Another $H$-super magic decompositions of the lexicographic product of graphs

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Abstract

Let $H$ and $G$ be two simple graphs. The concept of an $H$-magic decomposition of $G$ arises from the combination between graph decomposition and graph labeling. A decomposition of a graph $G$ into isomorphic copies of a graph $H$ is $H$-magic if there is a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, ..., |V(G) \cup E(G)|\}$ such that the sum of labels of edges and vertices of each copy of $H$ in the decomposition is constant. A lexicographic product of two graphs $G_1$ and $G_2$, denoted by $G_1[G_2]$, is a graph which arises from $G_1$ by replacing each vertex of $G_1$ by a copy of the $G_2$ and each edge of $G_1$ by all edges of the complete bipartite graph $K_{n,n}$ where $n$ is the order of $G_2$. In this paper we provide a sufficient condition for $C_n[K_m]$ in order to have a $P_t[K_m]$-magic decompositions, where $n > 3, m > 1$, and $t = 3, 4, n - 2$.

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1. Introduction

Let $G$ be a simple graph and $H$ be a subgraph of $G$. A decomposition of $G$ into isomorphic copies of $H$ is called $H$-magic if there is a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, ..., |V(G) \cup E(G)|\}$ such that the sum of labels of edges and vertices of each copy of $H$ in the decomposition is constant.
constant. A lexicographic product of two graphs $G_1$ and $G_2$ is defined as graph which constructed from the graph $G_1$ and then replacing each vertex of $G_1$ by a copy of $G_2$ and each edge of $G_1$ by edges of complete bipartite graph $K_{n,n}$, where $|V(G)| = n$. The lexicographic product of $G_1$ and $G_2$ with this construction is denoted by $G_1[G_2]$ [1].

A labeling of a graph $G = (V, E)$ is a bijection from a set of elements of graphs to a set of numbers. The edge magic and super edge magic labelings were first introduced by Kotzig and Roza [9] and Enomoto, Lladò, Nakamigawa, and Ringel [3], respectively. There are some results in edge magic and super edge magic, such as in [2, 3, 12, 13]. The notion of an $H$ (super) magic labeling was introduced by Gutiérrez and Lladó [5] in 2005. In 2010, Maryati and Salman [11] used multiset partition concept to obtain a super magic labeling of path amalgamation of isomorphic graphs. Inayah et al. [8] have improved the concept of labeling graphs became $H$-(anti) magic decomposition. In almost the same time, Liang [10] discussed cycle-supermagic decompositions of complete multipartite graphs and in 2015, Hendy [6] has discussed the $H$- super(anti)magic decompositions of antiprism graphs. For a complete results in graph labeling, see [4].

In this research we interest in decomposing the lexicographic product of graphs $C_n[K_m]$ then labeling of the edges and vertices of each isomorphic copies of $P_t[K_m]$ to obtain $P_t[K_m]$—magic decomposition, where $n > 3, m > 1$, and $t = 3, 4, n - 2$.

Preliminaries

Let $G$ be a simple graph. Complement of $G$, denoted by $\overline{G}$, is graph which $V(\overline{G}) = V(G)$ and $\forall u, v \in V(G)$ $uv$ is edge of $\overline{G}$ if and only if $uv$ is not edge of $G$. A family $\mathbb{B} = \{G_1, G_2, ..., G_t\}$ of subgraphs of $G$ is an $H$-decomposition of $G$ if all subgraphs are isomorphic to graph $H$, $E(G_i) \cap E(G_j) = \emptyset$, for $i \neq j$, and $\bigcup_{i=1}^t E(G_i) = E(G)$. In such case, we write $G = G_1 \oplus G_2 \oplus ... \oplus G_t$ and $G$ is said to be $H$-decomposable. if $G$ is an $H$-decomposable graph, then we also write $H|G$.

Let $\mathbb{B}$ be an $H$-decomposition of $G$. The graph $G$ is said to be $H$-magic if there exists a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, ..., |V(G) \cup E(G)|\}$ such that $\forall B \in \mathbb{B}, \sum_{v \in V(B)} f(v) + \sum_{e \in E(B)} f(e)$ is constant. Such a function $f$ is called an $H$-magic labeling of $G$. The sum of all the vertex and edges labels of $H$ (under a labeling $f$) is denoted by $\sum f(H)$. The constant value that every copy of $H$ takes under the labeling $f$ is denoted by $m(f)$.

The one of the concept of multi set partition, k-balance multi set, was presented by Maryati et al. [11]. In this paper, $\sum_{x \in X} x$, denoted by $\sum X$. Multi set is a set which may has the same elements. For positive integer $n$ and $k_i$ with $i \in [1, n]$, multi set $\{a_1^{k_1}, a_2^{k_2}, ..., a_n^{k_n}\}$ is a set which has $k_i$ elements $a_i$ for $i \in [1, n]$. Suppose $V$ and $W$ are two multi sets with $V = \{a_1^{k_1}, a_2^{k_2}, ..., a_n^{k_n}\}$ and $W = \{b_1^{k_1}, b_2^{k_2}, ..., b_m^{k_m}\}$. Defined by: $V \uplus W = \{a_1^{k_1}, a_2^{k_2}, ..., a_n^{k_n}, b_1^{k_1}, b_2^{k_2}, ..., b_m^{k_m}\}$. Let $k \in N$ and $Y$ is a multi set of positive integers. $Y$ is a $k$-balance multi set if there exists $k$ subsets of $Y$ such as: $Y_1, Y_2, ..., Y_k$, such that for all $i \in [1, k]$, $|Y_i| = \frac{|Y|}{k}$, $\sum Y_i = \sum_k Y_i \in N$ and $\{Y_1, Y_2, ..., Y_k\} = Y$.

**Lemma 1.1.** [7] $P_t[K_m]|C_n[K_m]$ if and only if $P_t|C_n$

**Lemma 1.2.** [7] Let $t$ be any integer with $t \geq 1$. If $P_t[K_m]|C_n[K_m]$ then $n(n-3) \equiv 0(\text{mod} 2(t-1))$

**Theorem 1.1.** [7] Let $n$ and $m$ be integers with $n > 3$ and $m > 1$. The graph $C_n[K_m]$ has $P_2[K_m]$-super magic decomposition if and only if $m$ is even or $m$ is odd and $n \equiv 1(\text{mod} 4)$, or $m$ is odd and $n \equiv 2(\text{mod} 4)$, or $m$ is odd and $n \equiv 3(\text{mod} 4)$.  

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2. Results

Lemma 2.1. $P_3[K_m]|C_n[K_m]$ if and only if $n \neq 4$, $n \equiv 0(\text{mod} 4)$ or $n \equiv 3(\text{mod} 4)$.

Proof. (⇒) Let $P_3[K_m]|C_n[K_m]$, then from Lemma 2.1 we have that $P_3|C_n$. From Lemma 2.2 we have that $n \equiv 0(\text{mod} 4)$ or $n \equiv 3(\text{mod} 4)$. Because of $C_4$ doesn’t have $P_3$, this is not occur for $n = 4$.

(⇐) Now let $n \neq 4$, $n \equiv 0(\text{mod} 4)$ dan $V(C_n) = \{v_1, ..., v_{4k}\}$, $k \in \mathbb{Z}^+$. Let $N(v_i) = V(C_n) \setminus \{v_{i-1}, v_{i+1}\}$. Follow this algorithm decompose $C_n$.

Algorithm 1:

1. Choose the path $P_1 : v_3 - v_1 - v_4$ and let $v_1$ be the center of the rotation. Rotate $P_1$ such that $v_1$ on $v_3$, $v_3$ on $v_5$ and $v_4$ on $v_6$, thus we have $P_2 : v_5 - v_3 - v_6$. Do the next rotation until $v_1$ on $v_5, v_{i-1}, ..., v_{4k-1}$. Then we have $2k$ of $P_3$-paths.

2. Choose the cycle $v_2 - v_4 - ... - v_{4k}$. Decompose this $2k$-cycle to $k$ of $P_3$-paths.

3. Do the rotation again ($v_1 \rightarrow v_3 \rightarrow v_5 \rightarrow ...$), with choosing two vertices which close with the vertices that is rotated in step 1. If this rotation is not the last rotation, do the rotation again until $v_1$ on position of $v_{4k-1}$, such that we have $2k$ of $P_3$-path. If this rotation is the last rotation, first do the rotation in step 1 until $v_1$ on position of $v_{2k-1}$ such that we have $k$ of $P_3$-path. Then rotate $P' = v_{n-2} - v_2 - v_{n-1}$ with $v_2$ as a center of this rotation until $v_2$ on position of $v_{2k}$ and we have $k P_3$-path.

From the Algorithm 1 above, we have that $P_3|C_n$. Then from Lemma 2.1 $P_3[K_m]|C_n[K_m]$ for $n \neq 4$, $n \equiv 0(\text{mod} 4)$.

Let $n \equiv 3(\text{mod} 4)$ dan $V(C_n) = \{v_1, ..., v_{4k+3}\}$, $k \in \mathbb{Z}^+$. Let $N(v_i) = V(C_n) \setminus \{v_{i-1}, v_{i+1}\}$. Decompose $C_n$ with the following steps.

Algorithm 2

Choose the path $Q_1 = v_3 - v_1 - v_4$ with $v_1$ is the center of rotation. Rotate $Q_1$ such that $v_1$ on $v_2$ and we have $Q_2 = v_4 - v_2 - v_5$. Do the next rotation such that $v_1$ on $v_3, v_4, v_{i,}, ..., v_{4k+3}$. Do the rotation such that we have $kn P_3$-path.

From Algorithm 2, it's clearly that $P_3|C_n$. Thus from Lemma 2.1 $P_3[K_m]|C_n[K_m]$ for $n \equiv 3(\text{mod} 4)$.

See Figure 1 to see graph $C_8$ can be decomposed into 10 $P_3$-path.

Theorem 2.1. Suppose $n, m \in \mathbb{Z}^+$ and $m > 1$. For $n \equiv 3(\text{mod} 4)$, or $(n \equiv 0(\text{mod} 4)$ and $m$ is even, Graph $C_n[K_m]$ have $P_3[K_m]$-magic decomposition.

Proof. Let $n \equiv 3(\text{mod} 4)$. From Lemma 2.1 we have for $n \equiv 3(\text{mod} 4)$, $P_3[K_m]|C_n[K_m]$. Let $m$ be even. Do the next vertex labeling steps and edge labeling steps such in case 1 in Theorem 2.1.

Let $V_1, V_2, ..., V_n$ be the partitions of $V(C_n[K_m])$, where $V(C_n[K_m]) = V_1 \cup V_2 \cup ... \cup V_n = \{v_{1,1}, v_{1,2}, ..., v_{1,m}\} \cup \{v_{2,1}, v_{2,2}, ..., v_{2,m}\} \cup ... \cup \{v_{n,1}, v_{n,2}, ..., v_{n,m}\}$. Consider the set $A^* = [1, mn] =$
[1, (2k)n], k \in \mathbb{Z}\). for every \(i \in [1, n]\), \(A^*_i = \{a^i_j / 1 \leq j \leq m\}\), where

\[
a^i_j = \begin{cases} 
  k(j - 1) + i, & \text{if } j \text{ is odd;} \\
  1 + nj - i, & \text{if } j \text{ is even.}
\end{cases}
\]

is a balance subset of \(A^*\).

Define a vertex labeling \(f_1\) of \(\overline{C_n[K_m]}\) which will label vertices of \(V_1, V_2, ..., V_n\) using elements of \(A^*_1, A^*_2, ..., A^*_n\) respectively.

Consider the set \(B^* = [mn + 1, mn + \frac{n(n-3)m^2}{2}]\). For every \(i \in [1, \frac{n(n-3)}{2}]\), \(B^*_i = \{b^i_j / 1 \leq j \leq m^2\}\), with \(b^i_j = \begin{cases} 
  mn + \frac{n(n-3)}{2}(j - 1) + i, & \text{if } j \text{ is odd;} \\
  (mn + 1) + (\frac{n(n-3)}{2})j - i, & \text{if } j \text{ is even.}
\end{cases}\)

\(B^*_i = \{b^i_j / 1 \leq j \leq m^2\}\) is a balance subset of \(B^*\). Define an edge labeling \(f_2\) of \(\overline{C_n[K_m]}\) with label all edges in \(P_3[\overline{K_m}], i \in [1, \frac{n(n-3)}{2}]\) with the elements in \(B^*_i\).

Since for all \(i \in [1, \frac{n(n-3)}{4}]\), \(m(f_1 + f_2)(P_3[\overline{K_m}]) = 3m(f_1) + 2m(f_2) = 3m^2n + m + 2(m^2(2mn + 1 + \frac{n(n-3)m^2}{2})) = 3m^2n + 3m + m^2(2mn + 1 + \frac{n(n-3)m^2}{2})\) then \(\overline{C_n[K_m]}\) has \(P_3[\overline{K_m}]\)-magic decomposition.

Now let \(m\) is odd. Do the vertex labeling steps and edge labeling steps such in case 4 in Theorem 2.1.

(a) Let \(m = 3\). Consider the set \(A = [1, m(n + \frac{n(n-3)}{2})] = [1, 3(n + \frac{n(n-3)}{2})]\). For every \(i \in [1, (n + \frac{n(n-3)}{2})]\), \(A_i = \{a_i, b_i, c_i\}\) where

\[
a_i = 1 + i; \\
b_i = \begin{cases} 
  (n + \frac{n(n-3)}{2}) + \frac{n(n-3)}{2} + i, & \text{for } i \in [1, \frac{n(n-3)}{2}]\]; \\
  (n + \frac{n(n-3)}{2}) - \frac{n+1}{2} + i, & \text{for } i \in [\lceil \frac{n+1}{2} \rceil, n + \frac{n(n-3)}{2}].
\end{cases}
\]

\[
c_i = \begin{cases} 
  3(n + \frac{n(n-3)}{2}) + 1 - 2i, & \text{for } i \in [1, \frac{n+3}{2}]; \\
  3(n + \frac{n(n-3)}{2}) + 2\lceil \frac{n+3}{2} \rceil - 2i, & \text{for } i \in [\lceil \frac{n+3}{2} \rceil, n + \frac{n(n-3)}{2}].
\end{cases}
\]
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$A_i = \{a_i, b_i, c_i\}$ is a balance subset of $A$. Consider the set $B = [3(n + \frac{n(n-3)}{2}) + 1, 3n + (\frac{n(n-3)}{2})m^2]$. For every $i \in [1, \frac{n(n-3)}{2}]$, $B_i = \{b_j/1 \leq j \leq m^2 - 3\}$, where

$$b_j = \begin{cases} 3(n + \frac{n(n-3)}{2}) + (n + \frac{n(n-3)}{2})(j - 1) + i, & \text{if } j \text{ is odd;} \\ 3(n + \frac{n(n-3)}{2}) + 1 + (n + \frac{n(n-3)}{2})j - i, & \text{if } j \text{ is even.} \end{cases}$$

$B_i = \{b_j/1 \leq j \leq m^2 - 3\}$ is a balance subset of $B$. Define a function $h_1 : V(C_n[K_m]) \rightarrow \{A_i, i \in [1, n]\} \subset A$ and label all vertices in every $V_i$ with the elements of $A_i$. Define a function $h_2 : E(C_n[K_m]) \rightarrow \{A_i, i \in [n + 1, (n + \frac{n(n-3)}{2})]\} \cup B$ and label all edges in every $P_2[K_m], i \in [1, \frac{n(n-3)}{2}]$ with the elements of $A_{n+i} \cup B_i$.

(b) Let $m > 3$ and $m$ be odd. Considering the set $A^* = [1, m(n + \frac{n(n-3)}{2})]$. Divide $A^*$ to be two sets.

$$A = [1, 3(n + \frac{n(n-3)}{2})];$$

$$E = [3(n + \frac{n(n-3)}{2}) + 1, m(n + \frac{n(n-3)}{2})].$$

Follow the same way with (a), for $m = 3$, $A$ is a $(n + \frac{n(n-3)}{2})$-balance multi set and for every $i \in [1, (n + \frac{n(n-3)}{2})]$, $A_i$ is a balance subset of $A$. Consider the set $E = [3(n + \frac{n(n-3)}{2})+1, m(n + \frac{n(n-3)}{2})]$. For every $i \in [1, (n + \frac{n(n-3)}{2})]$,

$$E_i = \{e_j/1 \leq j \leq m - 3\},$$

where

$$e_j = \begin{cases} 3(n + \frac{n(n-3)}{2}) + (n + \frac{n(n-3)}{2})(j - 1) + i, & \text{if } j \text{ is odd;} \\ 3(n + \frac{n(n-3)}{2}) + 1 + (n + \frac{n(n-3)}{2})j - i, & \text{if } j \text{ is even.} \end{cases}$$

$E_i = \{e_j/1 \leq j \leq m - 3\}$ is a balance subset of $E$. Considering the set $M = [m(n + \frac{n(n-3)}{2}) + 1, m^2(n + \frac{n(n-3)}{2}) + mn]$. For every $i \in [1, \frac{n(n-3)}{2}]$, $M_i = \{m_j/1 \leq j \leq m^2 - m\}$, where

$$m_j = \begin{cases} m(n + \frac{n(n-3)}{2}) + (\frac{n(n-3)}{2})(j - 1) + i, & \text{if } j \text{ is odd;} \\ m(n + \frac{n(n-3)}{2}) + 1 + (\frac{n(n-3)}{2})j - i, & \text{if } j \text{ is even.} \end{cases}$$

$M_i = \{m_j/1 \leq j \leq m^2 - m\}$ is a balance subset of $M$.

Define a function $q_1 : V(C_n[K_m]) \rightarrow \{A_i = A_i \cup E_i, i \in [1, n]\} \subset A^*$ and label all vertices in every $V_i$ with the elements of $\{A_i = A_i \cup E_i, i \in [1, n]\}$. Define a function $q_2 : E(C_n[K_m]) \rightarrow \{A_{n+i} = A_{n+i} \cup E_{n+i}\} \cup M$ and label all edges in every $P_2[K_m], i \in [1, \frac{n(n-3)}{2}]$ with the elements of $A_{n+i} \cup M_i$.

Since $\forall i \in [1, \frac{n(n-3)}{4}]$, $(q_1 + q_2)(P_3[K_m]) = 5 \sum A_i + 2 \sum M_i = 5(\sum A_i + \sum E_i) = 5((2 + 4n + 2n(n-3) + (\frac{n(n-3)}{2})) + (\frac{n(n-3)}{2})(3(n + \frac{n(n-3)}{2}) + 1 + m(n + \frac{n(n-3)}{2})) + 2(\frac{m^2-m}{2}) (m(n + \frac{n(n-3)}{2}) + 1 + m^2(n + \frac{n(n-3)}{2}) + mn))$ then $C_n[K_m]$ has $P_3[K_m]$-magic decomposition.

Now let $n \equiv 0(\text{mod}4)$ and $m$ be even. From Lemma 3, we have for $n \equiv 0(\text{mod}4), P_3[K_m]\mid C_n[K_m]$. Do the vertex labeling steps and edge labeling steps such in case 1 in Theorem 2.1. Since for all $i \in [1, \frac{n(n-3)}{4}]$, $m(f_1 + f_2)(P_3[K_m]) = 3m(f_1) + 2m(f_2) = 3m^2n + m + 2(\frac{m^2}{2})(2mn + 1 + \frac{n(n-3)m^2}{2}) = 3m^2n + 3m + m^2(2mn + 1 + \frac{n(n-3)m^2}{2})$, then $C_n[K_m]$ have $P_3[K_m]$-magic decomposition.

Figure 2 give an example that graph $C_3[K_2]$ have $P_3[K_2]$- super magic decomposition with the constant value $m(f_1 + f_2) = 503$. □

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Theorem 2.1. Since for all $i$

Now, let $m$

Algorithm 3
Choose the path $R_1: v_1 - v_3 - v_6 - v_4$ and let $v_1$ be the center of the rotation. Rotate $R_1$ such that $v_1$ on $v_2$, $v_3$ on $v_4$, $v_6$ on $v_1$ and $v_4$ on $v_5$, thus we have $R_2 = v_2 - v_4 - v_1 - v_5$. Do the next rotation such that $v_1$ on $v_3$, etc, and redo the process until $\left\lceil \frac{k-1}{2} \right\rceil$ rotations.

Figure 3 shows that graph $C_9$ can be decompose into 9 $P_4$-path.

**Theorem 2.2.** Let $n > 3$ and $m > 1$. For $n \equiv 3(mod12)$ or $n \equiv 6(mod12)$ or $n \equiv 9(mod12)$ or $n \equiv 0(mod12)$ and $m$ is even, Graph $C_n[K_m]$ have $P_4[K_m]$-magic decomposition

Proof. Let $n \equiv 3(mod12)$. From Lemma 2.2, we have that for $n \equiv 3(mod12)$, $P_4[K_m] \mid C_n[K_m]$. Now, let $m$ be even. Do the next vertex labeling steps and edge labeling steps such in case 1 in Theorem 2.1. Since for all $i \in [1, \frac{n(n-3)}{6}]$, $m(q_1 + q_2)(P_4[K_m]_i) = 4m(f_1) + 3m(f_2) = 4(m^2n + m) + 3(\frac{m^2}{2}2mn + 1 + \frac{n(n-3)m^2}{2})$ then $C_n[K_m]$ have $P_4[K_m]$-magic decomposition.

Let $m$ be odd. Do the next vertex labeling steps and edge labeling steps such in case 4 in Theorem 2.1. Since for all $i \in [1, \frac{n(n-3)}{6}]$, $m(q_1 + q_2)(P_4[K_m]_i) = 7\sum A_i^* + 3\sum M_i = 7(2 + 4n + 2n(n-3) + (\frac{2n(n-3)}{4})) + (\frac{m^2}{3})(3(n + \frac{n(n-3)}{2}) + 1 + m(n + \frac{n(n-3)}{2} + 3m^2 - 3m)(m(n + \frac{n(n-3)}{2} + 1 + m^2(n + \frac{n(n-3)}{2} + mn), then $C_n[K_m]$ has $P_4[K_m]$-magic decomposition.
Figure 3. $P_4$-decomposition of $C_9$

Let $n \equiv 6 (mod 12)$. From Lemma 2.2, we have that $n \equiv 6 (mod 12)$, $P_4[K_m] \mid C_n[K_m]$. Now let $m$ is even. Do the vertex labeling steps and edge labeling steps in case 1 Theorem 1. Because $\forall i \in [1, \frac{n(n-3)}{6}]$, $(f_1 + f_2)(P_4[K_{m_i}]) = 4 \sum Z_i + 3 \sum X_i$ then $C_n[K_m]$ have $P_4[K_m]$-magic decomposition. Let $m$ is odd. Do the vertex labeling steps and edge labeling steps such in case 3 in Theorem 2.1.

Let $m = 3$. Consider the set $D = [1, m(n + \frac{n(n-3)}{2})] = [1, 3(n + \frac{n(n-3)}{2})]$. For every $i \in [1, (n + \frac{n(n-3)}{2})]$, $D_i = \{a_i, b_i, c_i\}$ where:

$$a_i = 1 + i;$$

$$b_i = \begin{cases} (n + \frac{n(n-3)}{2}) + \left\lfloor \frac{n(n-3)}{2} \right\rfloor + i, & \text{for } i \in [1, \left\lfloor \frac{n(n-3)}{2} \right\rfloor]; \\ (n + \frac{n(n-3)}{2}) - \left\lceil \frac{n(n-3)}{2} \right\rceil + i, & \text{for } i \in [\left\lceil \frac{n(n-3)}{2} \right\rceil, (n + \frac{n(n-3)}{2})]. \end{cases}$$

$$c_i = \begin{cases} 3(n + \frac{n(n-3)}{2}) + 1 - 2i, & \text{for } i \in [1, \left\lceil \frac{n(n-3)}{2} \right\rceil]; \\ 3(n + \frac{n(n-3)}{2}) + 2\left\lceil \frac{n(n-3)}{2} \right\rceil - 2i, & \text{for } i \in [\left\lceil \frac{n(n-3)}{2} \right\rceil, n + \frac{n(n-3)}{2}]. \end{cases}$$

$D_i = \{a_i, b_i, c_i\}$ is a balance subset of $D$.

Considering the set $E = [3(n + \frac{n(n-3)}{2}) + 1, 3n + (\frac{n(n-3)}{2})m^2]$. For every $i \in [1, \frac{n(n-3)}{2}]$, $E_i = \{b_j^i | 1 \leq j \leq m^2 - 3\}$, with $b_j^i = \begin{cases} 3(n + \frac{n(n-3)}{2}) + (n + \frac{n(n-3)}{2})(j - 1) + i, & \text{if } j \text{ is odd}; \\ 3(n + \frac{n(n-3)}{2}) + 1 + (n + \frac{n(n-3)}{2})j - i, & \text{if } j \text{ is even}. \end{cases}$

$E_i$ is a balance subset of $E$.

Define a function $h_1 : V(C_n[K_m]) \to \{A_i, i \in [1, n]\} \subset A$ and label all vertices in every $V_i$ with the elements of $A_i$. Define a function $h_2 : E(C_n[K_m]) \to \{A_i, i \in [n + 1, (n + \frac{n(n-3)}{2})]\} \cup B$ and label all edges in $P_4[K_{m_i}]$, $i \in [1, \frac{n(n-3)}{2}]$ with the elements of $A_{n+i} \cup B_i$.

Let $m > 3$ and $m$ be odd. Consider the set $A^* = [1, m(n + \frac{n(n-3)}{2})]$. Divide $A^*$ to be the two
sets $A$ and $E$ where 

$$A = [1, 3(n + \frac{n(n-3)}{2})];$$

$$E = [3(n + \frac{n(n-3)}{2}) + 1, m(n + \frac{n(n-3)}{2})].$$

With the same way for $m = 3$, $A$ is $(n + \frac{n(n-3)}{2})$-balance set and for every $i \in [1, (n + \frac{n(n-3)}{2})]$, $A_i$ is a balance subset of $A$. Consider the set $E = [3(n + \frac{n(n-3)}{2}) + 1, m(n + \frac{n(n-3)}{2})]$. For every $i \in [1, (n + \frac{n(n-3)}{2})]$, $E_i = \{e_j^i/1 \leq j \leq m - 3\}$, where

$$e_j^i = \begin{cases} 3(n + \frac{n(n-3)}{2}) + (n + \frac{n(n-3)}{2})(j - 1) + i, & \text{if } j \text{ is odd;} \\ 3(n + \frac{n(n-3)}{2}) + 1 + (n + \frac{n(n-3)}{2})j - i, & \text{if } j \text{ is even.} \end{cases}$$

$E_i = \{e_j^i/1 \leq j \leq m - 3\}$ is a balance subset of $E$. Considering the set $M = [m(n + \frac{n(n-3)}{2}) + 1, m^2(n + \frac{n(n-3)}{2}) + mn]$. For every $i \in [1, (n + \frac{n(n-3)}{2})]$, $M_i = \{m_j^i/1 \leq j \leq m^2 - m\}$, where

$$m_j^i = \begin{cases} m(n + \frac{n(n-3)}{2}) + (n + \frac{n(n-3)}{2})(j - 1) + i, & \text{if } j \text{ is odd;} \\ m(n + \frac{n(n-3)}{2}) + 1 + (n + \frac{n(n-3)}{2})j - i, & \text{if } j \text{ is even.} \end{cases}$$

is a balance subset of $M$. Define a function $q_1 : V(C_n[K_m]) \to \{A_i^* = A_i \cup E_i, i \in [1, n]\} \subset A^*$ and label all vertices in every $V_i$ with the elements of $\{A_i^*, i \in [1, n]\}$.

Define a function $q_2 : E(C_n[K_m]) \to \{A_{n+i}^* = A_{n+i} \cup E_{n+i}, i \in [1, n]\} \cup M$ and label all edges in every $P_2[K_m]$, $i \in [1, (n + \frac{n(n-3)}{2})]$, with the elements of $A_{n+i}^* \cup M$. Since for all $i \in [1, (n + \frac{n(n-3)}{2})]$, $(f_1 + g)(P_4[K_m]) = 7 \sum A_i^* + 3 \sum M_i$ then $C_n[K_m]$ has $P_4[K_m]$-magic decomposition.

Now let $n \equiv 9(mod 12)$. From Lemma 2.2 we have that for $n \equiv 9(mod 12)$, $P_4[K_m] | C_n[K_m]$. Now, let $m$ be even. Do the vertex labeling steps and edge labeling steps such in case 1 in Theorem 2.1. Since for all $i \in [1, (n + \frac{n(n-3)}{2})]$, $(f_1 + g)(P_4[K_m]) = 4 \sum Z_i + 3 \sum X_i$ then $C_n[K_m]$ have $P_4[K_m]$-magic decomposition. Suppose $m$ is odd. Do the vertex labeling steps and edge labeling steps such in case 2 of Theorem 2.1. Since for all $i \in [1, (n + \frac{n(n-3)}{2})]$, $(f_2 + h)(P_4[K_m]) = 3 \sum Y_i + 2 \sum P_i$ and $(f_3 + h)(P_4[K_m]) = 3(3 \sum W_i + 3 \sum X_i) + 2 \sum P_i$ then $C_n[K_m]$ has $P_4[K_m]$-magic decomposition.

Now let $n \equiv 0(mod 12)$ and $m$ be even. Clearly from Lemma 2.2 that for $n \equiv 0(mod 12)$, $P_4[K_m] | C_n[K_m]$. Do the vertex labeling steps and edge labeling steps such in case 1 of Theorem 1. Because $\forall i \in [1, (n + \frac{n(n-3)}{2})]$, $(f_1 + g)(P_4[K_m]) = 4 \sum Z_i + 3 \sum X_i$ then $C_n[K_m]$ have $P_4[K_m]$-magic decomposition.

\begin{lemma}
\begin{proof}
Suppose $C_n$, where $n \equiv 1(mod 2)$ are $P_{n-2}$-decomposable graphs, then

$$\frac{|E(C_n)|}{3} = \frac{(2k+1)(2k-2)/2}{2k-2}, s \in Z^+$$

$$= k + \frac{1}{2} \notin Z^+. $$

(contradiction).

(\Rightarrow) Let $V(C_n) = \{v_1, ..., v_{2k}\}, k \in Z^+$ and $N(v_i) = V(C_n) \setminus \{v_{i-1}, v_{i+1}\}$. Do the next steps to decompose $C_n$. Choose the path $L_1 = v_1 - v_3 - v_n - v_4 - v_{n-1} - ...$ and let $v_1$ be the center of the rotation. Rotate $L1$ such that $v_1$ on $v_2$, $v_3$ on $v_4$, $v_n$ on $v_1$ etc. Do the next rotation such that $v_1$ on $v_3$, etc, and continue the process until all edge are used up.

\end{proof}
\end{lemma}
Another $H$-super magic decompositions ...

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Figure 4. $P_9$-decomposition of $C_{12}$

For example, $C_{12}$ in Figure 4 can be decomposed to be 6 $P_9$-path.

**Theorem 2.3.** Let $n > 3$ and $m > 1$. For $n \equiv 2 \pmod{4}$ or $(n \equiv 0 \pmod{4}$ and $m$ is even), $C_n[K_m]$ have $P_{n-2}[K_m]$-magic decomposition.

**Proof.** Let $n \equiv 2 \pmod{4}$. From Lemma 2.2 we have that for $n \equiv 2 \pmod{4}$, $P_{n-2}[K_m]|C_n[K_m]$. Now, let $m$ is even. Do the vertex labeling steps and edge labeling steps such in **case 1** of Theorem 2.1. Because of $\forall i \in [1, \frac{n}{2}]$, $(f_1 + f_2)(P_{n-2}[K_m]) = (n-2)m(f_1) + (n-3)m(f_2) = (n-2)(m^2 n + m) + (n-3)(\frac{m^2}{2}(2mn + 1 + \frac{n(n-3)m^2}{2})$. Thus $C_n[K_m]$ has $P_{n-2}[K_m]$-magic decomposition.

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