Indonesian Journal of Combinatorics **8** (2) (2024), 127–135

INDONESIAN JOURNAL OF COMBINATORICS

Locating Chromatic Number for Corona Operation of Path P_n and Cycle C_m, $(m = 3, 4)$

Nur Hamzah[a], Asmiati[a], Wahyu Dwi Amansyah[a]

[a]*Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Lampung, Bandar Lampung, Indonesia*

asmiati.1976@fmipa.unila.ac.id

Abstract

The locating chromatic number of a graph (*lcn*) is one of the topics in graph theory that is still interesting to research until now because there is no general theorem for determining the *lcn* of any graph. The corona operation of P_n and C_m, denoted by $P_n \odot C_m$ is defined as the graph obtained by taking one copy of P_n and $|V(P_n)|$ copies of C_m and then joining all the vertices of the k^{th} -copy of C_m with the k^{th}-vertex of P_n. In this paper, we discuss the *lcn* for the corona operation of path and cycle. The *lcn* of $P_n \odot C_3$ is 5 for $3 \leq n < 6$ and 6 for $n \geq 7$. Moreover, $P_n \odot C_4$ is 5 for $3 \leq n < 6$ and 6 for $n \geq 6$.

Keywords: locating chromatic number, corona operation, path, cycle
Mathematics Subject Classification : 05C12; 05C15.

1. Introduction

Chartand, et al.[13] in 2002 introduced the *lcn* by combining concept of coloring vertices and partition dimension of a graph. Let Z be a connected graph with $U(Z)$ is a set of vertices and $W(Z)$ is a set of edges. A $q-$coloring vertices of Z is a function $h : U(Z) \to \{1, 2, \ldots, q\}$ where $h(u) \neq h(v)$ for any two adjacent vertices u and v in Z. Let H_i be a set of vertices that are given a color i, hereinafter referred to as a color class, then partition $\prod = \{H_1,\ H_2,\ H_3,\ \ldots,\ H_q\}$ is a set of color classes from $U(Z)$. The color code for the vertex $v \in U(Z)$ denoted by $h_\Pi(v)$ are q-tuples $(d(v,\ H_1),\ d(v,\ H_2),\ d(v,\ H_3), \ldots,\ d(v,\ H_q)$ with $d(v,\ H_i) = \min\{d(v,\ x)|x \in H_i\}$

Received: 03 April 2024, Revised: 16 December 2024, Accepted: 30 December 2024.

for $1 \leq i \leq q$ where $d(v,\ x)$ is the minimum distance from vertex v to vertex x. If each vertices in Z has a different color code, then h is called a $q-$locating coloring (*q-lc*) of Z, denoted by $\chi(Z)$. While the minimum number of q colors used in coloring of Z is called the locating chromatic number (*lcn*) of Z, which is denoted by $\chi_L(Z)$.

The following theorem from chartrand [13] will be used to determine the *lcn* for corona operation of path and cycle.

Theorem 1.1. *The lcn of a cycle graph $C_n(n \geq 3)$ is 3 for odd n and 4 for even n.*

Chartrand, et al.[13] determined the *lcn* of some simple graphs, namely: complete graphs, cycles, paths, and caterpillars. Next, Chartrand et al.[14] obtained that there are trees T_n, $n \geq 5$ with *lcn* 1 varying n, except $n - 1$. In 2011, Asmiati et al.[1] studied the *lcn* for amalgamation of stars, other research results for several graph classes can be seen at [3], [4], [5], and [18]. Asmiati and Baskoro [2] characterized graphs having cycles with *lcn* 3. Then, Baskoro and Asmiati [11] in 2013 obtained the characterization of all tree with *lcn* 3.

Determining *lcn* of graph operations has been carried out. Asmiati et al.[6] determined *lcn* of barbell graphs contain Petersen graphs dan complete graphs. Next, Asmiati et al.[7] for *lcn* of subdivision of barbell graphs containing generalized Petersen graphs. Asmiati et al. [8] in 2021 succeeded the *lcn* of the path shadow graphs and its barbell, whereas Sudarsana et al.[20] for m-shadow of connected graphs. Recently, Asmiati et al.[10] investigated *lcn* for upper bounds for shadow cycle graphs. Asmiati et al. [9] determined the *lcn* for certain operation of origami graphs, whereas Behtoei and Omoomi[12] for cartesian product of graphs. Furuya and Matsumoto[16] obtained *lcn* of upper bound for trees, Ghanem et al.[17] for power of cycles and paths, Syofyan et al.[21] for homogeneous lobsters, and Welyyanti et al.[19] discussed the *lcn* of graphs contain dominant vertices.

The corona operation of P_n and C_m, denoted by $(P_n \odot C_m)$ is defined as the graph obtained by taking one copy of P_n and $|V(P_n)|$ copies of C_m and then joining all the vertices of the k^{th} -copy of C_m with the k^{th}-vertex of P_n [15]. We obtained $V(P_n \odot C_m) = \{v_k;\ k \in [1,\ n]\} \cup \{u_k^l;\ k \in [1,\ n],\ l \in [1,\ m]\}$ and $E(P_n \odot C_m) = \{v_k\, v_{k+1};\ k \in [1,\ n-1]\} \cup \{u_k^l\, u_k^{l+1};\ k \in [1,\ n],\ l \in [1,\ m-1]\} \cup \{u_k^1\, u_k^l;\ k \in [1,\ n],\ l = m\} \cup \{v_k\, u_k^l;\ k \in [1,\ n],\ l \in [1,\ m]\}$.

In this paper, we discuss the *lcn* for the corona operation of path and cycle $(P_n \odot C_m)$, where $n \geq 3$ and $m = 3, 4$.

2. Main Results

Theorem 2.1. *lcn for corona graph $P_n \odot C_3$ is 5 for $3 \leq n < 7$ and 6 for $n \geq 7$.*

PROOF.

Case 1. $(3 \leq n < 7)$ First, we determine the lower bound of the *lcn* of $P_n \odot C_3$. Since $P_n \odot C_3$ contains odd cycle graphs, based on Theorem 1.1, we have $\chi_L(P_n \odot C_3) \geq 3$. Suppose that h is a 3-*lc* (locating coloring) of $P_n \odot C_3$. It is clearly that there are two vertices with the same color, namely $h(u_k^a) = h(u_k^b)$ where $a \neq b$, so they have the same color code, a contradiction with the condition of *lcn*. Therefore, $\chi_L(P_n \odot C_3) \geq 4$. Suppose h is a 4-*lc* of $P_n \odot C_3$. Observe that $d(u_k^1,\ v) = d(u_k^3,\ v)$ where $v \notin \{u_k^1,\ u_k^3\}$ such that $\{h(u_k^a)\} = \{1, 2, 3, 4\} \backslash h(v_k)$. As a result, there are $h(u_a^m) = h(u_b^n)$ with $a \neq b,\ m \neq n$ and $h_\Pi(u_a^m) = h_\Pi(u_b^n)$, a contradiction with the condition of *lcn*. So, $\chi_L(P_n \odot C_3) \geq 5$.

Next, to determine the upper bound of the *lcn* of $P_n \odot C_3$ for $3 \leq n < 7$. Let h be 5-coloring in $P_3 \odot C_3$ as follows: $H_1 = \{u_1^1,\ u_3^1\},\ H_2 = \{u_1^2,\ u_2^1,\ u_3^2\},\ H_3 = \{u_1^3,\ u_2^2\},\ H_4 = \{u_2^3,\ v_1, v_3\},\ H_5 = \{u_3^3,\ v_2\}$. Then we have the following color code: $h_\Pi(u_1^1) = \{0,1,1,1,2\}$; $h_\Pi(u_1^2) = \{1,0,1,1,2\}$; $h_\Pi(u_1^3) = \{1,1,0,1,2\}$; $h_\Pi(u_2^1) = \{3,0,1,1,1\}$; $h_\Pi(u_2^2) = \{3,1,0,1,1\}$; $h_\Pi(u_2^3) = \{3,1,1,0,1\}$; $h_\Pi(u_3^1) = \{0,1,3,1,1\}$; $h_\Pi(u_3^2) = \{1,0,3,1,1\}$; $h_\Pi(u_3^3) = \{1,1,3,1,0\}$; $h_\Pi(v_1) = \{1,1,1,0,1\}$; $h_\Pi(v_2) = \{2,1,1,1,0\}$; $h_\Pi(v_3) = \{1,1,2,0,1\}$. Since all vertices in $P_3 \odot C_3$ have distinct color codes, then h is a 5-*lc*. So, $\chi_L(P_3 \odot C_3) \leq 5$. Therefore, we have $\chi_L(P_3 \odot C_3) = 5$.

Let h be 5-coloring in $P_4 \odot C_3$ as follows: $H_1 = \{u_1^1,\ u_3^1,\ u_4^1\},\ H_2 = \{u_1^2,\ u_2^1,\ u_3^2\},\ H_3 = \{u_1^3,\ u_2^2,\ u_4^2\},\ H_4 = \{u_2^3,\ u_4^3,\ v_1,\ v_3\},\ H_5 = \{u_3^3,\ v_2,\ v_4\}$. The color codes of vertices are: $h_\Pi(u_1^1) = \{0,\ 1,\ 1,\ 1,\ 2\}$; $h_\Pi(u_1^2) = \{1,\ 0,\ 1,\ 1,\ 2\}$; $h_\Pi(u_1^3) = \{1,\ 1,\ 0,\ 1,\ 2\}$; $h_\Pi(u_2^1) = \{3,0,1,1,1\}$; $h_\Pi(u_2^2) = \{3,1,0,1,1\}$; $h_\Pi(u_2^3) = \{3,1,1,0,1\}$; $h_\Pi(u_3^1) = \{0,1,3,1,1\}$; $h_\Pi(u_3^2) = \{1,0,3,1,1\}$; $h_\Pi(u_3^3) = \{1,1,3,1,0\}$; $h_\Pi(u_4^1) = \{0,3,1,1,1\}$; $h_\Pi(u_4^2) = \{1,3,0,1,1\}$; $h_\Pi(u_4^3) = \{1,3,1,0,1\}$; $h_\Pi(v_1) = \{1,1,1,0,1\}$; $h_\Pi(v_2) = \{2,1,1,1,0\}$; $h_\Pi(v_3) = \{1,1,2,0,1\}$; $h_\Pi(v_4) = \{1,2,1,1,0\}$. Since all vertices in $P_4 \odot C_3$ have distinct color codes, then h is a 5-*lc*. We have, $\chi_L(P_4 \odot C_3) \leq 5$. Thus, $\chi_L(P_4 \odot C_3) = 5$.

Let h be 5-coloring in $P_5 \odot C_3$ as follows: $H_1 = \{u_1^1,\ u_3^1,\ u_4^1,\ v_5\},\ H_2 = \{u_1^2,\ u_2^1,\ u_3^2,\ u_5^1\},\ H_3 = \{u_1^3,\ u_2^2,\ u_4^2,\ u_5^2\},\ H_4 = \{u_2^3, u_4^3, v_1,\ v_3\},\ H_5 = \{u_3^3,\ u_5^3,\ v_2,\ v_4\},$. Then,the color codes of vertices are: $h_\Pi(u_1^1) = \{0,1,1,1,2\}$; $h_\Pi(u_1^2) = \{1,0,1,1,2\}$; $h_\Pi(u_1^3) = \{1,1,0,1,2\}$; $h_\Pi(u_2^1) = \{3,0,1,1,1\}$; $h_\Pi(u_2^2) = \{3,1,0,1,1\}$; $h_\Pi(u_2^3) = \{3,1,1,0,1\}$; $h_\Pi(u_3^1) = \{0,1,3,1,1\}$; $h_\Pi(u_3^2) = \{1,0,3,1,1\}$; $h_\Pi(u_3^3) = \{1,1,3,1,0\}$; $h_\Pi(u_4^1) = \{0,3,1,1,1\}$; $h_\Pi(u_4^2) = \{1,3,0,1,1\}$; $h_\Pi(u_4^3) = \{1,3,1,0,1\}$; $h_\Pi(u_5^1) = \{1,0,1,3,1\}$; $h_\Pi(u_5^2) = \{1,1,0,3,1\}$; $h_\Pi(u_5^3) = \{1,1,1,3,0\}$; $h_\Pi(v_1) = \{1,1,1,0,1\}$; $h_\Pi(v_2) = \{2,\ 1,1,1,0\}$; $h_\Pi(v_3) = \{1,1,2,0,1\}$; $h_\Pi(v_4) = \{1,2,1,1,\ 0\}$; $h_\Pi(v_5) = \{0,1,1,2,1\}$. Since all vertices in $P_5 \odot C_3$ have distinct color codes. Then, h is a 5-*lc*. Therefore, $\chi_L(P_5 \odot C_3) \leq 5$. So, $\chi_L(P_5 \odot C_3) = 5$.

Let h be 5-coloring in $P_6 \odot C_3$ as follows: $H_1 = \{u_1^1,\ u_3^1,\ u_4^1,\ u_6^1,\ v_5\},\ H_2 = \{u_1^2,\ u_2^1,\ u_3^2,\ u_5^1,\ u_6^2\},\ H_3 = \{u_1^3,\ u_2^2,\ u_4^2,\ u_5^2,\ u_6^3\},\ H_4 = \{u_2^3,\ u_4^3,\ v_1,\ v_3,\ v_6\},\ H_5 = \{u_3^3,\ u_5^3,\ v_2,\ v_4\}$. Then the color codes of vertices are: $h_\Pi(u_1^1) = \{0,1,1,1,2\}$; $h_\Pi(u_1^2) = \{1,0,1,1,2\}$; $h_\Pi(u_1^3) = \{1,1,0,1,2\}$; $h_\Pi(u_2^1) = \{3,0,1,1,1\}$; $h_\Pi(u_2^2) = \{3,1,0,1,1\}$; $h_\Pi(u_2^3) = \{3,1,1,0,1\}$; $h_\Pi(u_3^1) = \{0,1,3,1,1\}$; $h_\Pi(u_3^2) = \{1,0,3,1,1\}$; $h_\Pi(u_3^3) = \{1,1,3,1,0\}$; $h_\Pi(u_4^1) = \{0,3,1,1,1\}$; $h_\Pi(u_4^2) = \{1,3,0,1,1\}$; $h_\Pi(u_4^3) = \{1,3,1,0,1\}$; $h_\Pi(u_5^1) = \{1,0,1,3,1\}$; $h_\Pi(u_5^2) = \{1,1,0,3,1\}$; $h_\Pi(u_5^3) = \{1,1,1,3,0\}$; $h_\Pi(u_6^1) = \{0,1,1,1,3\}$; $h_\Pi(u_6^2) = \{1,0,1,1,3\}$; $h_\Pi(u_6^3) = \{1,1,0,1,3\}$; $h_\Pi(v_1) = \{1,1,1,0,1\}$; $h_\Pi(v_2) = \{2,1,1,1,0\}$; $h_\Pi(v_3) = \{1,1,2,0,1\}$; $h_\Pi(v_4) = \{1,2,1,1,0\}$; $h_\Pi(v_5) = \{0,1,1,2,1\}$; $h_\Pi(v_6) = \{1,1,1,0,2\}$. Since all vertices in $P_6 \odot C_3$ have distinct color codes, then h is a 5-*lc*. Therefore, we have $\chi_L(P_6 \odot C_3) \leq 5$. So, $\chi_L(P_6 \odot C_3) = 5$.

Case 2. $(n \geq 7)$.

First, we determine the lower bound of *lcn* of $P_n \odot C_3$ for $n \geq 7$. Suppose h is a 5-*lc* of $P_n \odot C_3$ for $n \geq 7$. Then, there are $h(u_a^m) = h(u_b^n)$ with $a \neq b,\ m \neq n$ and $h_\Pi(u_a^m) = h_\Pi(u_b^n)$, contradiction with the condition of *lcn*. So, we have $\chi_L(P_n \odot C_4) \geq 6$ for $n \geq 7$.

Let h be a 6-coloring in $P_n \odot C_3$ for $n \geq 7$ as follows:

$$h(u_k^l) = \begin{cases} 1, & \text{for } k = 1, 2, \ldots, \ n \text{ and } l = 1. \\ 2, & \text{for } k = 1, 2, \ldots, \ n \text{ and } l = 2. \\ 3, & \text{for } k = 1, 2, \ldots, \ n-1 \text{ and } l = 3. \\ 6, & \text{for } k = n \text{ and } l = 3. \end{cases}$$

$$h(v_k) = \begin{cases} 4, & \text{for odd } k. \\ 5, & \text{for even } k. \end{cases}$$

Then the color code of vertices are:

$$h_\Pi(u_k^l) = \begin{cases} 0 & , 1^{st} \text{ tuple; } k = 1, 2, \ldots, n \text{ and } l = 1; \\ & , 2^{nd} \text{ tuple; } k = 1, 2, \ldots, n \text{ and } l = 2; \\ & , 3^{rd} \text{ tuple; } k = 1, 2, \ldots, n-1 \text{ and } l = 3; \\ & , 6^{th} \text{ tuple; } k = n \text{ and } l = 3. \\ 1 & , 1^{st} \text{ tuple; } k = 1, 2, \ldots, n \text{ and } \ l = 2, 3; \\ & , 2^{nd} \text{ tuple; } k = 1, 2, \ldots, n \text{ and } l = 1, \ 3; \\ & , 3^{rd} \text{ tuple; } k = 1, 2, \ldots, n \text{ and } l = 1, \ 2; \\ & , 4^{th} \text{ tuple; odd } k \text{ and } l = 1, \ 2, \ 3; \\ & , 5^{th} \text{ tuple; even } k \text{ and } l = 1, \ 2\ , 3; \\ & , 6^{th} \text{ tuple; } k = n \text{ and } l = 1, \ 2. \\ 2 & , 4^{th} \text{ tuple; even } k; \\ & , 5^{th} \text{ tuple; odd } k. \\ (n-k)+2 & , 6^{th} \text{ tuple; } k \leq n-1. \end{cases}$$

$$h_\Pi(v_k) = \begin{cases} 0 & , 4^{th} \text{ tuple; odd } k; \\ & , 5^{th} \text{ tuple; even } k. \\ 1 & , 1^{st} \text{ tuple; and } k = 1, 2, \ldots, n; \\ & , 2^{nd} \text{ tuple; } k = 1, 2, \ldots, n; \\ & , 3^{rd} \text{ tuple; } k = 1, 2, \ldots, n; \\ & , 4^{th} \text{ tuple; even } k; \\ & , 5^{th} \text{ tuple; odd } k. \\ (n-k)+1 & , 6^{th} \text{ tuple; } 1 \leq k \leq n. \end{cases}$$

Thus, since all vertices in $P_n \odot C_3$ have distinct color codes, then h is a 6-*lc*. Therefore, $\chi_L(P_n \odot C_3) = 6$ for $n \geq 7$. ■

Theorem 2.2. *lcn of corona graph* $P_n \odot C_4$ *is* 5 *for* $3 \leq n < 6$, *and* 6 *for* $n \geq 6$.

PROOF.

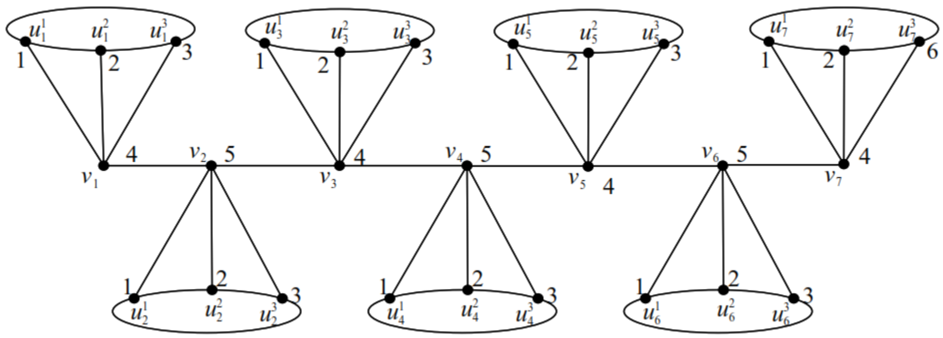

Figure 1. A minimum *lc* of $P_7 \odot C_3$

Case 1. $(3 \leq n < 6)$ First, we determine the lower bound of the *lcn* of $P_n \odot C_4$. Since $P_n \odot C_4$ contains cycle graphs, based on Theorem 1.1 we have $\chi_L(P_n \odot C_4) \geq 4$. For a contradiction, assume there exists a $4-$*lc* on $P_n \odot C_4$. It is clearly that there are two vertices with the same color, namely $h(u_k^a) = h(u_k^b)$ where $a \neq b$, so they have the same color code, a contradiction. Therefore, $\chi_L(P_n \odot C_3) \geq 5$.

Next, to determine the upper bound of the locating chromatic of $P_n \odot C_4$ for $3 \leq n < 5$. Let h be $5-$coloring in $P_3 \odot C_4$ as follows: $H_1 = \{u_1^1,\ u_2^1,\ u_3^3\}$, $H_2 = \{u_1^2,\ u_2^2,\ u_3^1\}$, $H_3 = \{u_1^3,\ u_2^4,\ v_3\}$, $H_4 = \{u_1^4,\ u_3^2,\ v_2\}$, $H_5 = \{u_2^3,\ u_3^4,\ v_1\}$. Then,The color codes of vertices are: $h_\Pi(u_1^1) = \{0,1,2,1,1\}$; $h_\Pi(u_1^2) = \{1,0,1,2,1\}$; $h_\Pi(u_1^3) = \{2,1,0,1,1\}$; $h_\Pi(u_1^4) = \{1,2,1,0,1\}$; $h_\Pi(u_2^1) = \{0,1,1,1,2\}$; $h_\Pi(u_2^2) = \{1,0,2,1,1\}$; $h_\Pi(u_2^3) = \{2,1,1,1,0\}$; $h_\Pi(u_2^4) = \{1,2,0,1,1\}$; $h_\Pi(u_3^1) = \{2,0,1,1,1\}$; $h_\Pi(u_3^2) = \{1,1,1,0,2\}$; $h_\Pi(u_3^3) = \{0,2,1,1,1\}$; $h_\Pi(u_3^4) = \{1,1,1,2,0\}$; $h_\Pi(v_1) = \{1,1,1,1,0\}$; $h_\Pi(v_2) = \{1,1,1,0,1\}$; $h_\Pi(v_3) = \{1,1,0,1,1\}$. Thus, since all vertices in $P_3 \odot C_4$ have distinct color code, then h is a $5-$*lc*. So, $\chi_L(P_3 \odot C_4) \leq 5$. Therefore, we have $\chi_L(P_3 \odot C_4) = 5$.

Let h be a $5-$coloring in $P_4 \odot C_4$ as follows: $H_1 = \{u_1^1,\ u_2^1,\ u_3^3\}$, $H_2 = \{u_1^2,\ u_2^2,\ u_3^1\}$, $H_3 = \{u_1^3,\ u_2^4,\ v_3\}$, $H_4 = \{u_1^4,\ u_3^2,\ v_2\}$, $H_5 = \{u_2^3,\ u_3^4,\ v_1\}$. Then, The color codes of vertices are: $h_\Pi(u_1^1) = \{0,1,2,1,1\}$; $h_\Pi(u_1^2) = \{1,0,1,2,1\}$; $h_\Pi(u_1^3) = \{2,1,0,1,1\}$; $h_\Pi(u_1^4) = \{1,2,1,0,1\}$; $h_\Pi(u_2^1) = \{0,1,1,1,2\}$; $h_\Pi(u_2^2) = \{1,0,2,1,1\}$; $h_\Pi(u_2^3) = \{2,1,1,1,0\}$; $h_\Pi(u_2^4) = \{1,2,0,1,1\}$; $h_\Pi(u_3^1) = \{2,0,1,1,1\}$; $h_\Pi(u_3^2) = \{1,1,1,0,2\}$; $h_\Pi(u_3^3) = \{0,2,1,1,1\}$; $h_\Pi(u_3^4) = \{1,1,1,2,0\}$; $h_\Pi(u_4^1) = \{0,1,1,2,1\}$; $h_\Pi(u_4^2) = \{1,1,0,1,2\}$; $h_\Pi(u_4^3) = \{2,1,1,0,1\}$; $h_\Pi(u_4^4) = \{1,1,2,1,0\}$; $h_\Pi(v_1) = \{1,1,1,1,0\}$; $h_\Pi(v_2) = \{1,1,1,0,1\}$; $h_\Pi(v_3) = \{1,1,0,1,1\}$; $h_\Pi(v_4) = \{1,0,1,1,1\}$. Thus, since all vertices in $P_4 \odot C_4$ have distinct color code, then h is a $5-$*lc*. So, $\chi_L(P_4 \odot C_4) \leq 5$ Therefore, we have $\chi_L(P_4 \odot C_4) = 5$.

Let h be $5-$coloring in $P_5 \odot C_4$ as follows: $H_1 = \{u_1^1,\ u_2^1,\ u_3^3,\ v_5\}$, $H_2 = \{u_1^2,\ u_2^2,\ u_3^1,\ u_5^1\}$, $H_3 = \{u_1^3,\ u_2^4,\ u_5^2,\ v_3\}$, $H_4 = \{u_1^4,\ u_3^2,\ u_5^4,\ v_2\}$, $H_5 = \{u_2^3,\ u_3^4,\ u_5^3, v_1\}$. Then, the color codes of vertices are: $h_\Pi(u_1^1) = \{0,1,2,1,1\}$; $h_\Pi(u_1^2) = \{1,0,1,2,1\}$; $h_\Pi(u_1^3) = \{2,1,0,1,1\}$; $h_\Pi(u_1^4) = \{1,2,1,0,1\}$; $h_\Pi(u_2^1) = \{0,1,1,1,2\}$; $h_\Pi(u_2^2) = \{1,0,2,1,1\}$; $h_\Pi(u_2^3) = \{2,1,1,1,0\}$; $h_\Pi(u_2^4) = \{1,2,0,1,1\}$; $h_\Pi(u_3^1) = \{2,0,1,1,1\}$; $h_\Pi(u_3^2) = \{1,1,1,0,2\}$; $h_\Pi(u_3^3) = \{0,2,1,1,1\}$; $h_\Pi(u_3^4) = \{1,1,1,2,0\}$; $h_\Pi(u_4^1) = \{0,1,1,2,1\}$; $h_\Pi(u_4^2) = \{1,1,0,1,2\}$; $h_\Pi(u_4^3) = \{2,1,1,0,1\}$; $h_\Pi(u_4^4) =$

$\{1,1,2,1,0\}$; $h_\Pi(u_5^1) = \{1,0,1,1,2\}$; $h_\Pi(u_4^2) = \{1,1,0,2,1\}$; $h_\Pi(u_4^3) = \{1,2,1,1,0\}$; $h_\Pi(u_4^4) = \{1,1,2,0,1\}$; $h_\Pi(v_1) = \{1,1,1,1,0\}$; $h_\Pi(v_2) = \{1,1,1,0,1\}$; $h_\Pi(v_3) = \{1,1,0,1,1\}$; $h_\Pi(v_4) = \{1,0,1,1,1\}$; $h_\Pi(v_5) = \{0,1,1,1,1\}$. Thus, since all vertices in $P_5 \odot C_4$ have distinct color code, then h is a $5-$*lc*. So, $\chi_L(P_5 \odot C_4) \le 5$. Therefore, we have $\chi_L(P_5 \odot C_4) = 5$.

Case 2. $(n \ge 6)$ First we determine the lower bound of the *lcn* of $P_n \odot C_4$. Suppose h is a $5-$*lc* of $P_6 \odot C_4$. Observe that $d(u_i^1,\ v) = d(u_i^4,\ v)$ where $v \notin \{u_i^1,\ u_i^4\}$ such that $\{h(u_i^k)\} = \{1,2,3,4\}\backslash h(v_i)$. As a result, there are $h(u_a^m) = h(u_b^n)$ with $a \ne b,\ m \ne n$ and $h_\Pi(u_a^m) = h_\Pi(u_b^n)$, a contradiction with the condition of *lcn*. So, $\chi_L(P_n \odot C_4) \ge 6$ for $n \ge 6$.

Next, to determine the upper bound of the *lcn* of $P_n \odot C_4$ for $n \ge 6$. Let h be a coloring using 6 colors as follow:

$$h(u_k^l) = \begin{cases} 1 & \text{, for } k = 1 \text{ and } l = 1. \\ 2 & \text{, for } k = 1 \text{ and } l = 2;\ k = 2,\ 3, \ldots,\ n \text{ and } l = 1. \\ 3 & \text{, for } k = 1 \text{ and } l = 3;\ k = 2,\ 3, \ldots,\ n \text{ and } l = 2. \\ 4 & \text{, for } k = 1 \text{ and } l = 4;\ k = 2,\ 3,\ \ldots,\ n \text{ and } l = 3. \\ 5 & \text{, for even } k \text{ and } l = 4. \\ 6 & \text{, for odd } k,\ k \ne 1 \text{ and } l = 4. \end{cases}$$

$$h(v_k) = \begin{cases} 5, & \text{for odd } k. \\ 6, & \text{for even } k. \end{cases}$$

Then, the color code of vertices are:

$$h_\Pi(v_k) = \begin{cases} 0 & ,5^{th} \text{ tuple; odd } k; \\ & ,6^{th} \text{ tuple; even } k. \\ 1 & ,2^{nd} \text{ tuple; } k = 1,2,\ldots,n; \\ & ,3^{rd} \text{ tuple; } k = 1,2,\ldots,n; \\ & ,4^{th} \text{ tuple; } k = 1,2,\ldots,n; \\ & ,5^{th} \text{ tuple; even } k; \\ & ,5^{th} \text{ tuple; odd } k. \\ (n-k)+1 & ,6^{st} \text{ tuple; } 1 \le k \le n. \end{cases}$$

$$h_\Pi(u_k^l) = \begin{cases} 0 & , 1^{st} \text{ tuple; } k = 1 \text{ and } l = 1; \\ & , 2^{nd} \text{ tuple; } k = 1 \text{ and } l = 2; \\ & , 2^{nd} \text{ tuple; } k \neq 1 \text{ and } l = 1; \\ & , 3^{rd} \text{ tuple; } k = 1 \text{ and } l = 3; \\ & , 3^{rd} \text{ tuple; } k \neq 1 \text{ and } l = 2; \\ & , 4^{th} \text{ tuple; } k = 1 \text{ and } l = 4; \\ & , 4^{th} \text{ tuple; } k \neq 1 \text{ and } l = 3; \\ & , 5^{th} \text{ tuple; even } k, k \neq 1, \text{ and } l = 4; \\ & , 5^{th} \text{ tuple; odd } k, k \neq 1, \text{ and } l = 4. \\ 1 & , 1^{st} \text{ tuple; } k = 1 \text{ and } l = 2, 4; \\ & , 2^{nd} \text{ tuple; } k = 1 \text{ and } l = 1, 3; \\ & , 2^{nd} \text{ tuple; } k \neq 1 \text{ and } l = 2, 4; \\ & , 3^{rd} \text{ tuple; } k = 1 \text{ and } l = 2, 4; \\ & , 3^{rd} \text{ tuple; } k \neq 1 \text{ and } l = 1, 3; \\ & , 4^{th} \text{ tuple; } k = 1 \text{ and } l = 1, 3; \\ & , 4^{th} \text{ tuple; } k \neq 1 \text{ and } l = 2, 4; \\ & , 5^{th} \text{ tuple; odd } k \text{ and } l = 1, 2, 3, 4; \\ & , 5^{th} \text{ tuple; even } k \text{ and } l = 2, 3; \\ & , 6^{th} \text{ tuple; even } k \text{ and } l = 1, 2, 3, 4; \\ & , 6^{th} \text{ tuple; odd } k, k \neq 1, \text{ and } l = 2, 3. \\ 2 & , 1^{st} \text{ tuple; } k = 1 \text{ and } l = 3; \\ & , 2^{nd} \text{ tuple; } k = 1 \text{ and } l = 4; \\ & , 2^{nd} \text{ tuple; } k \neq 1 \text{ and } l = 3; \\ & , 3^{rd} \text{ tuple; } k = 1 \text{ and } l = 1; \\ & , 3^{rd} \text{ tuple; } k \neq 1 \text{ and } l = 4; \\ & , 4^{th} \text{ tuple; } k = 1 \text{ and } l = 2; \\ & , 4^{th} \text{ tuple; } k \neq 1 \text{ and } l = 1; \\ & , 5^{th} \text{ tuple; even } k \text{ and } l = 2; \\ & , 6^{th} \text{ tuple; } k = 1 \text{ and } l = 1, 2, 3, 4; \\ & , 6^{th} \text{ tuple; odd } k, k \neq 1, \text{ and } l = 2. \\ k+1 & , 1^{st} \text{ tuple; } 1 < k \leq n \text{ and } l = 1, 2, 3, 4. \end{cases}$$

Since all vertices in $P_n \odot C_4$ have distinct color code, then h is a $6-lc$. So, $\chi_L(P_n \odot C_4) \leq 6$ for $n \geq 6$. Therefore $\chi_L(P_n \odot C_4) = 6$ for $n \geq 6$. ■

3. Conclusion

The *lcn* of $P_n \odot C_m$ is 5 for $m = 3$, $3 \leq n < 6$ and 6 for $n \geq 7$. Further, for $m = 4$ we have

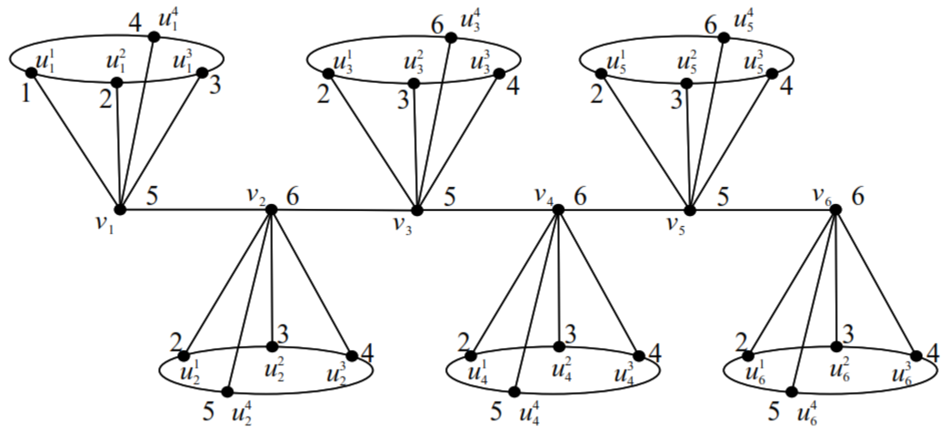

Figure 2. A minimum *lc* of $P_6 \odot C_4$

5, where $3 \leq n < 6$ and 6 for $n \geq 6$. This research can be continued to determine the locating chromatic number of $P_n \odot C_m$ for $m \geq 5$.

Acknowledgement

The research was supported by the Directorate of Higher Education, Ministry of Education and Culture, under contract No. 2624/UN26.21/PN/2023. The authors are thankful for the funding of the research.

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