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# On the local metric dimension of $t$-fold wheel, $P_n \odot K_m$, and generalized fan

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**Abstract**

Let $G$ be a connected graph and let $u, v \in V(G)$. For an ordered set $W = \{w_1, w_2, ..., w_n\}$ of $n$ distinct vertices in $G$, the representation of a vertex $v$ of $G$ with respect to $W$ is the $n$-vector $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_n))$, where $d(v, w_i)$ is the distance between $v$ and $w_i$ for $1 \le i \le n$. The set $W$ is a local metric set of $G$ if $r(u \mid W) \neq r(v \mid W)$ for every pair $u, v$ of adjacent vertices of $G$. The local metric set of $G$ with minimum cardinality is called a local metric basis for $G$ and its cardinality is called a local metric dimension, denoted by $lmd(G)$. In this paper we determine the local metric dimension of a $t$-fold wheel graph, $P_n \odot K_m$ graph, and generalized fan graph.

*Keywords:* local metric dimension, $t$-fold wheel graph, corona graph, generalized fan graph
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## 1. Introduction

One of the discussions in graph theory is the local metric dimension of graph which is the development of the metric dimension of graph. In 2010 Okamoto et al. [6] introduces the concept of a local metric dimension of a graph. The journal discusses about dimension metric local of a graph. Suppose the set $W$ is a subset of the vertex set in a graph $G$. The representation of one vertex in $G$ respect to set $W$ is a sequential pair whose element is the distance of a vertex to all vertex in the set $W$, where the distance on a graph is defined with the shortest path length of a

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vertex to the other vertex. The set $W$ is called a *local metric set* for $G$ (also called *local metric generator*) if every two adjacent vertices have distinct representations. A minimum *local metric set* is called a *local metric basis* for $G$ and its cardinality is called the *local metric dimension* of $G$ and denoted by $lmd(G)$.

Some authors have investigated the local metric dimension of some graph classes. In 2014 Kristina et al. [3] determined the local metric dimension of the comb product between cycle graph and star graph. In the same year, Ningsih et al. [2] observed the local metric dimension of comb product of cycle graph and path graph. In 2016 Rodríguez-Velázquez et al. [5] observed the local metric dimension of the corona product. Then in 2017 Rimadhany [4] found the local metric dimension of Circulant graph. In this paper, we determined the local metric dimension of $t$-fold wheel graph, $Pn \odot Km$ graph, and generalized fan graph.

## 2. Results

### Local Metric Dimension

The definitions of local metric dimension were taken from Okamoto et al [6], the $t$-fold wheel graph defined by Walis [7], the corona product of two graphs defined by Yero et al. [8], and the generalized fan graph defined by Intaja and Sitthiwarattham [1].

**Definition 2.1.** *Let $G$ be a connected graph. If an ordered set $W = \{w_1, w_2, w_3, \ldots, w_n\}$ of vertices in a connected graph $G$ and a vertex $v \in V(G)$, then the representation of $v$ with respect to $W$ is an ordered $n$-vector $r(v \mid W) = (d(v, w_1), d(v, w_2), d(v, w_3), \ldots, d(v, w_n))$, where $d(v, w_n)$ represents the distance between the vertices $v$ and $w_n$. The set $W$ is a local metric set of $G$ if $r(u \mid W) \neq r(v \mid W)$ for every pair $u, v$ of adjacent vertices of $G$. A minimum local metric set is called a local metric basis for $G$ and its cardinality the local metric dimension of $G$ and denoted by $lmd(G)$.*

We often use the following theorem given by Okamoto et al. [6]

**Theorem 2.1.** *Let $G$ be a nontrivial connected graph of order $n$. Then $lmd(G) = n - 1$ if and only if $G = K_n$ and $lmd(G) = 1$ if and only if $G$ is bipartite.*

### The Local Metric Dimension of t-fold wheel graph

The $t$-fold wheel $(W_n^t)$ graph is a graph that contains $t$ central vertices which each adjacent to all vertices of a cycle $C_n$, but not adjacent to each other. The t-fold wheel $(W_n^t)$ graph can be defined as a join of the cycle $C_n$ and the complement $K_t$, so it can be written as the graph $W_n^t = C_n + \bar{K}_t$ for $n \geq 3$ and $t \geq 1$. Let $(W_n^t)$ graph has a set of vertices $V(W_n^t) = \{u_0, u_1, \ldots, u_{t-1}, v_0, v_1, \ldots, v_{n-1}\}$ for $t \geq 1$ and $n \geq 3$ where $u_i$ is central vertices. Figure 1 is example of $t$-fold wheel graph with $t = 3$ and $n = 5$.

**Theorem 2.2.** *Let $W_n^t$ be a t-fold wheel graph with $t \geq 1$ and $n \geq 3$, then*

$$lmd(W_n^t) = \begin{cases} 3, & \text{for } t \geq 1 \quad \text{and } n = 3; \\ 2, & \text{for } t \geq 1 \quad \text{and } n = 4; \\ \lceil \frac{n}{4} \rceil, & \text{for } t \geq 1 \quad \text{and } n \geq 5. \end{cases}$$

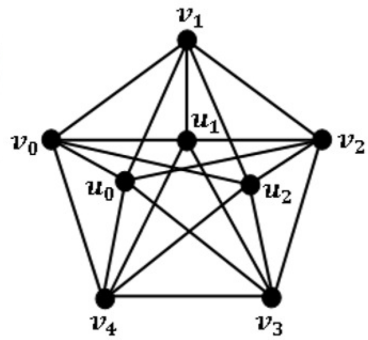


Figure 1. $t$-fold wheel graph with $t = 3$ and $n = 5$

*Proof.* Given a $t$-fold wheel graph $W_n^t$ with $t \geq 1$ dan $n \geq 3$ with the set of vertices $V(W_n^t) = \{u_0, u_1, \ldots, u_{t-1}, v_0, v_1, \ldots, v_{n-1}\}$. We prove for the local metric dimension of the $t$-fold wheel graph based on the values of $n$ and $t$.

Case 1. For $t \geq 1$ and $n = 3$.
The $W_n^t$ graph with $t = 1$ and $n = 3$ is a graph where each vertex of $W_3^t$ is in $C_3$. If $W = \{x\}$ with $x \in W_3^t$, $t \geq 1$, then there are vertices $y, z \in W_3^t$ which are adjacent each other and have the same representation. So, $r(y|W) = r(z|W) = 1$ and hence $lmd(W_3^t) \neq 1$. If we choose $W = \{x_1, y_1\}$ with $x_1, y_1 \in W_3^t$ then there are vertices $x_2, y_2 \in V(W_3^t)$ which have the same representations and adjacent each other, so that $lmd(W_3^t) \neq 2$. For example take $W = \{v_0, v_1, v_2\}$. The representations of each vertex with respect to $W$ are

$$\begin{array}{ll} r(v_0|W) = (0, 1, 1); & r(u_0|W) = (1, 1, 1); \\ r(v_1|W) = (1, 0, 1); & \quad\vdots \qquad\quad \vdots \\ r(v_2|W) = (1, 1, 0); & r(u_j|W) = (1, 1, 1). \end{array}$$

All vertices $v_i$ with $i = \{0, 1, 2\}$ of $W_3^t$ have a different representation respect to the local metric set $W$ and all vertices $u_j$ with $j = \{0, 1, \ldots, t-1\}$ have the same representation respect to the local metric set $W$ but not adjacent each other so, it can be concluded that $W$ is the local metric set.
Hence, $lmd(W_n^t) = 3$ for $t \geq 1$ and $n = 3$.

Case 2. For $t \geq 1$ and $n = 4$.
Same with previous explanation in case for $t \geq 1$ and $n = 3$. The $W_4^t$ graph is a graph where each vertex of $W_4^t$ is in $C_3$, so $lmd(W_4^t) \neq 1$. Suppose $W = \{v_0, v_1\}$, then there are two adjacent vertices have different representations with respect to $W$, so that $lmd(W_4^t) = 2$ for $t \geq 1$ and $n = 4$.

Case 3. For $t \geq 1$ dan $n \geq 5$
Let $W_n^t$ be a $t$-fold wheel graph with $t \geq 1$ and $n \geq 5$. We will show $lmd(W_n^t) \leq \lceil \frac{n}{4} \rceil$. Assume $W = \{v_{4i}\}$ where $i = \{0, 1, \ldots, \lfloor \frac{n}{4} \rfloor\}$, so, $|W| = \lceil \frac{n}{4} \rceil$. The representation of all vertices $W_n^t$ with

respect to $W$ are divided into two parts

1. For $n = 4k+1$, $n = 4k+3$ and $n = 4k+4$ with $k = 1, 2, \ldots$
   For $n = 4k+2$ with $k = 2, 3, \ldots$
   then

$$r(u_j|W) = (1, 1, 1, 1, \ldots, 1, 1) \qquad \text{for } j = 0, 1, \ldots, t-1;$$

$$r(v_i|W) = \begin{cases} (0, 2, 2, \ldots, 2, 1) & i = 0, \text{ for } n = a; \\ ((i - \lfloor \frac{i}{4} \rfloor)\text{mod } 3, 2, 2, \ldots, 2, 2), & i = 0, 1, 2; \\ (2, (i - \lfloor \frac{i}{4} \rfloor)\text{mod } 3, 2, \ldots, 2, 2), & i = 4, 5, 6; \\ \vdots & \vdots \\ (2, 2, 2, \ldots (i - \lfloor \frac{i}{4} \rfloor)\text{mod } 3, 2), & i = (4\lceil \frac{n}{4} \rceil - 8), (4\lceil \frac{n}{4} \rceil - 7), (4\lceil \frac{n}{4} \rceil - 6); \\ (2, 2, 2, \ldots, 2, (i - \lfloor \frac{i}{4} \rfloor)\text{mod } 3), & i = \begin{cases} (4\lceil \frac{n}{4} \rceil - 4); \text{for } n = b, \\ (4\lceil \frac{n}{4} \rceil - 4), (4\lceil \frac{n}{4} \rceil - 3); \\ \qquad \text{for } n = c, \\ (4\lceil \frac{n}{4} \rceil - 4), (4\lceil \frac{n}{4} \rceil - 3), (4\lceil \frac{n}{4} \rceil - 2); \\ \qquad \text{for } n = d; \end{cases} \\ (2, i \text{ mod } 2, 2, \ldots, 2, 2), & i = 3; \\ (2, 2, i \text{ mod } 2, \ldots, 2, 2), & i = 7; \\ \vdots & \vdots \\ (2, 2, 2, \ldots, 2, i \text{ mod } 2), & i = 4\lceil \frac{n}{4} \rceil - 5; \\ (1, 2, 2, \ldots, 2, 0), & i = n-1 \text{ for } n = a \\ (1, 2, 2, \ldots, 2, 1), & i = n-1 \text{ for } n = b \\ (1, 2, 2, \ldots, 2, 2), & i = n-1 \text{ for } n = c, d; \end{cases}$$

   with $a = 4k+1$, $b = 4k+2$, $c = 4k+3$, and $d = 4k+1$.
2. For $n = 6$
   Suppose $W = \{v_0, v_3\}$ for $n = 6$, the representation every vertices respect to $W$ are

$$r(u_j|W) = (1, 1, 1, 1, \ldots, 1, 1) \qquad \text{for } j = 0, 1, \ldots, t-1.$$

$$r(v_i|W) = \begin{cases} ((i - \lfloor \frac{i}{4} \rfloor)\text{mod } 2, 2) & \text{for } i = 0, 1; \\ (2, (i - \lfloor \frac{i}{4} \rfloor)\text{mod } 2) & \text{for } i = n-3, n-2; \\ (2, 1) & \text{for } i = 2; \\ (1, 2) & \text{for } i = n-1. \end{cases}$$

   Based on the two parts above, some vertices $v_i$ with $i = 0, 1, 2, \ldots, n-1$ and all vertices $u_j$ with $j = 0, 1, 2, \ldots, t-1$ have the same representation with respect to $W$ but not adjacent each other, so $W$ is the local metric set. Then $lmd(W_n^t) \leq \lceil \frac{n}{4} \rceil$.

   Next we show $lmd(W_n^t) \geq \lceil \frac{n}{4} \rceil$.
   Assume $W$ is a local metric set of a $t$-fold wheel graph $W_n^t$ with $|W| < \lceil \frac{n}{4} \rceil$. There are three possibilities to choose vertices of $W$.

(a) If all vertices of $W$ in $V(C_n) = \{v_i | \, 0 \leq i \leq n-1\} \subset V(W_n^t)$, then at least two vertices $x, y \in V(C_n)$ are adjacent such that
$r(x|W) = r(y|W) = (2, 2, \ldots, 2, 2)$.

(b) If some vertices of $W$ in $V(C_n) = \{v_i | \, 0 \leq i \leq n-1\} \subset V(W_t^n)$ and other vertices in $V(\overline{K_t}) = \{u_j | \, 0 \leq j \leq t-1\}$, then at least two vertices $x, y \in V(C_n)$ are adjacent such that
$d(x, v_i) = d(y, v_i) = 2; \quad \forall\, v_i \in W,$
$d(x, u_i) = d(y, u_i) = 1; \quad \forall\, u_j \in W.$

(c) If all vertices of $W$ in $V(\overline{K_t}) = \{u_j | \, 0 \leq j \leq t-1\} \subset V(W_t^n)$, then there are vertices $x_1, y_1 \in V(C_n)$ and $x_2, y_2 \in V(K_t)$ such that
$r(x_1|W) = r(y_1|W) = (1, 1, \ldots, 1); \quad x_1$ and $y_1$ are adjacent,
$r(x_2|W) = r(y_2|W) = (2, 2, \ldots, 2); \quad x_2$ and $y_2$ are adjacent.

From all possibilities to choose vertex of $W$ there are at least two adjacent vertices with the same representations, so $W$ is not local metric set. This contradicts with the fact that $W$ is a local metric set of $(W_n^t)$. Hence $lmd(W_n^t) \geq \lceil \frac{n}{4} \rceil$. This completes the proof of the theorem. □

**The Local Metric Dimension of $P_n \odot K_m$**

The corona product $P_n \odot K_m$ graph is a graph obtained from $P_n$ and $K_m$ by taking one copy of $P_n$ and $n$ copies of $K_m$ and joining by an edge each vertex from the $i^{th}$- copy of $K_m$ with the $i^{th}$- vertex of $P_n$. Let $P_n \odot K_m$ be a graph have a set of vertices $V(P_n \odot K_m) = \{u_1, \ldots, u_i, v_1^1, \ldots v_j^1, v_1^2, \ldots, v_j^2, \ldots, v_1^n, \ldots, v_j^n\}$ and vertices $u_i \in V(P_n)$, $v_j \in V(K_m)$ with $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots m$.

**Lemma 2.1.** *For $n, m \geq 2$, if $W$ is a local metric set for a $P_n \odot K_m$, then $|W| \geq n(m-1)$.*

*Proof.* By contradiction, we will show that $|W| \geq n(m-1)$. Assume that $W$ is a local metric set with $|W| < n(m-1)$. Let $W \subset V((P_n \odot K_m) - \{u_i, v_{m-1}^n, v_m^n\})$ with $i = 1, 2, \ldots n$. There are two vertices $v_{m-1}^n$ dan $v_m^n$ such that $r(v_{m-1}^n|W) = r(v_m^n|W) = \{n+1, n, n-1, \ldots, 5, 4, 3, 1\}$ where vertex $v_{m-1}^n$ dan $v_m^n$ adjacent each other. This contradicts with the fact that $W$ is a local metric set of $P_n \odot K_m$, so $|W| \geq n(m-1)$. □

**Lemma 2.2.** *For $n, m \geq 2$, if $W = \{v_j^i\} \subset V(P_n \odot K_m)$ with $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots, m-1$ then $W$ is a local metric set for a $P_n \odot K_m$ graph.*

*Proof.* The representations of all vertices of $P_n \odot K_m$ with respect to $W = \{v_j^i\} \subset V(P_n \odot K_m)$ with $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots, m-1$ are

$$\begin{array}{lllll}
d(u_1, v_j^1) = 1, & d(u_2, v_j^1) = 2, & d(u_3, v_j^1) = 3, & \ldots & d(u_n, v_j^1) = n; \\
d(u_1, v_j^2) = 2, & d(u_2, v_j^2) = 1, & d(u_3, v_j^2) = 2, & \ldots & d(u_n, v_j^2) = n-1; \\
d(u_1, v_j^3) = 3, & d(u_2, v_j^3) = 2, & d(u_3, v_j^3) = 1, & \ldots & d(u_n, v_j^3) = n-2; \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
d(u_1, v_j^n) = n, & d(u_2, v_j^n) = n-1, & d(u_3, v_j^n) = n-2, & \ldots & d(u_n, v_j^n) = 1;
\end{array}$$

and

$$\begin{array}{lllll}
d(v_m^1, v_j^1) = 1, & d(v_m^2, v_j^1) = 3, & d(v_m^3, v_j^1) = 4, & \dots & d(v_m^n, v_j^1) = n+1; \\
d(v_m^1, v_j^2) = 3, & d(v_m^2, v_j^2) = 1, & d(v_m^3, v_j^2) = 3, & \dots & d(v_m^n, v_j^2) = n; \\
d(v_m^1, v_j^3) = 4, & d(v_m^2, v_j^3) = 3, & d(v_m^3, v_j^3) = 1, & \dots & d(v_m^n, v_j^3) = n-1; \\
d(v_m^1, v_j^4) = 5, & d(v_m^2, v_j^4) = 4, & d(v_m^3, v_j^4) = 3, & \dots & d(v_m^n, v_j^4) = n-2; \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
d(v_m^1, v_j^n) = n+1, & d(v_m^2, v_j^n) = n, & d(v_m^3, v_j^n) = n-2, & \dots & d(v_m^n, v_j^n) = 1;
\end{array}$$

Every pair of adjacent vertices have distinct representations with respect to $W$, so that $W = \{v_j^i\}$ with $i = 1, 2, \dots n$ and $j = 1, 2, \dots, m-1$ is a local metric set on $P_n \odot K_m$ graph. $\square$

**Theorem 2.3.** *Let $P_n \odot K_m$ graph, then for $n \geq 1$ dan $m \geq 1$*

$$lmd(P_n \odot K_m) = \begin{cases} 1, & \textit{for } n \geq 1 \textit{ and } m = 1; \\ m, & \textit{for } n = 1 \textit{ and } m \geq 2; \\ n(m-1), & \textit{for } n, m \geq 2. \end{cases}$$

*Proof.* Given a $P_n \odot K_m$ graph with $n \geq 1$ and $m \geq 1$ and $V(P_n \odot K_m) = \{u_i, v_j^i\}$ with $i = 1, 2, \dots n$ and $j = 1, 2, \dots m$. We prove the local metric dimension of the $P_n \odot K_m$ graph according to the values of $n$ and $m$.

Case 1. For $n \geq 1$ and $m = 1$.
$P_n \odot K_m$ graph with $m = 1$ ia a tree graph (bipartite graph), based on the theorem 2.1 the local metric dimension of a graph is equal to one if and only if the graph is bipartite. So, $lmd(P_n \odot K_m) = 1$ for $m = 1$.

Case 2. For $n = 1$ and $m \geq 2$
$P_n \odot K_m$ with $n = 1$ and $m \geq 1$ is a complete graph with number of vertices is $m + 1$. Let $V(P_1 \odot K_m) = \{u_1, v_1^1, v_2^1, \dots, v_m^1\}$. Based on the theorem 2.1 the local metric dimension of the local metric graph is equal to $p - 1$ if and only if the graph is complete with $p$ order. So, it can be seen that $lmd(P_n \odot K_m) = (m + 1) - 1 = m$ for $n = 1$.
Let $W = \{v_1^1, v_2^1, \dots, v_m^1\} \subset V(P_1 \odot K_m)$ then $W$ is local metric set of the $P_1 \odot K_m$. The representations of all vertices $V(P_1 \odot K_m)$ with respect to W are $r(u_1|W) = (1, 1, 1, 1, \dots, 1, 1)$

$$r(v_i|W) = \begin{cases} (0, 1, 1, 1, \dots, 1, 1), & \text{for } i = 1; \\ \qquad \vdots & \qquad \vdots \\ (1, 1, 1, 1, \dots, 0, 1), & \text{for } i = m-1; \\ (1, 1, 1, 1, \dots, 1, 0), & \text{for } i = m. \end{cases}$$

All vertices in $V(P_1 \odot K_m)$ have distinct representations with respect to $W$ so that $W = \{v_1^1, v_2^1, \dots, v_m^1\}$ is a local metric set on $P_1 \odot K_m$.

Case 3. For $n \geq 2$ dan $m \geq 2$
Given a $P_n \odot K_m$ with $n \neq 1$ dan $m \neq 1$. By using Lemma 2.2, we have a set $W = \{v_j^i\} \subset$

$V(P_n \odot K_m)$ with $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots, m-1$ is a local metric set of $P_n \odot K_m$ graph. According to Lemma 2.1, $|W| \geq n(m-1)$ so that $W = \{v_j^i\}$ with $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots, m-1$ is a local metric basis of $P_n \odot K_m$ graph.
Hence $lmd(P_n \odot K_m) = n(m-1)$. □

**The Local Metric Dimension of Generalized Fan Graph**

Generalized fan graph $F_{m,n} \cong \overline{K}_m + P_n$ is a graph with $V(F_{m,n}) = V(\overline{K}_m) \cup V(P_n)$ and $E(F_{m,n}) = E(P_n) \cup \{uv | u \in V(\overline{K}_m), v \in V(P_n)\}$. Clearly $|V(F_{m,n})| = m + n$ and $|E(F_{m,n})| = mn + n - 1$. The generalized fan graph $F_{(m,n)}$ can be decipted as in Figure 2

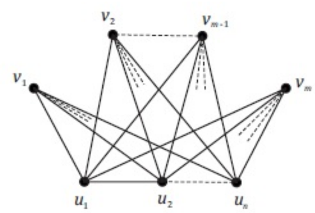


Figure 2. Generalized fan graph $F_{(m,n)}$

**Theorem 2.4.** *Let $F_{m,n}$ be a generalized fan graph with $m \geq 1$ and $n \geq 2$ then*

$$lmd(F_{(m,n)}) = \begin{cases} 2, & \textit{for } 2 \leq n \leq 5 \quad \textit{and } m \textit{ other}; \\ \lfloor \frac{n+2}{4} \rfloor, & \textit{for } n \geq 6 \qquad \textit{and } m \textit{ other}. \end{cases}$$

*Proof.* Given a generalized fan graph $F_{(m,n)}$ with $m \geq 1$ dan $n \geq 2$ with the set of vertices $V(F_{(m,n)}) = \{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n\}$. We prove for the local metric dimension of the generalized fan graph according to the values of $m$ and $n$.

Case 1. For $2 \leq n \leq 5$ and $m$ other.
The $F_{(m,n)}$ graph with $2 \leq n \leq 5$ and $m \geq 1$ is a graph where each vertex in $C_3$. If $W = \{x\}$ with $x \in F_{(m,n)}$ then there are vertices $y, z \in F_{(m,n)}$ which adjacent each other and have the same representation. So, $r(y|W) = r(z|W) = 1$ and hence $lmd(F_{(m,n)}) \neq 1$. If choose $W = \{u_1, u_k\}$ with $k = 2$ for $2 \leq n \leq 4$ and $k = 3$ for $n = 5$ then all vertices of $F_{(m,n)}$ with $2 \leq n \leq 5$ and $m \geq 1$ have different representation with respect to $W$, so $lmd(F_{(m,n)}) = 2$ for $2 \leq n \leq 5$ and $m$ other.

Case 2. For $n \geq 6$ and $m$ other.
We will shown $lmd\ (F_{(m,n)}) \leq \lfloor \frac{n+2}{4} \rfloor$. Assume $W = \{u_3, u_7, u_{11}, u_{15}, \ldots u_{n-2}\}$. Cardinality of $W$ is $\lfloor \frac{n+2}{4} \rfloor$. Then the representation of all vertices $F_{(m,n)}$ with respect to $W$ are

$$r(v_i|W) = (1, 1, 1, \ldots, 1, 1) \qquad \text{for } i = 0, 1, \ldots, t-1.$$

$$r(u_j|W) = \begin{cases} (2,2,2,\ldots,2,2), & j = \begin{cases} (1,5,9,\ldots n-5,n); \text{ for } n=a \\ (1,5,9,\ldots n-6,n); \text{ for } n=b \\ (1,5,9,\ldots n-7,n); \text{ for } n=c \\ (1,5,9,\ldots n-4,n); \text{ for } n=d \end{cases} \\ (1,2,2,\ldots,2,2), & j=2,4 \\ (2,1,2,\ldots,2,2), & j=6,8 \\ (2,2,1,\ldots,2,2), & j=10,12 \\ \vdots & \vdots \\ (2,2,2,\ldots,1,2), & j = \begin{cases} (n-4); \text{ for } n=a \\ (n-5); \text{ for } n=b \\ (n-4,n-6); \text{ for } n=c \\ (n-5,n-7); \text{ for } n=d \end{cases} \\ (2,2,2,\ldots,1,0), & j=n-2 \text{ for } n=a \\ (2,2,2,\ldots,1,1), & j=n-3 \text{ for } n=b \\ (2,2,2,\ldots,2,1), & j = \begin{cases} (n-1); \text{for } n=a \text{ and } b \\ (n-1,n-3); \text{for } n=c \text{ and } d \end{cases} \\ (0,2,2,\ldots,2,2), & j=3 \\ (2,0,2,\ldots,2,2), & j=7 \\ (2,2,0,\ldots,2,2), & j=11 \\ \vdots & \vdots \\ (2,2,2,\ldots,0,2), & j = \begin{cases} (n-4); \text{ for } n=b \\ (n-5); \text{ for } n=c \\ (n-6); \text{ for } n=d \end{cases} \\ (2,2,2,\ldots,0,1), & j=n-3 \text{ for } a \\ (2,2,2,\ldots,2,0), & j=n-2 \text{ for } a,b,c \\ (2,2,2,\ldots,1,0), & j=n-2 \text{ for } a \end{cases}$$

with $a = \{6, 10, 14, 18, \ldots\}$, $b = \{7, 11, 15, 19, \ldots\}$,
$c = \{8, 12, 16, 20, \ldots\}$, and $d = \{9, 13, 17, 21, \ldots\}$.

All vertices $v_i$ and some vertices $u_j$ with $i = 1, 2, \ldots m$ and $j = 1, 2, \ldots n$ have the same representation with respect to $W$ but not adjacent each other, so $W$ is the local metric set. Then $lmd(F_{(m,n)}) \leq \lfloor \frac{n+2}{4} \rfloor$.

Next we show $lmd(F_{(m,n)}) \geq \lfloor \frac{n+2}{4} \rfloor$. Assume $W$ is a local metric set of a generalized fan graph $F_{(m,n)}$ with $|W| < \lfloor \frac{n+2}{4} \rfloor$. There are three possibilities to choose vertex of $W$

1. If all vertices of $W$ in $V(\overline{K}_m) = \{v_i | 1 \leq i \leq m\} \subset V((F_{(m,n)})$ then there are vertices $x, y \in V(\overline{K}_m)$ are adjacent such that
   $r(x|W) = r(y|W) = (1, 1, \ldots, 1, 1)$
2. If some vertices of $W$ in $V(\overline{K}_m) = \{v_i | 1 \leq i \leq m\} \subset V(F_{(m,n)})$ and other vertices in $V(P_n) = \{u_j | 1 \leq j \leq n\}$, then there are vertices $x, y \in V(\overline{K}_m)$ are adjacent such that
   $r(x|W) = r(y|W) = (2, 2, \ldots, 1, 1)$
3. If all vertices of $V(P_n) = \{u_j | 1 \leq i \leq n\} \subset V((F_{(m,n)})$, then there are vertices $x, y \in$

$V(\overline{K}_m)$ are adjacent such that
$r(x|W) = r(y|W) = (2, 2, \ldots, 1, 1)$

From all possibilities to choose vertex of $W$ there are at least two adjacent vertices with the same representations, so $W$ is not local metric set. This contradicts with the fact that $W$ is a local metric set of $(W_n^t)$. Hence $lmd(F_{(m,n)}) \geq \lfloor \frac{n+2}{4} \rfloor$. These complete the proof of the theorem. □

## 3. Conclusion

According to the discussion above it can be concluded that the local metric dimension of a $t$-fold wheel graph, $P_n \odot K_m$ graph, and a generalized fan graph are as stated in Theorem 2.2, Theorem 2.3, and Theorem 2.4 respectively.

**Problem 1.** *Determine the total metric dimension of $P_n \odot^k K_m$.*

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