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Graceful labeling on torch graph

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Abstract

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. An injective function $f : V \to \{0, 1, 2, ..., |E|\}$ is called graceful labeling if f induces a function $f^*(uv) = |f(u)-f(v)|$ which is a bijection from $E(G)$ to the set $\{1, 2, 3, ..., |E|\}$. A graph which admits a graceful labeling is called a graceful graph. In this paper, we show that torch graph O_n is a graceful graph.

Keywords: graceful graph, graceful labeling, torch graph
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1. Introduction

In this paper we consider only finite simple undirected graphs. The set of vertices and edges of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. A labeling of a graph G is a function that carries a set of graph elements, usually the vertices and/or edges, into a set of numbers, usually integers, called labels. A graceful labeling of a graph $G = (V, E)$ is an injective function f from the vertices of G to the set $\{0, 1, 2, ..., |E|\}$ such that when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge label are distinct. Rosa [3] has identified essentially three reasons why a graph fails to be graceful: (1) G has too many vertices and not enough edges, (2) G has too many edges, and (3) G has the wrong parity. A graph which admits a graceful labeling is called a graceful graph. Many results of graceful labelings can be found in [2].

In [1] Rachmandika and Sugeng proved that Torch graph O_n has super edge-magic total labeling. The aim of this paper is to prove that torch graph O_n is graceful graph. In the next section we give the definition of Torch graph O_n. We give main result in Section 3.

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2. Torch Graph

Torch Graph O_n has $n+4$ vertices and $2n+3$ edges. The set of vertices and edges, respectively, are :

$V(O_n) = \{v_i | 1 \leq i \leq n+4\}$

$E(O_n) = \{v_i v_{n+1} | 2 \leq i \leq n-2\} \cup \{v_i v_{n+3} | 2 \leq i \leq n-2\} \cup \{v_1 v_i | n \leq i \leq n+4\} \cup \{v_{n-1}v_n, v_n v_{n+2}, v_n v_{n+4}, v_{n+1}v_{n+3}\}$

The following Figure 1 shows a Torch Graph O_3.

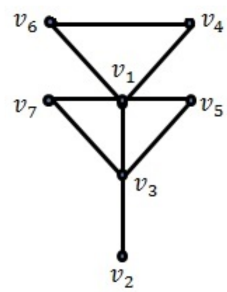

Figure 1. Torch Graph O_3

The following Figure 2 shows a Torch Graph O_n for $n \geq 4$.

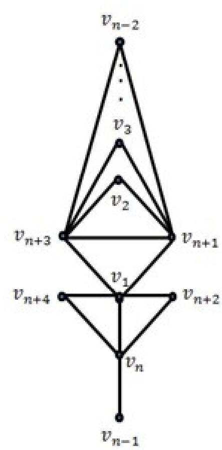

Figure 2. Torch Graph O_n for $n \geq 4$

3. Main Result

In this section we give a proof that Torch graph O_n is a graceful Graph for $n \geq 3$.

Theorem 3.1. *Torch graph O_n is a graceful graph for $n \geq 3$.*

Proof. The number of vertices in a torch graph O_n is $n + 4$ and the number of edges is $2n + 3$. We will show that there exists injective function f from $V(O_n)$ to $\{0, 1, 2, ..., 2n + 3\}$ such that f induces bijective function f^* from $E(O_n)$ to $\{1, 2, ..., 2n + 3\}$.
Let f be the function on $V(O_n)$ defined by :

$$f(v_i) \quad = \begin{cases} 0 & \text{for } i \in \{1\}, \\ 2i - 1 & \text{for } i \in \{2, 3, 4, ..., n + 2\}, \\ 2(i - 3) & \text{for } i \in \{n + 3, n + 4\}. \end{cases}$$

Since all values of $f(v_i)$ are distinct and $0 \leq f(v_i) \leq 2n+3$, we conclude that f is an injective function from $V(O_n)$ to $\{0, 1, 2, ..., 2n + 3\}$.

Now we will proof that f^* is bijective. Note that :

- For $v_iv_j \in \{v_iv_j | i = 2, 3, ..., n - 2 \text{ and } j = n + 3\}$ we have :

$$\begin{aligned} f^*(v_iv_j) &= |f(v_i) - f(v_j)| \\ &= |(2i - 1) - 2(n + 3 - 3)| \\ &= |2i - 2n - 1| \\ &= 2n - 2i + 1. \end{aligned}$$

- For $v_iv_j \in \{v_iv_j | i = 1 \text{ and } j = n, n + 1, n + 2\}$ we have :

$$\begin{aligned} f^*(v_iv_j) &= |f(v_i) - f(v_j)| \\ &= |0 - (2j - 1)| \\ &= | - 2j + 1| \\ &= 2j - 1. \end{aligned}$$

- For $v_iv_j \in \{v_iv_j | i = 1 \text{ and } j = n + 3, n + 4\}$ we have :

$$\begin{aligned} f^*(v_iv_j) &= |f(v_i) - f(v_j)| \\ &= |0 - 2(j - 3)| \\ &= | - 2j + 6| \\ &= 2j - 6. \end{aligned}$$

- For another v_iv_j we have :

$$\begin{aligned} f^*(v_{n-1}v_n) &= |f(v_{n-1}) - f(v_n)| \\ &= |2(n - 1) - 1 - (2n - 1)| \\ &= |2n - 2 - 1 - 2n + 1| \\ &= 2. \end{aligned}$$

$$\begin{aligned} f^*(v_nv_{n+2}) &= |f(v_n) - f(v_{n+2})| \\ &= |2n-1-(2(n+2)-1)| \\ &= |2n-1-2n-3| \\ &= 4. \end{aligned}$$

$$\begin{aligned} f^*(v_nv_{n+4}) &= |f(v_n) - f(v_{n+4})| \\ &= |2n-1-(2(n+4-3))| \\ &= |2n-1-2n-8+6| \\ &= 3. \end{aligned}$$

$$\begin{aligned} f^*(v_{n+1}v_{n+3}) &= |f(v_{n+1}) - f(v_{n+3})| \\ &= |2(n+1)-1-(2(n+3-3))| \\ &= |2n+2-1-2n| \\ &= 1. \end{aligned}$$

Thus we obtain :

$$f^*(v_iv_j) = \begin{cases} 2n-2i+2 & \text{for } \{v_iv_j | i = 2,3,...,n-2 \text{ and } j = n+1\} \\ 2n-2i+1 & \text{for } \{v_iv_j | i = 2,3,...,n-2 \text{ and } j = n+3\} \\ 2j-1 & \text{for } \{v_iv_j | i = 1 \text{ and } j = n, n+1, n+2\} \\ 2j-6 & \text{for } \{v_iv_j | i = 1 \text{ and } j = n+3, n+4\} \\ 1 & \text{for } i = n+1 \text{ and } j = n+3 \\ 2 & \text{for } i = n-1 \text{ and } j = n \\ 3 & \text{for } i = n \text{ and } j = n+4 \\ 4 & \text{for } i = n \text{ and } j = n+2. \end{cases}$$

We can see that :
$f^*(\{v_iv_j | i = 2,3,...,n-2 \text{ and } j = n+1\}) = \{6, 8, 10, ..., 2n-2\}$.
$f^*(\{v_iv_j | i = 2,3,...,n-2 \text{ and } j = n+3\}) = \{5, 7, 9, ..., 2n-3\}$.
$f^*(\{v_iv_j | i = 1 \text{ and } j = n, n+1, n+2\}) = \{2n-1, 2n+1, 2n+3\}$.
$f^*(\{v_iv_j | i = 1 \text{ and } j = n+3, n+4\}) = \{2n, 2n+2\}$.
And for another v_iv_j we have $f^*(\{v_iv_j\}) = \{1, 2, 3, 4\}$.

Then we have range of f^*
$= f^*(\{v_iv_j | i = 2,3,...,n-2 \text{ and } j = n+1\}) \cup f^*(\{v_iv_j | i = 2,3,...,n-2 \text{ and } j = n+3\}) \cup$
$f^*(\{v_iv_j | i = 1 \text{ and } j = n, n+1, n+2\}) \cup f^*(\{v_iv_j | i = 1 \text{ and } j = n+3, n+4\})$
$= \{6, 8, 10, ..., 2n-2\} \cup \{5, 7, 9, ..., 2n-3\} \cup \{2n-1, 2n+1, 2n+3\} \cup \{2n, 2n+2\} \cup \{1, 2, 3, 4\}$
$= \{1, 2, 3, ..., 2n+3\}$
$= \{1, 2, 3, ..., |E(O_n)|\}$.

This implies that f^* is onto and one-one, so we conclude that f^* is a bijection from $E(O_n)$ to $\{1, 2, 3, ..., |E(O_n)|\}$. Since we have proved that there exist an injective function $f : V(O_n) \to \{0, 1, 2, ..., |E(O_n)|\}$ such that f induces a function $f^*(uv) = |f(u) - f(v)|$ which is a bijection from $E(O_n)$ to the set $\{1, 2, 3, ..., |E(O_n)|\}$, then we conclude that O_n is graceful graph. □

The following Figure 3 shows a graceful labeling for O_3.

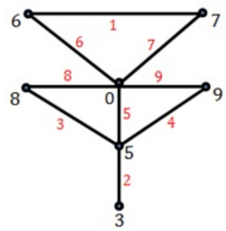

Figure 3. Graceful labeling for O_3

The following Figure 4 is an example of graceful labeling for O_7. □

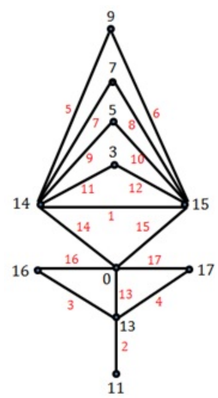

Figure 4. Graceful labeling for O_7

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