Further results on locating chromatic number for amalgamation of stars linking by one path

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Abstract

Let $G = (V,E)$ be a connected graph. Let $c$ be a proper coloring using $k$ colors, namely $1, 2, \cdots, k$. Let $\Pi = \{S_1, S_2, \cdots, S_k\}$ be a partition of $V(G)$ induced by $c$ and let $S_i$ be the color class that receives the color $i$. The color code, $c_{\Pi}(v) = (d(v, S_1), d(v, S_2), \cdots, d(v, S_k))$, where $d(v, S_i) = \min\{d(v, x) | x \in S_i\}$ for $i \in [1, k]$. If all vertices in $V(G)$ have different color codes, then $c$ is called as the locating-chromatic $k$-coloring of $G$. Minimum $k$ such that $G$ has the locating-chromatic $k$-coloring is called the locating-chromatic number, denoted by $\chi_L(G)$. In this paper, we discuss the locating-chromatic number for $n$ certain amalgamation of stars linking a path, denoted by $nS_k,m$, for $n \geq 1$, $m \geq 2$, $k \geq 3$, and $k > m$.

Keywords: locating chromatic number, amalgamation of stars
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1. Introduction

The locating chromatic number is a topic in graph theory, derived from the vertex-coloring and partition dimension of a graph [11]. Many paper discussed about the locating chromatic number since Chartrand et al. [9] introduced the concept in 2002.

All graphs considered are finite, undirected and simple. Let $G = (V,E)$ be a connected graph. Let $c$ be a proper coloring using $k$ colors, namely $1, 2, \cdots, k$. Let $\Pi = \{S_1, S_2, \cdots, S_k\}$ be a partition of $V(G)$ induced by $c$ and let $S_i$ be the color class that receives the color $i$. The
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color code, \( c_\Pi(v) = (d(v, S_1), d(v, S_2), \ldots, d(v, S_k)) \), where \( d(v, S_i) = \min\{d(v, x)|x \in S_i\} \) for \( i \in [1, k] \). If all vertices in \( V(G) \) have different color codes, then \( c \) is called as the locating-chromatic \( k \)-coloring of \( G \). Minimum \( k \) such that \( G \) has the locating-chromatic \( k \)-coloring is called the locating-chromatic number, denoted by \( \chi_L(G) \).

**Theorem 1.1.** [10] Let \( G \) be a simple connected graph and \( c \) be a locating coloring of \( G \). If \( v, w \in V(G) \) and \( v \neq w \) such that \( d(v, x) = d(w, x) \) for all \( x \in V(G) - \{v, w\} \), then \( c(v) \neq c(w) \). In particular, if \( v \) and \( w \) are non adjacent vertices of \( G \) such that neighborhood of \( v \) is equal to neighborhood of \( w \), then \( c(v) \neq c(w) \).

**Corollary 1.1.** [10] If \( G \) is a simple connected graph containing a vertex that is adjacent to \( k \) leaves of \( G \), then \( \chi_L(G) \geq k + 1 \).

Chartrand et al. [9][10] obtained the locating chromatic number of some classes of graphs such that: paths, stars, double stars, caterpillars, complete graphs, bipartite graphs, and the characterization of graphs having locating chromatic number \( n \), \((n-1)\), or \((n-2)\). Next, Asmiati et al. investigated locating chromatic number for special kind of trees, namely: amalgamation of stars [1], [4], firecracker graphs [2], banana trees [5]. Moreover, Baskoro at al. [8] determined the locating chromatic number for corona product of some graphs. Beside that, Asmiati et al. [3] characterized graphs containing cycle having locating chromatic number tree and Baskoro et al. [7] characterized all trees having locating chromatic number three.

Let \( S_{m+2} \) be a star with \((m+2)\) vertices. The amalgamation of stars, denoted by \( S_{k,m} \), where \( k \geq 2 \), is obtained from \((k-1)\) stars \( S_{m+2} \), by identifying one leaf of every stars \( S_{m+2} \). The identified vertex is denoted as the center of \( S_{k,m} \). Graph \( nS_{k,m} \) is obtained from \( n \) copies \( S_{k,m} \) and every center of them, denoted by \( x_i \), for \( i = 1, 2, \ldots, n \) is linked by one path, and \((n-1)\) new vertices denoted \( y_i, i = 1, 2, \ldots, n-1 \) are the subdivision vertices in \( x_i x_{i+1} \), \( i = 1, 2, \ldots, n-1 \). Next, the vertices of distance 1 from the center \( x_i \) are defined as the intermediate vertices, denoted by \( l_i^j, i = 1, 2, \ldots, n, j = 1, 2, \ldots, k-1 \) and the \( t \)-th leaf of the intermediate vertices \( l_i^j \) are denoted by \( l_i^j t (t = 1, 2, \ldots, m) \).

In [6], Asmiati et al. determined the locating chromatic number of \( nS_{k,m} \) for \( k \leq m \), where \( k \geq 3 \) and \( m \geq 2 \), as follows.

\[
\chi_L(nS_{k,m}) = \begin{cases} 
  m + 1, & 1 \leq n \leq \left\lfloor \frac{m}{k-1} \right\rfloor; \\
  m + 2, & \text{otherwise}.
\end{cases}
\]

In this paper we will discuss the locating chromatic number of \( nS_{k,m} \) for \( k > m \), where \( k \geq 3 \) and \( m \geq 2 \).

2. Main Results

In this section, we will discuss about the locating chromatic number of \( nS_{k,m} \) for \( n \geq 1 \) and \( k > m, k \geq 3, m \geq 2 \).

**Lemma 2.1.** Let \( c \) be a coloring on \( nS_{k,m} \) using \((k-a)\) colors, where \( k > m, k \geq 3, m \geq 2 \), \( a \geq 0 \), \( a = k - m - 1 \). Coloring \( c \) is a locating coloring if and only if \( c(l_i^j) = c(l_s^t) \), \( j \neq n \) and \( i \neq s \) such that \( \{c(l_i^j)\} | t = 1, 2, 3, \ldots, m \} \) and \( \{c(l_n^j)\} | t = 1, 2, 3, \ldots, m \} \) are two different sets.
Proof. Consider \( P = \{ c(l_{jt}^i) \mid t = 1, 2, 3, \ldots, m \} \) and \( Q = \{ c(l_{nt}^i) \mid t = 1, 2, 3, \ldots, m \} \). Let \( c \) be a locating coloring of \( nS_{k,m}, k > m, k \geq 3, m \geq 2, a \geq 0 \), dan \( c(l_{jt}^i) = c(l_{nt}^i) \), for some \( j \neq n \), and \( i \neq s \). Suppose that \( P = Q \). Since \( d(l_{jt}^i, u) = d(l_{nt}^i, u) \) for each \( u \in V \setminus \{ l_{jt}^i \cup l_{nt}^i \} \), then the color codes of \( l_{jt}^i \) and \( l_{nt}^i \) are the same. So, \( c \) is not a locating coloring, a contrary. As the result, \( P \neq Q \).

Let \( \Pi \) be a partition of \( V(G) \) with \( |\Pi| \geq m \). Consider \( c(l_{jt}^i) = c(l_{nt}^i), j \neq n, \) dan \( i \neq s \). Since \( P \neq Q \), then there are two colors, namely \( x \) and \( y \) such that \( (x \in P, x \notin Q) \) or \( (y \in P, y \notin Q) \). Next, we will show that every \( v \in V(nS_{k,m}) \) have different color codes.

- It is clear that \( c_{\Pi}(l_{jt}^i) \neq c_{\Pi}(l_{nt}^i) \), since their color codes are different in the \( x \)-ordinat or \( y \)-ordinat.
- If \( c(l_{jt}^i) = c(l_{nt}^i) \), for each \( l_{jt}^i \neq l_{nt}^i \), then we divide two cases to show that \( c_{\Pi}(l_{jt}^i) \neq c_{\Pi}(l_{nt}^i) \)
  - Case 1: If \( c(l_{jt}^i) = c(l_{nt}^i) \), then based on the previous proof \( P \neq Q \). So, \( c_{\Pi}(l_{jt}^i) \neq c_{\Pi}(l_{nt}^i) \).
  - Case 2: Consider \( c(l_{jt}^i) = p_1 \) and \( c(l_{nt}^i) = p_2 \), where \( p_1 \neq p_2 \). Then \( c_{\Pi}(l_{jt}^i) \neq c_{\Pi}(l_{nt}^i) \) because their color codes are different at least in the \( p_1 \)-ordinat and \( p_2 \)-ordinat.
- If \( c(x_i) = c(l_{jt}^i) \), then the color code of \( c_{\Pi}(x_i) \) contains at least two components with value 1, whereas in \( c_{\Pi}(l_{jt}^i) \) contains exactly one component with value 1. So, \( c_{\Pi}(x_i) \neq c_{\Pi}(l_{jt}^i) \).
- If \( c(y_i) = c(l_{jt}^i) \), then the color code of \( c_{\Pi}(y_i) \) contains at least two components with values 1, whereas in \( c_{\Pi}(l_{jt}^i) \) contains exactly one component with value 1. So, \( c_{\Pi}(y_i) \neq c_{\Pi}(l_{jt}^i) \).

From all cases, we can see that all vertices in \( nS_{k,m} \) have different color codes, so \( c \) is a locating coloring. \( \square \)

Lemma 2.2. Let \( n \geq 1, k > m, k \geq 3, m \geq 2, a \geq 0, \) and \( a = k - m - 1 \). If \( c \) is a locating coloring of \( nS_{k,m} \) using \( k - a \) colors and \( H(a) = \left[ \frac{(k - a - 1)(k - a - 1)}{m} \right] \), then \( n \leq H(a) \).

Proof. Let \( c \) be a \((k - a)\)-locating coloring of \( nS_{k,m} \). For some \( j \), consider \( c(l_{jt}^i) \) as the color of \( l_{jt}^i \), then the color combination of \( \{ l_{jt}^i \mid t = 1, 2, 3, \ldots, m \} \) is \((k - a - 1)\). Since one color has been used for the central vertex \( x \), then there are \((k - a - 1)\) colors left to be assigned to \( l_{jt}^i \), for each \( i = 1, 2, \ldots, n \) and \( j = 1, 2, 3, \ldots, k - 1 \). By Lemma 2.1, the maximum number for \( n \) is 
\[
\left[ \frac{(k - a - 1)(k - a - 1)}{m} \right] = H(a), a \geq 0. \square
\]

Theorem 2.1. Let \( nS_{k,m} \) be some certain amalgamation of stars for \( a \geq 0, k > m, k \geq 3, m \geq 2, a = k - m - 1 \). Then
\[
\chi_L(nS_{k,m}) = \begin{cases} 
  k - a, & \text{if } 1 \leq n \leq H(a), \\
  k - a + 1, & \text{otherwise}.
\end{cases}
\]
Proof. First, we determine the lower bound of $\chi_L(nS_{k,m})$ for $1 \leq n \leq H(a) = \left\lceil \frac{(k-a-1)(k-a-1)^{m-1}}{k-1} \right\rceil$. Since every vertex $l_j^i$ for $i = 1, 2, 3, \ldots, n$ and $j = 1, 2, 3, \ldots, k-1$ are adjacent to $m = k-a-1$ leaves, then by Corollary 1.1, we have $\chi_L(nS_{k,m}) \geq k-a$.

To determine the upper bound of $\chi_L(nS_{k,m})$ for $1 \leq n \leq H(a) = \left\lceil \frac{(k-a-1)(k-a-1)^{m-1}}{k-1} \right\rceil$, let $c$ be a coloring of $V(nS_{k,m})$ using $(k-a)$ colors. We assign the coloring as follows.

- $c(x_i) = 1$, for $i = 1, 2, 3, \ldots, n$.
- $c(y_i) = 2$, for odd $i$ and 3 for even $i = 1, 2, 3, \ldots, n$.
- Color of $l_i^j$ for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, (k-1)$ given color $2, 3, \ldots, (k-a)$, respectively.
- $\{c(l_i^j)\} = \{1, 2, 3, \ldots, k-a\} \setminus \{c(l_i^j)\}$ for $t = 1, 2, 3, \ldots, m$.

Next, we will show that all vertices in $V(nS_{k,m})$ have different color codes. Consider $u, v \in V(nS_{k,m})$ and $c(u) = c(v)$. Then we have the following cases.

- If $u = x_i, v = x_k$ for some $i, k$ and $i \neq k$, then $c_{\Pi}(u) \neq c_{\Pi}(v)$ because $c(l_i^j) \neq c(l_k^j)$ for each $i = 1, 2, \ldots, (k-1)$.
- If $u = x_i, v = l_i^hj$ for some $i, h, j, t$, then in $c_{\Pi}(u)$ does not have component value four, whereas in $c_{\Pi}(v)$, exactly one component has value 4. So, $c_{\Pi}(u) \neq c_{\Pi}(v)$.
- If $u = y_i, v = l_j^i$ for some $i, j$, then in $c_{\Pi}(u)$ exactly two components have value 1, whereas in $c(v)$, at least three components have value 1. So, $c_{\Pi}(u) \neq c_{\Pi}(v)$.
- If $u = y_i, v = l_j^k$ for some $i, k, j$ and $i \neq k$, then in $c_{\Pi}(u)$ exactly two components have value 1, whereas in $c(v)$, at least three components have value 1. So, $c_{\Pi}(u) \neq c_{\Pi}(v)$.
- If $u = y_i, v = l_j^t$ for some $i, j, t$, then in $c_{\Pi}(u)$, exactly two components have value 1, whereas in $c(v)$, exactly one component has value 1. As a result, $c_{\Pi}(u) \neq c_{\Pi}(v)$.
- If $u = y_i, v = l_j^k$ for some $i, k, j, t$ and $i \neq k$, then in $c_{\Pi}(u)$ at least two components have value 1, whereas in $c(v)$, exactly one component has value 1. So, $c_{\Pi}(u) \neq c_{\Pi}(v)$.
- If $u = l_i^j, v = l_j^i$ for some $i, j, t$, then in $c_{\Pi}(u)$ at least two components have value 1, whereas in $c(v)$, exactly one component has value 1. As a result, $c_{\Pi}(u) \neq c_{\Pi}(v)$.
- If $u = l_i^j, v = l_j^ht$ for some $i, j, k, h, t$ and $i \neq k, j \neq h$, then in $c_{\Pi}(u)$, at least two components have value 1, whereas in $c(v)$, exactly one component has value 1. So, $c_{\Pi}(u) \neq c_{\Pi}(v)$
- If $u = l_i^j, v = l_j^ht$ for some $i, j, h, t, j \neq h$. Since $\{c(l_i^j)\} \neq \{c(l_j^h)\}$, then $c_{\Pi}(u) \neq c_{\Pi}(v)$.
Since all vertices have different color codes, then $c$ is a locating coloring on $nS_{k,m}$. Thus, $\chi_L(nS_{k,m}) \leq k - a$ for $n \leq H(a)$.

Next, we discuss the locating chromatic number of $nS_{k,m}$ for $n > H(a)$.

By Corollary 1.1, we have the trivial lower bound, $\chi_L(S_{k,m}) \geq k - a$ for $H(a)$. Suppose that $c$ is a locating coloring using $(k-a)$ colors on $nS_{k,m}$ for $k \geq m$, $k \geq 3$, $m \geq 2$, and $n \geq H(a)$. Since $n \geq H(a)$, then there are $i,j,k,t, i \neq k$ and \{c($l^i_j$), c($l^j_t$)\} = \{c($l^k_j$), c($l^k_t$)\} = \{1,2,3,\ldots,k-a\} such that $c(l^i_j) = c(l^j_t)$ for some $j = 1, 2, 3, \ldots, k-1, t = 1, 2, 3, \ldots, m$, a contrary. Thus, $\chi_L(S_{k,m}) \geq k - a + 1$ for $n > H(a)$.

Let $c$ be a coloring on $nS_{k,m}$ using $(k-a+1)$ colors. We assign the coloring as follows.

- $c(x_i) = 1$, for $i = 1, 2, 3, \ldots, n$.
- $c(y_i) = 2$, for odd $i$ and 3 for even $i = 1, 2, 3, \ldots, n$.
- For $j = 1, 2, 3, \ldots, (k-1)$, $c(l^j) = 2$, for odd $i$ and 3 for even $i = 1, 2, 3, \ldots, n$.
- If $A = \{1, 2, \ldots, k-a+1\}$, define:

$$\{c(l^j_{ij}) | t = 1, 2, \ldots, m)\} = \begin{cases} A \setminus \{1, k-a\} & \text{if } i = 1, \\ A \setminus \{k-a+1\} & \text{otherwise.} \end{cases}$$

The maximum number of colored $p$ is $\binom{k-a-1}{m}$ for any $p$. We can do that because $n \geq H(a)$. So, $c(l^i_j) = c(l^s_n)$, $j \neq n$, dan $i \neq s$. Thus, we can arrange such that \{c($l^j_{ij}$) | $t = 1, 2, 3, \ldots, m)\} \neq \{c(l^s_{nt}) | t = 1, 2, 3, \ldots, m)\}. As the result, by Lemma 2.1, $c$ is a locating coloring. Thus, $\chi_L(nS_{k,m}) \leq k - a + 1$ for $n > H(a)$.

As the conclusion, we obtain that $\chi_L(nS_{k,m}) = k - a + 1$. □

For an illustration, we give the locating-chromatic coloring of $nS_{5,3}$ for $1 \leq n \leq 4$ in Figure 1 and $nS_{5,3}$ for $n > 4$ in Figure 2.

![Figure 1. A minimum locating coloring of 4S_{5,3}](image-url)
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Figure 2. A minimum locating coloring of $nS_{5,3}$ for $n > 4$, $a = 0$

References


