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Seidel spectrum for splitting V-vertex join and S-vertex join of graphs

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Abstract

Splitting V-vertex join and S-vertex join of graphs are the graph structures obtained from splitting graph of a graph. Present work concentrates on the study of Seidel spectrum for the vertex join structures considered.

Keywords: Splitting graph of a graph, join of two graphs, Seidel spectrum
Mathematics Subject Classification : 05C50

1. Introduction

In recent times, scholarly articles are all about spectral properties of graph structures resulted through graph operations [1, 2, 3, 5, 7, 8, 9, 11, 13, 14]. Motivated by definitions of various graph operations involving join structres, Z. Lu in 2023 [12] defined two more new graph operations namely, splitting V-vertex join and splitting S-vertex join which are obtained from splitting graph [15] and hence studied adjacency, Laplacian and signless Laplacian spectrum for these defined graph operations containing regular graphs. Here, in the current study we focus on the Seidel spectrum for these vertex join operations containing regular graphs.

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2. Preliminaries

For the present study finite, simple, connected and undirected graphs are accounted. An ordered pair $(V(G), E(G))$ comprising of vertex set $V(G)$ and edge set $E(G)$ together with an incidence relation between them, is defined to be a graph G. $V(G)$'s cardinality accounts to order of G. Vertices which are linked by an edge are adjacent (neighbors). Number of edges at a vertex v corresponds to its degree d_i, if $d_i = r$ (a constant) for every vertex v_i then G is a graph with regularity r. Graph generated from G, possessing same number of vertices as G, where two vertices are adjacent if and only if they are not adjacent in G, is the complement $\overline{G}$. For a graph G of order n, adjacency matrix is $\mathcal{A}(G) = [a_{ij}]_{n\times n}$ with $a_{ij} = 1$ if there is adjacecy between vertices v_i and v_j and $a_{ij} = 0$, in other case. Eigenvalues associated to $A(G)$ are adjacency eigenvalues (noted as $\lambda_i(G)$). Seidel matrix [10, 16] for a graph of order n is $\mathcal{S}(G) = [s_{ij}]_{n\times n}$ in which $s_{ij} = -1$ if there is adjacency among v_i and v_j, $s_{ij} = 1$ if v_i and v_j are not adjacent and $s_{ij} = 0$, in other case. Eigenvalues associated to $\mathcal{S}(G)$ are Seidel eigenvalues, their collection is Seidel spectrum. For a graph G, $S(G) = A(\overline{G}) - A(G)$. J denote all 1's matrix, I is the unit matrix. We adhere to the book [6] for undefined graph theoretical terms and notations.

Definition 1. *[15] Splitting graph $SP(G)$ of a graph G is the graph resulted from G by inserting a new vertex u' for every $u \in V(G)$ and joining u' to those vertices of G which are neighbors of u.*

Definition 2. *[4] Join $G_1 \vee G_2$ of vertex-disjoint graphs G_1 and G_2 is resulted by inserting all possible edges between G_1 and G_2.*

Definition 3. *[12] For graphs G_1 and G_2 having vertices n_1 and n_2, respectively.*

i. Splitting V-vertex join given as $G_1 \underline{\vee} G_2$ is a consequence of $SP(G_1)$ by connecting every vertex of G_1 to those of G_2.

ii. Splitting S-vertex join given as $G_1 \underline{\triangle} G_2$ is a consequence of $SP(G_1)$ by connecting vertices of $S(G_1)$ to those of G_2, where $S(G_1) = V\big(SP(G_1)\big) - V(G_1)$.

Definition 4. *[4] Partition $V_1 \bigcup V_2 \bigcup \ldots \bigcup V_k$ of vertices in graph G is equitable if any two vertices in V_i have the same number of neighbors in V_j for all i, j.*

Reduced matrix obtained due to an equitable partition is called quotient matrix (Q).

Proposition 2.1. *[4] If $\lambda_i(G)$ for $i = 1, 2, \ldots n$ ($\lambda_1(G) = r$) are A-eigenvalues of a graph G of order n and regularity r, eigenvalues of $A(\overline{G})$ are $n - r - 1$ and $-1 - \lambda_i(G), i = 2, 3, \ldots n$.*

Theorem 2.1. *[4] For the quotient matrix Q corresponding to an equitable partition of any square matrix M, the spectrum of M contains the spectrum of Q.*

3. Seidel spectrum for splitting V-vertex join of graphs

Theorem 3.1. *For graphs G_1, G_2 on n_1, n_2 vertices having r_1, r_2 regularities, respectively. Seidel spectrum of the spitting V-vertex join involves the following:*

i. $-1-2\lambda_i(G_2)$ *for* $i=2,3,\ldots,n_2$

ii. roots of the equation

$$x^2+2\big(\lambda_i(G_1)+1\big)x-\big(4\lambda_i^2(G_1)-2\lambda_i(G_1)-1\big)$$

for $i=2,3,\ldots,n_1$

iii. roots of the equation

$$\begin{aligned}&x^3+(2r_1+2r_2-2n_1-n_2+3)x^2\\&\quad+(2n_1r_1-2n_2r_1-4n_1r_2+4r_1r_2-4r_1^2-4n_1-2n_2+4r_1+4r_2+3)x\\&\quad+(4n_1^2n_2+4n_2r_1^2-8r_1^2r_2-8n_1n_2r_1+4n_1r_1r_2+2n_1r_1-2n_2r_1-4n_1r_2\\&\qquad\qquad+4r_1r_2-4r_1^2-2n_1-n_2+2r_1+2r_2+1).\end{aligned}$$

Proof. General Seidel matrix due to the vertex join structure with a desired labeling according to the ordering of $V(G_1), S(G_1)$ and $V(G_2)$ is:

$$\mathcal{S}(G_1\underline{\vee}G_2)=\begin{pmatrix}A(\overline{G_1})-A(G_1) & I_{n_1}+A(\overline{G_1})-A(G_1) & -J_{n_1\times n_2}\\ I_{n_1}+A(\overline{G_1})-A(G_1) & (J-I)_{n_1} & J_{n_1\times n_2}\\ -J_{n_2\times n_1} & J_{n_2\times n_1} & A(\overline{G_2})-A(G_2)\end{pmatrix}.$$

Graphs G_1 and G_2 are regular, partition $V(G_1)\bigcup V\Big(S(G_1)\Big)\bigcup V(G_1)$ of the vertex set $V\Big(G_1\underline{\vee}G_2\Big)$ will be equitable. As in the vertex join structure

i. elements of $V(G_1)$ have r_1 neighbors among themselves, r_1 neighbors with those of $S(G_1)$ and n_2 neighbors with $V(G_2)$

ii. elements of $S(G_1)$ have no neighbors among themselves, r_1 neighbors with those of $V(G_1)$ and no neighbors with $V(G_2)$

iii. elements of $V(G_2)$ have r_2 neighbors among themselves, no neighbors with those of $S(G_1)$ and n_1 neighbors with $V(G_1)$.

Thus, reduced matrix due the equitable partition considered will be

$$Q_{\mathcal{S}}(G_1\underline{\vee}G_2)=\begin{pmatrix}n_1-2r_1-1 & n_1-2r_1 & -n_2\\ n_1-2r_1 & n_1-1 & n_2\\ -n_1 & n_1 & n_2-2r_2-1\end{pmatrix}.$$

Polynomial due to $Q_{\mathcal{S}}(G_1\underline{\vee}G_2)$ is,

$$\begin{aligned}\phi_{Q_\mathcal{S}} = x^3 &+ (2r_1 + 2r_2 - 2n_1 - n_2 + 3)x^2 \\ &+(2n_1r_1 - 2n_2r_1 - 4n_1r_2 + 4r_1r_2 - 4r_1^2 - 4n_1 - 2n_2 + 4r_1 + 4r_2 + 3)x \\ &+(4n_1^2n_2 + 4n_2r_1^2 - 8r_1^2r_2 - 8n_1n_2r_1 + 4n_1r_1r_2 + 2n_1r_1 - 2n_2r_1 - 4n_1r_2 \\ &\qquad +4r_1r_2 - 4r_1^2 - 2n_1 - n_2 + 2r_1 + 2r_2 + 1).\end{aligned} \tag{1}$$

Roots of $Q_\mathcal{S}(G_1 \underline{\vee} G_2)$ will be a part of the $\mathcal{S}$-spectrum (from Theorem 2.1).
Residual eigenvalues are examined by observing $SP(G_1)$ and G_2.
Due to G_2 (applying Proposition 2.1), we have

$$-1 - 2\lambda_i(G_2) \ \text{ for } \ i = 2, 3, \ldots, n_2 \tag{2}$$

obtained from $A(\overline{G_1}) - A(G_1)$.
From the structure of $SP(G_1)$, we take part of the matrix $\mathcal{S}(G_1 \underline{\vee} G_2)$, i.e.,

$$\begin{pmatrix} A(\overline{G_1}) - A(G_1) & I_{n_1} + A(\overline{G_1}) - A(G_1) \\ I_{n_1} + A(\overline{G_1}) - A(G_1) & (J - I)_{n_1} \end{pmatrix}.$$

As quotient matrix involves entries due to regularities, for eigenvalues other than regularity, above matrix will be

$$\begin{pmatrix} -1 - 2\lambda_i(G_1) & -2\lambda_i(G_1) \\ -2\lambda_i(G_1) & -1 \end{pmatrix}.$$

Polynomial associated to above matrix will be

$$x^2 + 2\big(\lambda_i(G_1) + 1\big)x - \big(4\lambda_i^2(G_1) - 2\lambda_i(G_1) - 1\big) \tag{3}$$

for $i = 2, 3, \ldots, n_1$, whose roots will constitute part of $\mathcal{S}$-spectrum.
Hence, result follows from (1), (2) and (3). □

4. Seidel spectrum for splitting S-vertex join of graphs

Theorem 4.1. *For graphs G_1, G_2 on n_1, n_2 vertices having r_1, r_2 regularities, respectively. Seidel spectrum of the spitting S-vertex join involves the following:*

i. $-1 - 2\lambda_i(G_2)$ *for* $i = 2, 3, \ldots, n_2$

ii. roots of the equation

$$x^2 + 2\big(\lambda_i(G_1) + 1\big)x - \big(4\lambda_i^2(G_1) - 2\lambda_i(G_1) - 1\big)$$

for $i = 2, 3, \ldots, n_1$

iii. roots of the equation

$$\begin{aligned}
&x^3+(2r_1+2r_2-2n_1-n_2+3)x^2\\
&\quad+(2n_1r_1-2n_2r_1-4n_1r_2+4r_1r_2-4r_1^2-4n_1-2n_2+4r_1+4r_2+3)x\\
&\quad+(4n_1^2n_2+4n_2r_1^2-8r_1^2r_2-8n_1n_2r_1+4n_1r_1r_2+2n_1r_1-2n_2r_1-4n_1r_2\\
&\qquad\qquad+4r_1r_2-4r_1^2-2n_1-n_2+2r_1+2r_2+1).
\end{aligned}$$

Proof. Proof follows in the same procedure as did for Theorem 3.1. Here, general Seidel matrix is:

$$\mathcal{S}(G_1\underline{\vee}G_2)=\begin{pmatrix} A(\overline{G_1})-A(G_1) & I_{n_1}+A(\overline{G_1})-A(G_1) & J_{n_1\times n_2}\\ I_{n_1}+A(\overline{G_1})-A(G_1) & (J-I)_{n_1} & -J_{n_1\times n_2}\\ J_{n_2\times n_1} & -J_{n_2\times n_1} & A(\overline{G_2})-A(G_2)\end{pmatrix}.$$

Equitable partition $V(G_1)\bigcup V\Big(S(G_1)\Big)\bigcup V(G_2)$ is also alike and quotient matrix is

$$Q_{\mathcal{S}}(G_1\triangle G_2)=\begin{pmatrix} n_1-2r_1-1 & n_1-2r_1 & n_2\\ n_1-2r_1 & n_1-1 & -n_2\\ n_1 & -n_1 & n_2-2r_2-1\end{pmatrix}.$$

Hence, result follows from the structure formed in $V(G_2)$ and $SP(G_1)$ together with the polynomial due to $Q_{\mathcal{S}}(G_1\triangle G_2)$ (from Theorem 2.1). □

5. Conclusion

Study of splitting V-vertex join and S-vertex join of graphs is made easy by reducing the size of the matrix with the help of partition of the vertex set.

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