,q}$ and $u_pu_{p,q}$ $1 \leq p \leq n$ and $1 \leq q \leq m$,

$$\lambda(v_pv_{p,q}) = \begin{cases} p, & \text{for } p = 1, 2, \text{ and } 3, \\ (\frac{2pm-6m+2p-3}{3}) + q - 1, & \text{for } p \equiv 0 \pmod 3, m \geq 1, p \neq 3 \text{ and} \\ & p \equiv 1, 2 \pmod 3, m \equiv 2 \pmod 3, p \neq 1, 2, \\ (\frac{2pm-6m+2p+1}{3}) + q - 1, & \text{for } p \equiv 1 \pmod 3, m \equiv 0 \pmod 3, p \neq 1 \\ & \text{and } p \equiv 2 \pmod 3, m \equiv 1 \pmod 3, p \neq 2, \\ (\frac{2pm-6m+2p-7}{3}) + q - 1, & \text{for } p, m \equiv 1 \pmod 3, p \neq 1 \text{ and} \\ & p \equiv 2 \pmod 3, m \equiv 0 \pmod 3, p \neq 2. \end{cases}$$

$$\lambda(u_p u_{p,q}) = \begin{cases} q, & \text{for } p = 1,\, m \text{ odd,} \\ q+1, & \text{for } p = 1,\, m \text{ even,} \\ (\frac{2pm-3m+2p-3}{3}) + q, & \text{for } p \equiv 0 \pmod 3,\, m \geq 1, \\ (\frac{2pm-3m+2p+4}{3}) + q - 1, & \text{for } p = 2,\, m \equiv 1 \pmod 3; \\ & p \equiv 1 \pmod 3,\, m \equiv 0 \pmod 3,\, p \neq 1; \\ & \text{and } p \equiv 2 \pmod 3,\, m \equiv 1 \pmod 3,\, p \neq 2, \\ (\frac{2pm-3m+2p-4}{3}) + q - 1, & \text{for } p = 2,\, m \equiv 0 \pmod 3; \\ & p, m \equiv 1 \pmod 3,\, p \neq 1; \text{ and} \\ & p \equiv 2 \pmod 3,\, m \equiv 0 \pmod 3,\, p \neq 2, \\ (\frac{2pm-3m+2p}{3}) + q - 1, & \text{for } p = 2,\, m \equiv 2 \pmod 3 \text{ and} \\ & p \equiv 1, 2 \pmod 3,\, m \equiv 2 \pmod 3,\, p \neq 1, 2. \end{cases}$$

According to the labeling, the largest label, which represents the upper bound of $res(Cp_n \odot N_m)$, is found at vertex u_p where $p = n$.

$$\lambda(u_n) = \begin{cases} m, & \text{for } n = 1,\, m \text{ even,} \\ m - 1, & \text{for } n = 1,\, m \text{ odd,} \\ \frac{2nm+2n}{3}, & \text{for } n \equiv 0 \pmod 3,\, m \geq 1 \text{ and} \\ & n \equiv 1, 2 \pmod 3,\, m \equiv 2 \pmod 3,\, n \neq 1, \\ \frac{2nm+2n-2}{3}, & \text{for } n \equiv 1 \pmod 3,\, m \equiv 0 \pmod 3,\, n \neq 1 \\ & \text{and } n \equiv 2 \pmod 3,\, m \equiv 1 \pmod 3, \\ \frac{2nm+2n+2}{3}, & \text{for } n, m \equiv 1 \pmod 3,\, n \neq 1 \text{ and} \\ & n \equiv 2 \pmod 3,\, m \equiv 0 \pmod 3. \end{cases} \tag{6}$$

It is shown that the upper bound of $res(Cp_n \odot N_m)$ in (6) satisfies the right side of (5) as follows: For $\lambda(u_n) = \frac{2nm+2n}{3}$, it is evident that $\frac{2nm+2n}{3}$ satisfies $\lceil \frac{2nm+2n-1}{3} \rceil$. When $n \equiv 0 \pmod 3$ and $m \geq 1$, the size of the corona of the centipede and null graph $Cp_n \odot N_m$ is $2nm + 2n - 1$. Since $n \equiv 0 \pmod 3$ and $m \geq 1$, we have $2nm \equiv 0 \pmod 6$ and $2n \equiv 0 \pmod 6$. Let $0 \pmod 6 \equiv 6k$ for some integer k, so $2nm + 2n - 1 = (6k) + (6k) - 1 = 12k - 1 \equiv 5 \pmod 6$. Thus, it is concluded that $\frac{2nm+2n}{3}$ for $n \equiv 0 \pmod 3$ and $m \geq 1$ satisfies $\lceil \frac{2nm+2n-1}{3} \rceil$ for $2nm+2n-1 \not\equiv 2, 3 \pmod 6$. Meanwhile, the condition for $n \equiv 1, 2 \pmod 3$, $m \equiv 2 \pmod 3$, and $n \neq 1$ can be proven in the same way. Moreover, for $\lambda(u_n) = \frac{2nm+2n+2}{3}$, it is evident that $\frac{2nm+2n+2}{3}$ satisfies $\lceil \frac{2nm+2n-1}{3} \rceil + 1$. When $n, m \equiv 1 \pmod 3$ and $n \neq 1$, the size of the corona of $Cp_n \odot N_m$ is $2nm + 2n - 1$. Since $n, m \equiv 1 \pmod 3$, we have $2nm \equiv 2 \pmod 6$ and $2n \equiv 2 \pmod 6$. Let $2 \pmod 6 \equiv 6k + 2$ for some integer k, so $2nm + 2n - 1 = (6k + 2) + (6k + 2) - 1 = 12k + 3 \equiv 3 \pmod 6$. Thus, it is concluded that $\frac{2nm+2n+2}{3}$ for $n, m \equiv 1 \pmod 3$ satisfies $\lceil \frac{2nm+2n-1}{3} \rceil + 1$ for $2nm + 2n - 1 \equiv 2, 3 \pmod 6$. The condition for $n \equiv 2 \pmod 3$ and $m \equiv 0 \pmod 3$ can also be proven in the same way.

Based on the two cases discussed above, it can be concluded that if the upper bound satisfies the right side of (5), we obtain

$$res(Cp_n \odot N_m) \leq \begin{cases} \left\lceil \frac{2nm+2n-1}{3} \right\rceil, & \text{for } 2nm+2n-1 \not\equiv 2,3 \pmod 6, \\ \left\lceil \frac{2nm+2n-1}{3} \right\rceil + 1, & \text{for } 2nm+2n-1 \equiv 2,3 \pmod 6. \end{cases}$$

After determining the lower and upper bounds of $res(Cp_n \odot N_m)$, we find that the largest labels for the vertices and edges are k. The edge weights in $Cp_n \odot N_m$ are given as follows.

$$w_t(v_p v_{p+1}) = 2pm + 2p, \text{ for } 1 \leq p \leq n-1.$$
$$w_t(v_p u_p) = 2pm - m + 2p - 1, \text{ for } 1 \leq p \leq n.$$
$$w_t(v_p v_{p,q}) = \begin{cases} q, & \text{for } p = 1, 1 \leq q \leq m, \\ 2pm - 2m + 2p - 2 + q, & \text{for } 2 \leq p \leq n, 1 \leq q \leq m. \end{cases}$$
$$w_t(u_p u_{p,q}) = 2pm - m + 2p - 1 + q, \text{ for } 1 \leq p \leq n, 1 \leq q \leq m.$$

It can be concluded from this that every edge in the $Cp_n \odot N_m$ has a unique weight. The lower bound of $Cp_n \odot N_m$ is equal to its upper bound. Thus, Theorem 2.2 has been proven. □

The edge irregular reflexive 4-labeling for $Cp_n \odot N_m$, with $n, m = 2$, is illustrated in Figure 4.

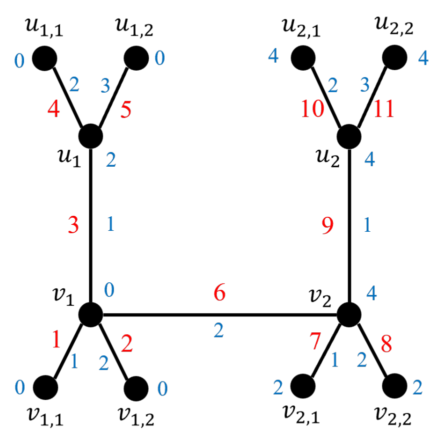

Figure 4. The edge irregular reflexive 4-labeling of $Cp_n \odot N_m$ graph for $n, m = 2$

Based on Figure 4, the edge and vertex labels are indicated by blue numbers, the edge weights are indicated by red numbers, and the vertex names are indicated by black letters. From Theorem 2.2, we obtain $res(Cp_2 \odot N_2) = 4$.

3. Conclusion

According to the results, the following conclusions can be drawn:

1. Reflexive edge strength on SL_m for $m \geq 2$ is

$$res(SL_m) = \begin{cases} \left\lceil \frac{3m-3}{3} \right\rceil, & \text{for } 3m-3 \not\equiv 2,3 \pmod 6, \\ \left\lceil \frac{3m-3}{3} \right\rceil + 1, & \text{for } 3m-3 \equiv 2,3 \pmod 6. \end{cases}$$

2. Reflexive edge strength on $Cp_n \odot N_m$ for $n \geq 2$ and $m \geq 1$ is

$$res(Cp_n \odot N_m) = \begin{cases} \left\lceil \frac{2nm+2n-1}{3} \right\rceil, & \text{for } 2nm+2n-1 \not\equiv 2,3 \pmod 6, \\ \left\lceil \frac{2nm+2n-1}{3} \right\rceil + 1, & \text{for } 2nm+2n-1 \equiv 2,3 \pmod 6. \end{cases}$$

Open problem: How is the reflexive edge strength defined in the corona of a centipede graph and another graph, such as the corona of centipede and path graph?

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