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Hamming index of graphs with respect to its incidence matrix

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Abstract

Let $B(G)$ be the incidence matrix of a graph G. The row in $B(G)$ corresponding to a vertex v, denoted by $s(v)$ is the string which belongs to $\mathbb{Z}_2^m$, a set of m-tuples over a field of order two. The Hamming distance between the strings $s(u)$ and $s(v)$ is the number of positions in which $s(u)$ and $s(v)$ differ. In this paper we obtain the Hamming distance between the strings generated by the incidence matrix of a graph. The sum of Hamming distances between all pairs of strings, called Hamming index of a graph is obtained.

Keywords: Hamming distance, strings, incidence matrix, Hamming index
Mathematics Subject Classification: 05C99, 05C85

1. Introduction

The basic unit of information, called message, is a finite sequence of characters. Every character or symbol that is to be transmitted is represented as a sequence of m elements from the set $\mathbb{Z}_2 = \{0, 1\}$. The set $\mathbb{Z}_2$ is a group under binary operation $\oplus$ with addition modulo 2.

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Therefore for any positive integer m, $\mathbb{Z}_2^m = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (m factors) is a group under the operation $\oplus$ defined by

$$(x_1, x_2, \ldots, x_m) \oplus (y_1, y_2, \ldots, y_m) = (x_1 + y_1, x_2 + y_2, \ldots, x_m + y_m).$$

Element of $\mathbb{Z}_2^m$ is an m-tuple $(x_1, x_2, \ldots, x_m)$ written as $x = x_1 x_2 \ldots x_m$ where every x_i is either 0 or 1 and is called a *string* or *word*. The number of 1's in $x = x_1 x_2 \ldots x_m$ is called the *weight* of x and is denoted by $wt(x)$.

Let $x = x_1 x_2 \ldots x_m$ and $y = y_1 y_2 \ldots y_m$ be the elements of $\mathbb{Z}_2^m$. Then the sum $x \oplus y$ is computed by adding the corresponding components of x and y under addition modulo 2. That is, $x_i + y_i = 0$ if $x_i = y_i$ and $x_i + y_i = 1$ if $x_i \neq y_i$, $i = 1, 2, \ldots, m$.

The *Hamming distance* $H_d(x, y)$ between the strings $x = x_1 x_2 \ldots x_m$ and $y = y_1 y_2 \ldots y_m$ is the number of i's such that $x_i \neq y_i$, $1 \leq i \leq m$.

Thus $H_d(x, y) =$ Number of positions in which x and y differ $= wt(x \oplus y)$ [5].

Let $x = 01001$ and $y = 11010$. Therefore $x \oplus y = 10011$. Hence $H_d(x, y) = wt(x \oplus y) = 3$.

Lemma 1.1. [5] *For all $x, y, z \in \mathbb{Z}_2^m$, the following conditions are satisfied.*
(i) $H_d(x, y) = H_d(y, x)$
(ii) $H_d(x, y) \geq 0$
(iii) $H_d(x, y) = 0$ *if and only if* $x = y$
(iv) $H_d(x, z) \leq H_d(x, y) + H_d(y, z)$.

Let G be a simple, undirected graph with n vertices and m edges. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G and $E(G) = \{e_1, e_2, \ldots, e_m\}$ be the edge set of G. If the vertices u and v are adjacent then we write $u \sim v$ and if they are not adjacent then we write $u \nsim v$. The edge and its end vertex are said to be *incident* to each other. The *degree* of a vertex v, denoted by $deg_G(v)$ is the number of edges incident to it. A graph is said to be r-regular graph if all its vertices have same degree equal to r. A path on n vertices denoted by P_n is a graph with vertices $v_1, v_2, \ldots, v_n$, where v_i is adjacent to v_{i+1}, $i = 1, 2, \ldots, n-1$.

The *incidence matrix* of G is a matrix $B(G) = [b_{ij}]$ of order $n \times m$, in which $b_{ij} = 1$ if the vertex v_i is incident to the edge e_j and $b_{ij} = 0$, otherwise. Denote by $s(v)$, the row of the incidence matrix corresponding to the vertex v. It is a string in the set $\mathbb{Z}_2^m$ of all m-tuples over the field of order two.

Sum of Hamming distances between all pairs of strings generated by the incidence matrix of a graph G is denoted by $H_B(G)$ and is called the *Hamming index* of G. Thus,

$$H_B(G) = \sum_{\{u,v\} \subseteq V(G)} H_d(s(u), s(v)).$$

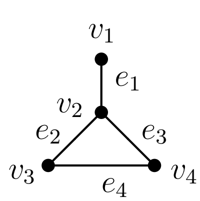

Fig. 1: Graph G

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For a graph G given in Fig. 1, the incidence matrix is

$$B(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

and the strings are $s(v_1) = 1000$, $s(v_2) = 1110$, $s(v_3) = 0101$, $s(v_4) = 0011$.

$$H_d(s(v_1), s(v_2)) = 2, \quad H_d(s(v_1), s(v_3)) = 3, \quad H_d(s(v_1), s(v_4)) = 3,$$
$$H_d(s(v_2), s(v_3)) = 3, \quad H_d(s(v_2), s(v_4)) = 3, \quad H_d(s(v_3), s(v_4)) = 2.$$

Therefore $H_B(G) = 2 + 3 + 3 + 3 + 3 + 2 = 16$.

A graph G with vertex set $V(G)$ is called a *Hamming graph* [3, 4] if each vertex $v \in V(G)$ can be labeled by a string $s(v)$ of a fixed length such that $H_d(s(u), s(v)) = d_G(u, v)$ for all $u, v \in V(G)$, where $d_G(u, v)$ is the length of shortest path joining u and v in G [2].

The Hamming distance between the strings generated by the adjacency matrix of a graph is obtained in [1, 6]. Hamming distance between the strings generated by edge-vertex incidence matrix of a graph is reported in [7]. In this paper we obtain the Hamming distance between the strings generated by the vertex-edge incidence matrix of a graph and also we obtain the Hamming index of graphs. In the sequel we develop an algorithm to obtain the Hamming distance between the strings and Hamming index.

2. Hamming Distance Between Strings

In this section we obtain the Hamming distance between the strings generated by the incidence matrix of a graph.

Theorem 2.1. *Let G be a graph with n vertices and m edges. Let u and v be the vertices of G and l be the number of edges which are neither incident to u nor incident to v. Then*

$$H_d(s(u), s(v)) = \begin{cases} m - 1 - l & \text{if } u \sim v \\ m - l & \text{if } u \nsim v. \end{cases} \tag{1}$$

Proof. Let k be the number of edges which are incident to booth u and v simultaneously and l be the number of edges which are neither incident to u nor to v. Therefore the remaining $m - k - l$ edges are incident to either u or v, but not to both simultaneously. Therefore the strings of u and v from $B(G)$ will be in the form

$$s(u) = x_1 x_2 \ldots x_k x_{k+1} \ldots x_{k+l} x_{k+l+1} \ldots x_m$$

and

$$s(v) = y_1 y_2 \ldots y_k y_{k+1} \ldots y_{k+l} y_{k+l+1} \ldots y_m$$

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where $x_i = y_i = 1$ for $i = 1, 2, \ldots, k$,
$x_i = y_i = 0$ for $i = k+1, k+2, \ldots, k+l$ and
$x_i \neq y_i$ for $i = k+l+1, k+l+2, \ldots, m$.
Therefore $s(u)$ and $s(v)$ differ at $m-k-l$ places.
Hence

$$H_d(s(u), s(v)) = m - k - l \tag{2}$$

If u and v are adjacent then $k = 1$.
Therefore, by Eq. (2), $H_d(s(u), s(v)) = m - 1 - l$.

If u and v are non adjacent then $k = 0$.
Therefore, by Eq. (2), $H_d(s(u), s(v)) = m - l$. □

Theorem 2.2. *Let u and v be the vertices of G. Then*

$$H_d(s(u), s(v)) = \begin{cases} deg_G(u) + deg_G(v) - 2 & \text{if } u \sim v \\ deg_G(u) + deg_G(v) & \text{if } u \nsim v. \end{cases} \tag{3}$$

Proof. Let m be the number of edges of G and let l be the number of edges which are neither incident to u nor to v.

If u and v are adjacent then $l = m - (deg_G(u) + deg_G(v) - 1)$. Substituting this in Eq. (1), we get $H_d(s(u), s(v)) = deg_G(u) + deg_G(v) - 2$.

If u and v are non adjacent then $l = m - (deg_G(u) + deg_G(v))$. Substituting this in Eq. (1), we get $H_d(s(u), s(v)) = deg_G(u) + deg_G(v)$. □

Theorem 2.3. *For a connected graph G, $H_d(s(u), s(v)) = d_G(u, v)$ if and only if $G = P_3$.*

Proof. Let $d_G(u, v) = H_d(s(u), s(v))$.

Case (i): If $u \sim v$, then $d_G(u, v) = 1$. Therefore by Eq. (3), $deg_G(u) + deg_G(v) - 2 = 1$. It gives that $deg_G(u) + deg_G(v) = 3$. Thus it is possible only when $G = P_3$.

Case (ii): If $u \nsim v$, then $d_G(u, v) \geq 2$. Therefore by Eq. (3), $deg_G(u) + deg_G(v) \geq 2$. It gives that $deg_G(u) \geq 1$ and $deg_G(v) \geq 1$. Thus in this case $H_d(s(u), s(v)) = d_G(u, v) = deg_G(u) + deg_G(v)$ is possible only when $G = P_3$.

Converse is obvious. □

3. Hamming Index

In this section we obtain the Hamming index of some graphs.

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Theorem 3.1. *If G is an r-regular graph with n vertices, then*

$$H_B(G) = nr(n-2).$$

Proof. The degree of each vertex of G is r. By Theorem 2.2, if u and v are adjacent vertices, then $H_d(s(u), s(v)) = 2r - 2$ and if u and v are non-adjacent vertices then $H_d(s(u), s(v)) = 2r$. In any graph with n vertices and m edges, there are m pairs of adjacent vertices and $\binom{n}{2} - m$ pairs of non adjacent vertices. Therefore

$$\begin{aligned} H_B(G) &= \sum_{u\sim v} H_d(s(u), s(v)) + \sum_{u\nsim v} H_d(s(u), s(v)) \\ &= \sum_{m} (2r-2) + \sum_{\binom{n}{2}-m} 2r \\ &= m(2r-2) + \left[\binom{n}{2} - m\right] 2r \\ &= nr(n-2), \qquad \text{since } \; m = nr/2. \end{aligned}$$

□

Corollary 3.1. *For a complete graph K_n, $H_B(K_n) = n(n-1)(n-2)$.*

Corollary 3.2. *For a cycle C_n, $H_B(C_n) = 2n(n-2)$.*

Theorem 3.2. *For a complete bipartite graph $K_{p,q}$,*

$$H_B(K_{p,q}) = 2pq(p+q-2).$$

Proof. The graph $G = K_{p,q}$ has $n = p + q$ vertices and $m = pq$ edges. If the vertices u and v are adjacent then $deg_G(u) = p$ and $deg_G(v) = q$ or vice versa. Therefore by Theorem 2.2, we have

$$H_d(s(u), s(v)) = deg_G(u) + deg_G(v) - 2 = p + q - 2.$$

Let V_1 and V_2 be the partite sets of the vertices of a graph $K_{p,q}$, where $|V_1| = p$ and $|V_2| = q$. Let u and v be non-adjacent vertices. If $u, v \in V_1$ then $deg_G(u) = deg_G(v) = q$. Therefore by Theorem 2.2, $H_d(s(u), s(v)) = 2q$. Similarly, if $u, v \in V_2$, then $H_d(s(u), s(v)) = 2p$. In the graph $K_{p,q}$, there are pq pairs of adjacent vertices and $\left[\binom{p}{2} + \binom{q}{2}\right]$ pairs of non adjacent vertices.

Therefore,

$$\begin{aligned} H_B(K_{p,q}) &= \sum_{u\sim v} H_d(s(u), s(v)) + \sum_{u\nsim v} H_d(s(u), s(v)) \\ &= pq(p+q-2) + \binom{p}{2}(2q) + \binom{q}{2}(2p) \\ &= 2pq(p+q-2). \end{aligned}$$

□

Theorem 3.3. *Let G be a graph with n vertices and m edges and $\overline{G}$ be the complement of G. Then $H_B(G) + H_B(\overline{G}) = n(n-1)(n-2)$.*

Proof.

$$\begin{aligned} H_B(G) &= \sum_{\{u,v\}\subseteq V(G)} H_d(s(u), s(v)) \\ &= \sum_{u\sim v \text{ in } G} H_d(s(u), s(v)) + \sum_{u\nsim v \text{ in } G} H_d(s(u), s(v)) \\ &= \sum_{u\sim v \text{ in } G} (deg_G(u) + deg_G(v) - 2) \\ &\quad + \sum_{u\nsim v \text{ in } G} (deg_G(u) + deg_G(v)). \end{aligned} \tag{4}$$

If the vertices u and v are adjacent (respectively non-adjacent) in G, then they are non-adjacent (respectively adjacent) in $\overline{G}$. Further $deg_{\overline{G}}(u) = n - 1 - deg_G(u)$. Therefore

$$\begin{aligned} H_B(\overline{G}) &= \sum_{u\sim v \text{ in } \overline{G}} H_d(s(u), s(v)) + \sum_{u\nsim v \text{ in } \overline{G}} H_d(s(u), s(v)) \\ &= \sum_{u\sim v \text{ in } \overline{G}} (deg_{\overline{G}}(u) + deg_{\overline{G}}(v) - 2) + \sum_{u\nsim v \text{ in } \overline{G}} (deg_{\overline{G}}(u) + deg_{\overline{G}}(v)) \\ &= \sum_{u\nsim v \text{ in } G} (n - 1 - deg_G(u) + n - 1 - deg_G(v) - 2) \\ &\quad + \sum_{u\sim v \text{ in } G} (n - 1 - deg_G(u) + n - 1 - deg_G(v)) \\ &= \left[\binom{n}{2} - m\right](2n-4) - \sum_{u\nsim v \text{ in } G} (deg_G(u) + deg_G(v)) \\ &\quad + m(2n-4) - \sum_{u\sim v \text{ in } G} (deg_G(u) + deg_G(v) - 2). \end{aligned} \tag{5}$$

Adding Eqs. (4) and (5), the result follows.

□

Theorem 3.4. *If G is a self-complementary graph with n vertices, then*

$$H_B(G) = \frac{n(n-1)(n-2)}{2}.$$

Proof. Proof follows by Theorem 3.5 as $G \cong \overline{G}$. □

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4. Algorithm

Algorithm: Hamming_Index(G):

```
1. for i = 1 to n increment by 1
2.     SVi[m]
3.     Deg(Vi) = 0
4. IM[n][m]
5. for i = 1 to n increment by 1
6.     for j = 1 to m increment by 1
7.         if (Vi is source or destination of Ej)
8.             IM[i − 1][j − 1] = 1
9.             SVi[j − 1] = 1
10.        else
11.            IM[i − 1][j − 1] = 0
12.            SVi[j − 1] = 0
13. temp = 0
14. for i = 1 to n increment by 1
15.     for j = 1 to m increment by 1
16.         temp = SVi[j − 1]
17.         if (temp = 1)
18.             Deg(Vi) = Deg(Vi) + 1
19. Hamming_Index = 0
20. for i = 1 to n − 1 increment by 1
21.     for j = i + 1 to n increment by 1
22.         if (there exists edge (Vi,Vj)) then
23.             HDij = [Deg(Vi) + Deg(Vj)] − 2
24.         else
25.             HDij = [Deg(Vi) + Deg(Vj)]
26. Hamming_Index = Hamming_Index + HDij
```

Notations:

n - Total number of vertices in a given graph.
m - Total number of edges in a given graph.
IM - Incidence Matrix
HDij - Hamming distance between the vertices i and j.

Working of an algorithm:

- The for loop in lines 1 to 3 creates n number of arrays, where in each array is of length m and they are named as SV1, SV2, ..., SVn. This loop also initializes the degree of all vertices in the graph with 0.

- In line 4 a matrix **IM** of order $n \times m$ is created which will become the incidence matrix (giving the vertex – edge adjacency) of a given graph.

- The 2 nested for loops from line 5 to 12 are used to create the Incidence Matrix and Strings corresponding to each vertex in the given graph.

  - For each vertex represented by i, every edge represented by j is checked for adjacency using the condition in line 7 as - if (Vi is source or destination of Ej).

  - If the conidtion in line 7 is true then the edge Ej is incident to the vertex Vi and hence the corresponding entry in the incidence matrix will be set to 1 and also 1 will be entered into the string corresponding to the vertex Vi. Otherwise, if the condition in line 7 is false then the edge Ej is not incident to the vertex Vi and hence the corresponding entry in the incidence matrix will be set to 0 and also 0 will be entered into the string corresponding to the vertex Vi.

- A temporary variable "temp" is created in the line 13 and is initialized with 0, which will hold the values extracted from the strings corresponding to each vertex.

- The 2 nested for loops in line 14 to 18 are used to calculate the degrees of each vertex in the graph based on the strings corresponding to each vertex.

  - In these loops, for each vertex Vi the entries from its corresponding string is extracted one by one and checked whether that entry is equal to 1 in the condition of line 17. If this condition is true then the degree of that particular vertex Vi is incremented by 1 in line 18.

- In line 19 a variable "Hamming_Index" is created and initialized with 0. This variable gives the total Hamming distance between the strings corresponding to all the vertices of a graph.

- The 2 nested for loops in line 20 to 26 calculate the Hamming distance between every pair of vertices and also the Hamming index of a given graph.

  - In these loops, we check for every possible pair of vertices whether they both are adjacent using the condition in line 22. If this condition is true then the vertices Vi and Vj are adjacent and hence the Hamming distance between them is calculated using the formula HDij = [Deg(Vi) + Deg(Vj)] $-$ 2. Otherwise, if the condition in line 22 is false then the vertices Vi and Vj are not adjacent and hence the Hamming distance between them is calculated using the formula HDij = [Deg(Vi) + Deg(Vj)].

– In line 26, the Hamming distance calculated in line 23 or 25 is added to the Hamming index.

5. Conclusion

The Hamming distance between the strings generated by the incidence matrix of a graph is obtained. Thus the Hamming index of some graphs are reported. Theorem 2.3 provides the graph in which $H_d(s(u), s(v)) = d_G(u, v)$ for every pair of vertices. In general the cases $H_d(s(u), s(v)) > d_G(u, v)$ and $H_d(s(u), s(v)) < d_G(u, v)$ required further study.

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