Indonesian Journal of Combinatorics **8** (1) (2024), 26–32

INDONESIAN JOURNAL OF COMBINATORICS

On the generating graph of a finite group

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Abstract

In this paper, we study the generating graph for some finite groups which are semi-direct product $\mathbb{Z}_n \rtimes \mathbb{Z}_m$ (direct product $\mathbb{Z}_n \times \mathbb{Z}_m$) of cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$. We show that the generating graphs of them are regular (bi-regular, tri-regular) connected graph with diameter 2 and girth 3 if n and m are prime numbers. Several graph properties are obtained. Furthermore, the probability that 2-randomly elements that generate a finite group G is $P(G) = \frac{|\{(a,b)\in G\times G \mid G=\langle a,b\rangle\}|}{|G|^2}$. We find the general formula for $P(G)$ of given groups. Our computations are done with the aid of GAP and the YAGs package.

Keywords: Semi-Direct Product Group, Generating Graph, Probability
Mathematics Subject Classification : 20B40

1. Introduction

The concept of a generating graph of a finite group was first introduced in [5]. In this paper, they studiy a finite non abelian groups. This work considers the problems of a finite group G with 2-generators. First, what is the generating graph look like. Second, what is the probability that 2-randomly chosen elements will generate the whole group G. We show that there is a relationship between them. Here, we investigate these problems for some finite semi-direct product and direct product groups. Several examples for these groups are provided.

We will only consider the following finite groups:

- $G \cong \mathbb{Z}_n \rtimes \mathbb{Z}_m$ (the semi-direct product of $\mathbb{Z}_n$ by $\mathbb{Z}_m$).
- $G \cong \mathbb{Z}_n \times \mathbb{Z}_m$ (the direct product of $\mathbb{Z}_n$ by $\mathbb{Z}_m$).

Received: 09 February 2024, Revised 06 June 2024, Accepted: 30 June 2024.

- $G \cong \mathbb{Z}_n \times \mathbb{Z}_n$.

where $m, n \geq 2$.

Throughout this paper, we consider a finite simple un-directed graph Γ and G is a finite group. The eccentricity $ecc(v)$ of a vertex v in a graph Γ is the greatest distance from v to any other vertex. The radius $rad(\Gamma)$ of Γ is the value of the smallest eccentricity. The diameter $diam(\Gamma)$ of Γ is the value of the greatest eccentricity. The center of Γ is the set of all vertices v such that $ecc(v) = rad(\Gamma)$. The girth of a graph, denoted by $g(\Gamma)$ is the length of the short cycle in the graph Γ. A graph Γ is said to be connected if there is a path between every pair of vertices. Two simple graphs Γ and Σ are isomorphic if exists a bijection $f\colon \Gamma \to \Sigma$ between vertices of the two graphs such that if ab is an edge in Γ then $f(a)f(b)$ is an edge in Σ. The biregular graph and the triregular graph are defined as follows. Let a, b be positive integers, $1 \leq a < b$. A graph Γ is said to be (a, b)-biregular if it possesses vertices of degree a and b. Let a, b and c be integers, $1 \leq a < b < c$. A graph is said to be (a, b, c)-triregular if the degrees of its vertices assume exactly three different values: a, b and c. A planar graph is a graph that can be drawn on the plane in such a way that its edges intersect only at their endpoints. We will use Kuratowski's Theorem, which states that

Theorem 1.1. *A graph is planar if and only if it does not contained a subdivision of the complete graph K_5 or of the complete bipartite graph $K_{3,3}$.*

Clearly, the directed generating graph $\Gamma(G)$ is planar if and only if the undirected generating graph Γ is planar.

Lemma 1.1. *In any given graph, the sum of degrees of all the vertices is twice the number of edges contained in it.*

Theorem 1.2. *A connected graph Γ is Eulerian if and only if the degree of each vertex of Γ is even.*

Corollary 1.1. *If Γ is a simple graph with $n \geq 3$ vertices, and if $deg(v) \geq \frac{n}{2}$ for each vertex v, then Γ is Hamiltonian.*

Definition 1.1. *Let H, N be subgroups of a group G. We say that G is semidirect product of N by H if H is normal in G, $G = NH$, and $N \cap H = \{e\}$.*

Definition 1.2. *Let H, N be groups. The cartesion product of H and N, $N \times H = \{(n, h) : n \in N$ and $h \in H\}$ is a group with respect to $(n_1, n_2)(h_1, h_2) = (n_1h_1, n_2h_2)$. This group is the direct product of groups N and H.*

In [2], it was studied the generating graph $\Gamma(G)$ for non abelian simple groups. Here, we study the generating graph for some semi-direct product and direct product of groups. The aim of this paper is to determine these groups G with the property that $\Gamma(G)$ is connected. We will achieve that if G is isomorphic to one of the following: $\mathbb{Z}_p \times \mathbb{Z}_q, \mathbb{Z}_p \rtimes \mathbb{Z}_q$ $\mathbb{Z}_p \times \mathbb{Z}_p$, where p, q are prime numbers. Furthermore, we find the general formula for the probability of each of them.

Definition 1.3. *[5] Let G be a finite group. The generating graph $\Gamma(G)$ is a graph on the non-identity elements of G so that two distinct vertices x, y are adjacent if they generate G, that is $G = \langle x, y \rangle$.*

Throughout this papar $V(\Gamma(G))$ and $E(\Gamma(G))$ denote the set of vertices and edges of the generating graph $\Gamma(G)$ respectively.

Definition 1.4. *Let G be a finite group and let $\Gamma(G)$ be its generating graph. The probability that 2-randomly elements that generate G is $P(G) = \frac{|E(\Gamma(G))|}{|G|^2}$*

Remark 1.1. *If G is an abelian group and its probability denoted by $P_a(G)$, then $P(G) = P_a(G) = P_n(G)$. Otherwise, $P_a(G) = P_n(G)$ and $P(G) = P_a(G) + P_n(G)$, where $P_n(G)$ denoted the probability for non- abelian group.*

Example 1.1. *The generating graph of $D_6 := \mathbb{Z}_2 \rtimes \mathbb{Z}_3$ is*

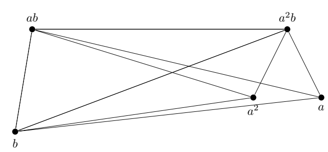

Figure 1. The generating graph of D_6

and the probability of G is $P_a(G) = \frac{1}{4}$ ($P_n(G) = \frac{1}{4}$). So $P(G) = \frac{1}{2}$.

2. Main Results

Proposition 2.1. *The generating graph $\Gamma(G)$ of $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$ has $\frac{(p^2-p)(p^2-1)}{2}$ edges.*

Proof. *For $a, b, c, d \in \mathbb{Z}_p$, $\langle (a, b), (c, d) \rangle$ generates $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$ if and only if there exist $r_1, r_2, r_3, r_4 \in \mathbb{Z}$ such that $r_1(a, b) + r_2(c, d) = (1, 0)$ and $r_3(a, b) + r_4(c, d) = (0, 1)$. It can be written in this form.*

$$A \begin{pmatrix} r_1 & r_3 \\ r_2 & r_4 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} r_1 & r_3 \\ r_2 & r_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

To solve this system, we need to take the inverse of A, which exists if and only if the column of A are independent. The first column (a, b) can be any thing except the zero vector. So, there are $p^2 - 1$ choice for (a, b). The next column must be chosen so that it is not a multiple of (a, b). Hence, given (a, b), there are $p^2 - p$ choices for the column (c, d). Thus the number of choices for the pair $(a, b), (c, d)$ is $\frac{1}{2}(p^2 - p)(p^2 - 1)$. We divide by 2 because it is abelian. Hence, there are $\frac{1}{2}(p^2 - p)(p^2 - 1)$ ways to choose (a, b) and (c, d) such that $\langle (a, b), (c, d) \rangle$ generate $\mathbb{Z}_p \times \mathbb{Z}_p$ and we get the result.

Proposition 2.2. *Let $\Gamma(G)$ be the generating graph of $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Then for any $v \in V(\Gamma(G)) \setminus \{0\}$, we have $deg(v) = p^2 - p$.*

Proof. *The proof similar as Proposition 2.1*

Corollary 2.1. *The probability of 2-randomly chosen elements generate $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$ is $\frac{1}{2}(1 - \frac{1}{p} - \frac{1}{p^2} + \frac{1}{p^3})$.*

Proof. *The proof follows from Proposition 2.1 and Definition 1.4.*

Let $n = pq$ where p, q are distinct odd prime numbers. Without loss of generality we may assume that $p < q$. It is well known that if $q \cong 1 \mod p$, then there are two groups of order $n = pq$ up to isomorphism, one is cyclic and one is non-abelain. First, we consider the cyclic case.

Proposition 2.3. *The generating graph $\Gamma(G)$ of $G \cong \mathbb{Z}_p \times \mathbb{Z}_q \equiv \mathbb{Z}_{pq}$ has $\frac{1}{2}(p^2q^2 - p^2 - q^2 - 3pq + 3p + 3q - 2)$ edges.*

Proof. *For $a, c \in \mathbb{Z}_p$ and $b, d \in \mathbb{Z}_q$, $\langle (a, b), (c, d)\rangle$ generates $G \cong \mathbb{Z}_p \times \mathbb{Z}_q$ if and only if there exist $r_1, r_2, r_3, r_4 \in \mathbb{Z}$ such that $r_1(a, b) + r_2(c, d) = (1, 0)$ and $r_3(a, b) + r_4(c, d) = (0, 1)$. It can be written in this form.*

$$A\begin{pmatrix} r_1 & r_3 \\ r_2 & r_4 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} r_1 & r_3 \\ r_2 & r_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

To solve this system, we need to take the inverse of A, which exists if and only the column of A are independent. The first coulmn (a, b) can be any thing except the zero. So, there are $(p-1)^2q$ choice for (a, b). The next column must be chosen so that it is not a multiple of (a, b). Hence, given (a, b), there are $(q - 1)^2p$ choices for the column (c, d). Also, there are $(pq - 2)(pq - p - q + 1)$ choices for $(a, 0), (0, d)$. Thus the number of choices for the pair $(a, b), (c, d)$ is $\frac{1}{2}((p - 1)^2q + (q - 1)^2p + (pq - 2)(pq - p - q + 1))$. We divide by 2 because it is abelian. Hence, there are $\frac{1}{2}((p - 1)^2q + (q - 1)^2p + (pq - 2)(pq - p - q + 1))$ ways to choose (a, b) and (c, d) such that $\langle (a, b), (c, d)\rangle$ generate $\mathbb{Z}_p \times \mathbb{Z}_q$ and we get the result.

Proposition 2.4. *Let $G \cong \mathbb{Z}_{pq}$ and let $\Gamma(G)$ be its generating graph. Then for any $v \in V(\Gamma(G)) \setminus \{0\}$, we have* $deg(v) = \begin{cases} n-p & if\ v \in \mathbb{Z}_p \\ n-q & if\ v \in \mathbb{Z}_q \\ n-2 & otherwise \end{cases}$

Proof. *It is clear that $|V(\Gamma(G))| = n - 1$. We can partition $V(\Gamma(G))$ into three sets as follows: $A = \{a, ..., a^{p-1}\}$, $B = \{b, ..., b^{q-1}\}$ and $AB = \{ab, ..., a^{p-1}b^{q-1}\}$. Now we know that any element in A is adjacent with any element in B or in AB, because they generate G. However the pair of elements in A cannot generate G. So $deg(v) = n-p$. We can use the same arguments for elements of B. Finally, it is clear that any element in AB is adjacent with any element except zero and itself, they generate G and the result holds.*

Corollary 2.2. *The probability of 2-randomly chosen elements generate $G \cong \mathbb{Z}_{pq}$ is $\frac{1}{2}(1 - (\frac{1}{p^2} + \frac{1}{q^2}) + \frac{1}{pq}(\frac{3}{p} + \frac{3}{q} - \frac{2}{pq}))$.*

Proof. *The proof follows from Proposition 2.3 and Definition 1.4.*

Here, we consider a non-abelian group G of order pq. That is, $xy \neq yx$ for some $x, y \in G$. In our situation, $G = \langle x, y\rangle = \langle y, x\rangle$.

Proposition 2.5. *Let $\Gamma(G)$ be the generating graph of $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$. Then for any $v \in V(\Gamma(G)) \setminus \{0\}$, we have* $deg(v) = \begin{cases} n-q & if\ v \in \mathbb{Z}_q \\ n-p & otherwise \end{cases}$

Proof. It is clear that $|V(\Gamma(G))| = n - 1$. We can partition $V(\Gamma(G))$ into two sets as follows: $B = \{b, ..., b^{q-1}\}$ and $AB = \{ab, ..., a^{p-1}b^{q-1}, a, a^2\}$. Now if we take any element in B and any elements in AB, they generate G. However the pair of elements in B cannot generate G. So $deg(v) = n - q$. Similarly, $deg(v) = n - p$ for $v \in AB$. This completes the proof. □

Proposition 2.6. *The generating graph $\Gamma(G)$ of $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ has $\frac{1}{2}(p^2q^2 - p^2q - q^2 + q)$ edges.*

Proof. *By Proposition 2.5, we have $deg(v) = n - q$ for $v \in \mathbb{Z}_q$ and there are $q - 1$ vertices of this degree. Also, we have $deg(v) = n - p$ for $v \in \mathbb{Z}_p$ and there are $n - q$ vertices of degree $n - p$. The rest follows from Lemma 1.1.*

The following result can be seen as the answer for the open question which exists in [4]. So the probability that two elements randomly chosen from $(\mathbb{Z}_p \rtimes \mathbb{Z}_q)$ generate it, is $(1 - \frac{1}{p^2})(1 - \frac{1}{q})$.

Corollary 2.3. *The right probability of 2-randomly chosen elements generate $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ is $\frac{1}{2}((1 - \frac{1}{p^2})(1 - \frac{1}{q}))$.*

Proof. *The proof follows from Proposition 2.6 and Definition 1.4.*

The following result shows that non-isomorphic groups have the same generating graph.

Proposition 2.7. *The generating graphs of $G \cong \mathbb{Z}_2 \times \mathbb{Z}_q$ and $H \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_q$ are isomorphic.*

Proof. *Let $G = \langle a, b | a^2 = 1 = b^q,\ ab = ba\rangle$ and $H = \langle x, y | x^2 = 1 = y^q,\ xyx = y^{-1}\rangle$. Define $f : G \setminus \{1\} \to H \setminus \{1\}$ by $f(a) = x$ and $f(b) = y$. It is clear that f is bijective and f preserves adjacency of vertices. Therefore, the generating graphs of $G \cong \mathbb{Z}_2 \times \mathbb{Z}_q$ and $H \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_q$ are isomorphic.*

In general, the generating graphs of $G \cong \mathbb{Z}_p \times \mathbb{Z}_q$ and $H \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ are not isomorphic if p and q are distinct odd prime numbers. For instance $\Gamma(\mathbb{Z}_7 \times \mathbb{Z}_3) \ncong \Gamma(\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$, because they have different number of edges.

Example 2.1. *Let $G = D_{2p} \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_p$. By Proposition 3.1, the generating graph is connected graph with $p(p-1)$ edges and its right probability $P_r(G) = \frac{3(p-1)}{8p} = P_l(G)$. Thus $P(G) = \frac{3(p-1)}{4p}$.*

Example 2.2. *Let $G = \mathbb{Z}_{2^m}$. There are 2^{m-1} vertices of degree $2^m - 2$. Also there are $2^{m-1} - 1$ vertices of degree 2^{m-1}. The generating graph is bi-regular connected graph with $3(2^{m-2})(2^{m-1} - 1)$ edges and its probability $P(G) = \frac{3(2^{m-1}-1)}{2^{m+1}}$.*

3. Some Properties of $\Gamma(G)$

In this section, we study some basic properties of $\Gamma(G)$, such as connectivity, radius, diameter and planarity.

Proposition 3.1. *The generating graph $\Gamma(G)$ of $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ is bi-regular connected with diameter and radius two. It has girth 3.*

Proof. *The bi-regularity follows from Proposition 2.5. Let* $u, v \in \Gamma(G)$. *If* $u, v \in \mathbb{Z}_q$, *then there exists* $z \in \Gamma(G) \setminus \mathbb{Z}_q$ *such that* $G = \langle u, z \rangle$ *and* $G = \langle v, z \rangle$. *So there exists a path between of them. If* $u \in \mathbb{Z}_q$ *and* v *is not, then* u *is adjacent to* v. *We are done. If* u *and* v *are adjacent and they are not in* $\mathbb{Z}_q$. *Then there is* x *in* $\mathbb{Z}_q$ *which is adjacent to* u *and* v. *Hence there is a path from* u *and* v. *For any* $u \in \Gamma(G)$. *Then* $ecc(v) = 2$ *by previous arguments. So* $rad(G) = diam(G) = 2$. *Finally, take* $a^k b^l$ *and* $a^m b^n$ *which are adjacent where* $1 \leq k, m \leq p-1$ *and* $1 \leq l, n \leq q-1$. *These two vertices are adjacent with* b^d *for some* d, *where* $1 \leq d \leq q-1$. *It gives* K_3 *in* $\Gamma(G)$. *So it has girth 3. This completes the proof.*

Proposition 3.2. *The generating graph* $\Gamma(G)$ *of* $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$ *is a regular connected with diameter and radius 2. It has girth 3.*

Proof. *The regularity follows from Proposition 2.2 and the rest is similar as Proposition 3.1.*

Proposition 3.3. *The generating graph* $\Gamma(G)$ *of* $G \cong \mathbb{Z}_p \times \mathbb{Z}_q$ *is tri-regular connected with diameter 2 and radius 1. It has girth 3.*

Proof. *The proof is similar to Proposition 3.1.*

Proposition 3.4. *If* p, q *are odd prime numbers and* $G = \mathbb{Z}_p \rtimes \mathbb{Z}_q$, *then the generating graph* $\Gamma(G)$ *is Eulerian.*

Proof. *The proof follows from Proposition 2.5 and Theorem 1.2.*

Proposition 3.5. *If* $G = \mathbb{Z}_p \rtimes \mathbb{Z}_q, \mathbb{Z}_p \times \mathbb{Z}_q, \mathbb{Z}_p \times \mathbb{Z}_p$, *then the generating graph* $\Gamma(G)$ *is Hamiltonian.*

Proof. *The proof follows from Propositions 2.5, 2.4, 2.2 and Theorem 1.2.*

Proposition 3.6. *If* p, q *are odd distinct prime numbers and* $G = \mathbb{Z}_p \rtimes \mathbb{Z}_q$, *then the generating graph* $\Gamma(G)$ *is not planar.*

Proof. *Since* q *and* p *are odd prime numbers then all* $pq - q$ *vertices of degree* $pq - p$ *are adjacent together which contains a subgraph* K_5. *So by Theorem 1.1, the generating graph* $\Gamma(G)$ *is not planar.*

Proposition 3.7. *If* p, q *are odd distinct prime numbers and* $G = \mathbb{Z}_p \times \mathbb{Z}_q$, *then the generating graph* $\Gamma(G)$ *is not planar.*

Proof. *Since* q *and* p *are odd prime numbers then all* $pq - p - q + 1$ *vertices of degree* $pq - 2$ *are adjacent together which contains a subgraph* K_5. *So by Theorem 1.1, the generating graph* $\Gamma(G)$ *is not planar.*

Proposition 3.8. *If* $p \geq 5$ *and* $G = D_{2p}$, *then the generating graph* $\Gamma(G)$ *is not planar.*

Proof. *Since* $q \geq 5$, *then all* $2p - q$ *vertices of degree* $2p - 2$ *are adjacent together which contains a subgraph* K_5. *So by Theorem 1.1, the generating graph* $\Gamma(G)$ *is not planar.*

We end up this work with the following remark.

Remark 3.1. *In general, if* n *and* m *are not both odd prime numbers, then* $\Gamma(G)$ *is not a connected graph. It can be seen in Figure 2.*

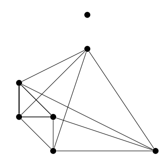

Figure 2. The generating graph of $\mathbb{Z}_4 \times \mathbb{Z}_2$

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