On the local metric dimension of $t$-fold wheel, $P_n \odot K_m$, and generalized fan

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Abstract
Let $G$ be a connected graph and let $u, v \in V(G)$. For an ordered set $W = \{w_1, w_2, ..., w_n\}$ of $n$ distinct vertices in $G$, the representation of a vertex $v$ of $G$ with respect to $W$ is the $n$-vector $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_n))$, where $d(v, w_i)$ is the distance between $v$ and $w_i$ for $1 \leq i \leq n$. The set $W$ is a local metric set of $G$ if $r(u|W) \neq r(v|W)$ for every pair $u, v$ of adjacent vertices of $G$. The local metric set of $G$ with minimum cardinality is called a local metric basis for $G$ and its cardinality is called a local metric dimension, denoted by $\text{lmd}(G)$. In this paper we determine the local metric dimension of a $t$-fold wheel graph, $P_n \odot K_m$ graph, and generalized fan graph.

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1. Introduction

One of the discussions in graph theory is the local metric dimension of graph which is the development of the metric dimension of graph. In 2010 Okamoto et al. [6] introduces the concept of a local metric dimension of a graph. The journal discusses about dimension metric local of a graph. Suppose the set $W$ is a subset of the vertex set in a graph $G$. The representation of one vertex in $G$ respect to set $W$ is a sequential pair whose element is the distance of a vertex to all vertex in the set $W$, where the distance on a graph is defined with the shortest path length of a
vertex to the other vertex. The set \( W \) is called a local metric set for \( G \) (also called local metric generator) if every two adjacent vertices have distinct representations. A minimum local metric set is called a local metric basis for \( G \) and its cardinality is called the local metric dimension of \( G \) and denoted by \( \text{lmd}(G) \).

Some authors have investigated the local metric dimension of some graph classes. In 2014 Kristina et al. [3] determined the local metric dimension of the comb product between cycle graph and star graph. In the same year, Ningsih et al. [2] observed the local metric dimension of comb product of cycle graph and path graph. In 2016 Rodríguez-Velázquez et al. [5] observed the local metric dimension of the corona product. Then in 2017 Rimadhany [4] found the local metric dimension of \( P_n \odot K_m \) graph, and generalized fan graph. In this paper, we determined the local metric dimension of \( t \)-fold wheel graph, \( P_n \odot K_m \) graph, and generalized fan graph.

2. Results

Local Metric Dimension

The definitions of local metric dimension were taken from Okamoto et al [6], the \( t \)-fold wheel graph defined by Walis [7], the corona product of two graphs defined by Yero et al. [8], and the generalized fan graph defined by Intaja and Sitthiwartatham [1].

**Definition 2.1.** Let \( G \) be a connected graph. If an ordered set \( W = \{w_1, w_2, w_3, \ldots, w_n\} \) of vertices in a connected graph \( G \) and a vertex \( v \in V(G) \), then the representation of \( v \) with respect to \( W \) is an ordered \( n \)-vector \( r(v \mid W) = (d(v, w_1), d(v, w_2), d(v, w_3), \ldots, d(v, w_n)) \), where \( d(v, w_n) \) represents the distance between the vertices \( v \) and \( w_n \). The set \( W \) is a local metric set of \( G \) if \( r(u \mid W) \neq r(v \mid W) \) for every pair \( u, v \) of adjacent vertices of \( G \). A minimum local metric set is called a local metric basis for \( G \) and its cardinality the local metric dimension of \( G \) and denoted by \( \text{lmd}(G) \).

We often use the following theorem given by Okamoto et al. [6]

**Theorem 2.1.** Let \( G \) be a nontrivial connected graph of order \( n \). Then \( \text{lmd}(G) = n - 1 \) if and only if \( G = K_n \) and \( \text{lmd}(G) = 1 \) if and only if \( G \) is bipartite.

**The Local Metric Dimension of \( t \)-fold wheel graph**

The \( t \)-fold wheel \( (W^t_n) \) graph is a graph that contains \( t \) central vertices which each adjacent to all vertices of a cycle \( C_n \), but not adjacent to each other. The \( t \)-fold wheel \( (W^t_n) \) graph can be defined as a join of the cycle \( C_n \) and the complement \( K_1 \), so it can be written as the graph \( W^t_n = C_n + K_t \) for \( n \geq 3 \) and \( t \geq 1 \). Let \( (W^t_n) \) graph has a set of vertices \( V(W^t_n) = \{u_0, u_1, \ldots, u_{t-1}, v_0, v_1, \ldots, v_{n-1}\} \) for \( t \geq 1 \) and \( n \geq 3 \) where \( u_i \) is central vertices. Figure 1 is example of \( t \)-fold wheel graph with \( t = 3 \) and \( n = 5 \).

**Theorem 2.2.** Let \( W^t_n \) be a \( t \)-fold wheel graph with \( t \geq 1 \) and \( n \geq 3 \), then

\[
\text{lmd}(W^t_n) = \begin{cases} 
3, & \text{for } t \geq 1 \text{ and } n = 3; \\
2, & \text{for } t \geq 1 \text{ and } n = 4; \\
\left\lceil \frac{n}{4} \right\rceil, & \text{for } t \geq 1 \text{ and } n \geq 5.
\end{cases}
\]
On the local metric dimension of of t-fold wheel graph,...  |  R. A. Solekhah and T.A. Kusmayadi

Figure 1. t-fold wheel graph with $t = 3$ and $n = 5$

**Proof.** Given a $t$-fold wheel graph $W_t^n$ with $t \geq 1$ dan $n \geq 3$ with the set of vertices $V(W_t^n) = \{u_0, u_1, \ldots, u_{t-1}, v_0, v_1, \ldots, v_{n-1}\}$. We prove for the local metric dimension of the $t$-fold wheel graph based on the values of $n$ and $t$.

Case 1. For $t \geq 1$ and $n = 3$.

The $W_t^n$ graph with $t = 1$ and $n = 3$ is a graph where each vertex of $W_3^t$ is in $C_3$. If $W = \{x\}$ with $x \in W_3^t$, $t \geq 1$, then there are vertices $y, z \in W_3^t$ which are adjacent each other and have the same representation. So, $r(y|W) = r(z|W) = 1$ and hence $lmd(W_3^t) \neq 1$. If we choose $W = \{x_1, y_1\}$ with $x_1, y_1 \in W_3^t$ then there are vertices $x_2, y_2 \in V(W_3^t)$ which have the same representations and adjacent each other, so that $lmd(W_3^t) \neq 2$. For example take $W = \{v_0, v_1, v_2\}$. The representations of each vertex with respect to $W$ are

$$
\begin{align*}
r(v_0|W) &= (0, 1, 1); & r(u_0|W) &= (1, 1, 1); \\
r(v_1|W) &= (1, 0, 1); & \vdots & \vdots \\
r(v_2|W) &= (1, 1, 0); & r(u_j|W) &= (1, 1, 1).
\end{align*}
$$

All vertices $v_i$ with $i = \{0, 1, 2\}$ of $W_3^t$ have a different representation respect to the local metric set $W$ and all vertices $u_j$ with $j = \{0, 1, \ldots, t - 1\}$ have the same representation respect to the local metric set $W$ but not adjacent each other so, it can be concluded that $W$ is the local metric set.

Hence, $lmd(W_3^t) = 3$ for $t \geq 1$ and $n = 3$.

Case 2. For $t \geq 1$ and $n = 4$.

Same with previous explanation in case for $t \geq 1$ and $n = 3$. The $W_4^t$ graph is a graph where each vertex of $W_4^t$ is in $C_3$, so $lmd(W_4^t) \neq 1$. Suppose $W = \{v_0, v_1\}$, then there are two adjacent vertices have different representations with respect to $W$, so that $lmd(W_4^t) = 2$ for $t \geq 1$ and $n = 4$.

Case 3. For $t \geq 1$ dan $n \geq 5$

Let $W_n^t$ be a $t$-fold wheel graph with $t \geq 1$ and $n \geq 5$. We will show $lmd(W_n^t) \leq \lceil \frac{n}{4} \rceil$. Assume $W = \{v_i\}$ where $i = \{0, 1, \ldots, \lceil \frac{n}{4} \rceil\}$, so, $|W| = \lceil \frac{n}{4} \rceil$. The representation of all vertices $W_n^t$ with
respect to $W$ are divided into two parts

1. For $n = 4k + 1$, $n = 4k + 3$ and $n = 4k + 4$ with $k = 1, 2, \ldots$
   For $n = 4k + 2$ with $k = 2, 3, \ldots$
   then
   \[
   r(u_j|W) = (1, 1, 1, 1, \ldots, 1, 1) \quad \text{for } j = 0, 1, \ldots, t - 1;
   \]
   \[
   r(v_i|W) = \begin{cases}
   (0, 2, 2, \ldots, 2, 1) & i = 0, \text{ for } n = a; \\
   ((i - \lfloor \frac{i}{4} \rfloor) \mod 3, 2, 2, \ldots, 2, 2), & i = 0, 1, 2; \\
   (2, (i - \lfloor \frac{i}{4} \rfloor) \mod 3, 2, \ldots, 2, 2), & i = 4, 5, 6; \\
   \vdots & \\
   (2, 2, 2, \ldots, (i - \lfloor \frac{i}{4} \rfloor) \mod 3, 2), & i = (4\lceil \frac{n}{4} \rceil - 8), (4\lceil \frac{n}{4} \rceil - 7), (4\lceil \frac{n}{4} \rceil - 6); \\
   (2, 2, 2, \ldots, 2, (i - \lfloor \frac{i}{4} \rfloor) \mod 3), & i = \begin{cases}
   (4\lceil \frac{n}{4} \rceil - 4); & \text{for } n = b, \\
   (4\lceil \frac{n}{4} \rceil - 4), (4\lceil \frac{n}{4} \rceil - 3); & \text{for } n = c, \\
   (4\lceil \frac{n}{4} \rceil - 4), (4\lceil \frac{n}{4} \rceil - 3), (4\lceil \frac{n}{4} \rceil - 2); & \text{for } n = d;
   \end{cases}
   \end{cases}
   \]
   \[
   (2, i \mod 2, 2, \ldots, 2, 2), \quad i = 3; \\
   (2, 2, i \mod 2, \ldots, 2, 2), \quad i = 7; \\
   \vdots \\
   (2, 2, \ldots, 2, i \mod 2), \quad i = 4\lceil \frac{n}{4} \rceil - 5; \\
   (1, 2, 2, \ldots, 2, 0), \quad i = n - 1 \text{ for } n = a \\
   (1, 2, 2, \ldots, 2, 1), \quad i = n - 1 \text{ for } n = b \\
   (1, 2, 2, \ldots, 2, 2), \quad i = n - 1 \text{ for } n = c, d;
   \]
   with $a = 4k + 1, b = 4k + 2, c = 4k + 3,$ and $d = 4k + 1$.

2. For $n = 6$
   Suppose $W = \{v_0, v_3\}$ for $n = 6$, the representation every vertices respect to $W$ are
   \[
   r(u_j|W) = (1, 1, 1, 1, \ldots, 1, 1) \quad \text{for } j = 0, 1, \ldots, t - 1.
   \]
   \[
   r(v_i|W) = \begin{cases}
   ((i - \lfloor \frac{i}{4} \rfloor) \mod 2, 2) & \text{for } i = 0, 1; \\
   (2, (i - \lfloor \frac{i}{4} \rfloor) \mod 2) & \text{for } i = n - 3, n - 2; \\
   (2, 1) & \text{for } i = 2; \\
   (1, 2) & \text{for } i = n - 1.
   \end{cases}
   \]
   Based on the two parts above, some vertices $v_i$ with $i = 0, 1, 2, \ldots, n - 1$ and all vertices $u_j$ with $j = 0, 1, 2, \ldots, t - 1$ have the same representation with respect to $W$ but not adjacent each other, so $W$ is the local metric set. Then $lmd(W_n^t) \leq \lceil \frac{n}{4} \rceil$.

Next we show $lmd(W_n^t) \geq \lceil \frac{n}{4} \rceil$.
Assume $W$ is a local metric set of a $t$-fold wheel graph $W_n^t$ with $|W| < \lceil \frac{n}{4} \rceil$. There are three possibilities to choose vertices of $W$. 

The local metric dimension of $P_n \odot K_m$

The corona product $P_n \odot K_m$ graph is a graph obtained from $P_n$ and $K_m$ by taking one copy of $P_n$ and $n$ copies of $K_m$ and joining by an edge each vertex from the $i$th copy of $K_m$ with the $i$th vertex of $P_n$. Let $P_n \odot K_m$ be a graph have a set of vertices $V(P_n \odot K_m) = \{u_1, \ldots, u_i, v^1_i, \ldots, v^m_i, \ldots, v^1_j, \ldots, v^m_j, \ldots, v^1_n, \ldots, v^m_n\}$ and vertices $u_i \in V(P_n)$, $v_j \in V(K_m)$ with $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

**Lemma 2.1.** For $n, m \geq 2$, if $W$ is a local metric set for a $P_n \odot K_m$, then $|W| \geq n(m - 1)$.

**Proof.** By contradiction, we will show that $|W| \geq n(m - 1)$. Assume that $W$ is a local metric set with $|W| < n(m - 1)$. Let $W \subset V((P_n \odot K_m) - \{u_i, v^m_{m-1}, v^1_m\})$ with $i = 1, 2, \ldots, n$. There are two vertices $v^m_{m-1}$ dan $v^1_m$ such that $r(v^m_{m-1}|W) = r(v^1_m|W) = \{n + 1, n, n - 1, \ldots, 5, 4, 3, 1\}$ where vertex $v^m_{m-1}$ dan $v^1_m$ adjacent each other. This contradicts with the fact that $W$ is a local metric set of $P_n \odot K_m$, so $|W| \geq n(m - 1)$.

**Lemma 2.2.** For $n, m \geq 2$, if $W = \{v^j_i\} \subset V(P_n \odot K_m)$ with $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m - 1$ then $W$ is a local metric set for a $P_n \odot K_m$ graph.

**Proof.** The representations of all vertices of $P_n \odot K_m$ with respect to $W = \{v^j_i\} \subset V(P_n \odot K_m)$ with $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m - 1$ are

\[
\begin{align*}
d(u_1, v^1_i) & = 1, & d(u_2, v^1_i) & = 2, & d(u_3, v^1_i) & = 3, & \ldots & d(u_n, v^1_i) & = n; \\
d(u_1, v^2_i) & = 2, & d(u_2, v^2_i) & = 1, & d(u_3, v^2_i) & = 2, & \ldots & d(u_n, v^2_i) & = n - 1; \\
d(u_1, v^3_i) & = 3, & d(u_2, v^3_i) & = 2, & d(u_3, v^3_i) & = 1, & \ldots & d(u_n, v^3_i) & = n - 2; \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
d(u_1, v^m_i) & = n, & d(u_2, v^m_i) & = n - 1, & d(u_3, v^m_i) & = n - 2, & \ldots & d(u_n, v^m_i) & = 1;
\end{align*}
\]
and

\[
\begin{align*}
d(v_1^m, v_j^1) &= 1, & d(v_2^m, v_j^1) &= 3, & d(v_3^m, v_j^1) &= 4, & \ldots & d(v_n^m, v_j^1) &= n + 1; \\
d(v_1^m, v_j^2) &= 3, & d(v_2^m, v_j^2) &= 1, & d(v_3^m, v_j^2) &= 3, & \ldots & d(v_n^m, v_j^2) &= n; \\
d(v_1^m, v_j^3) &= 4, & d(v_2^m, v_j^3) &= 3, & d(v_3^m, v_j^3) &= 1, & \ldots & d(v_n^m, v_j^3) &= n - 1; \\
d(v_1^m, v_j^4) &= 5, & d(v_2^m, v_j^4) &= 4, & d(v_3^m, v_j^4) &= 3, & \ldots & d(v_n^m, v_j^4) &= n - 2; \\
& \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
d(v_1^m, v_j^n) &= n + 1, & d(v_2^m, v_j^n) &= n, & d(v_3^m, v_j^n) &= n - 2, & \ldots & d(v_n^m, v_j^n) &= 1;
\end{align*}
\]

Every pair of adjacent vertices have distinct representations with respect to \(W\), so that \(W = \{v_j^i\}\) with \(i = 1, 2, \ldots n\) and \(j = 1, 2, \ldots, m - 1\) is a local metric set on \(P_n \odot K_m\) graph.

**Theorem 2.3.** Let \(P_n \odot K_m\) graph, then for \(n \geq 1\) dan \(m \geq 1\)

\[\text{lmd}(P_n \odot K_m) = \begin{cases} 
1, & \text{for } n \geq 1 \text{ and } m = 1; \\
1, & \text{for } n = 1 \text{ and } m \geq 2; \\
1, & \text{for } n, m \geq 2.
\end{cases}\]

**Proof.** Given a \(P_n \odot K_m\) graph with \(n \geq 1\) and \(m \geq 1\) and \(V(P_n \odot K_m) = \{u_i, v_j^i\}\) with \(i = 1, 2, \ldots n\) and \(j = 1, 2, \ldots, m\). We prove the local metric dimension of the \(P_n \odot K_m\) graph according to the values of \(n\) and \(m\).

Case 1. For \(n \geq 1\) and \(m = 1\).

\(P_n \odot K_m\) graph with \(m = 1\) is a tree graph (bipartite graph), based on the theorem 2.1 the local metric dimension of a graph is equal to one if and only if the graph is bipartite. So, \(\text{lmd}(P_n \odot K_m) = 1\) for \(m = 1\).

Case 2. For \(n = 1\) and \(m \geq 2\)

\(P_n \odot K_m\) with \(n = 1\) and \(m \geq 1\) is a complete graph with number of vertices is \(m + 1\). Let \(V(P_1 \odot K_m) = \{u_1, v_1^1, v_1^2, \ldots, v_1^m\}\). Based on the theorem 2.1 the local metric dimension of the local metric graph is equal to \(p - 1\) if and only if the graph is complete with \(p\) order. So, it can be seen that \(\text{lmd}(P_n \odot K_m) = (m + 1) - 1 = m\) for \(n = 1\).

Let \(W = \{v_1^1, v_2^1, \ldots, v_m^1\} \subset V(P_1 \odot K_m)\) then \(W\) is local metric set of the \(P_1 \odot K_m\). The representations of all vertices \(V(P_1 \odot K_m)\) with respect to \(W\) are \(r(u_i|W) = (1, 1, 1, \ldots, 1, 1)\)

\[
r(v_i|W) = \begin{cases} 
(0, 1, 1, 1, \ldots, 1, 1), & \text{for } i = 1; \\
\vdots & \vdots \\
(1, 1, 1, 1, \ldots, 0, 1), & \text{for } i = m - 1; \\
(1, 1, 1, 1, \ldots, 1, 0), & \text{for } i = m.
\end{cases}
\]

All vertices in \(V(P_1 \odot K_m)\) have distinct representations with respect to \(W\) so that \(W = \{v_1^1, v_2^1, \ldots, v_m^1\}\) is a local metric set on \(P_1 \odot K_m\).

Case 3. For \(n \geq 2\) dan \(m \geq 2\)

Given a \(P_n \odot K_m\) with \(n \neq 1\) dan \(m \neq 1\). By using Lemma 2.2, we have a set \(W = \{v_j^i\} \subset \)
V(P_n \odot K_m) with i = 1, 2, \ldots n and j = 1, 2, \ldots m - 1 is a local metric set of \( P_n \odot K_m \) graph. According to Lemma 2.1, \(|W| \geq n(m-1)\) so that \( W = \{v_j\} \) with \( i = 1, 2, \ldots n \) and \( j = 1, 2, \ldots m - 1 \) is a local metric basis of \( P_n \odot K_m \) graph. Hence \( lmd(P_n \odot K_m) = n(m-1) \).

The Local Metric Dimension of Generalized Fan Graph

Generalized fan graph \( F_{m,n} \cong K_m + P_n \) is a graph with \( V(F_{m,n}) = V(K_m) \cup V(P_n) \) and \( E(F_{m,n}) = E(P_n) \cup \{uv \mid u \in V(K_m), v \in V(P_n)\} \). Clearly \( |V(F_{m,n})| = m + n \) and \( |E(F_{m,n})| = mn + n - 1 \). The generalized fan graph \( F_{m,n} \) can be depicted as in Figure 2.

![Figure 2. Generalized fan graph \( F_{m,n} \)](image)

**Theorem 2.4.** Let \( F_{m,n} \) be a generalized fan graph with \( m \geq 1 \) and \( n \geq 2 \) then

\[
lmd(F_{m,n}) = \begin{cases} 
2, & \text{for } 2 \leq n \leq 5 \text{ and } m \text{ other}; \\
\lceil \frac{n+2}{4} \rceil, & \text{for } n \geq 6 \text{ and } m \text{ other}.
\end{cases}
\]

**Proof.** Given a generalized fan graph \( F_{m,n} \) with \( m \geq 1 \) dan \( n \geq 2 \) with the set of vertices

\[V(F_{m,n}) = \{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n\}.\]

We prove for the local metric dimension of the generalized fan graph according to the values of \( m \) and \( n \).

Case 1. For \( 2 \leq n \leq 5 \) and \( m \) other.

The \( F_{m,n} \) graph with \( 2 \leq n \leq 5 \) and \( m \geq 1 \) is a graph where each vertex in \( C_3 \). If \( W = \{x\} \) with \( x \in F_{m,n} \) then there are vertices \( y, z \in F_{m,n} \) which adjacent each other and have the same representation. So, \( r(y|W) = r(z|W) = 1 \) and hence \( lmd(F_{m,n}) \neq 1 \). If choose \( W = \{u_1, u_2\} \) with \( k = 2 \) for \( 2 \leq n \leq 4 \) and \( k = 3 \) for \( n = 5 \) then all vertices of \( F_{m,n} \) with \( 2 \leq n \leq 5 \) and \( m \geq 1 \) have different representation with respect to \( W \), so \( lmd(F_{m,n}) = 2 \) for \( 2 \leq n \leq 5 \) and \( m \) other.

Case 2. For \( n \geq 6 \) and \( m \) other.

We will shown \( lmd(F_{m,n}) \leq \lceil \frac{n+2}{4} \rceil \). Assume \( W = \{u_3, u_7, u_{11}, u_{15}, \ldots, u_{n-2}\} \). Cardinality of \( W \) is \( \lceil \frac{n+2}{4} \rceil \). Then the representation of all vertices \( F_{m,n} \) with respect to \( W \) are

\[r(v_i|W) = (1, 1, 1, \ldots, 1, 1) \quad \text{for } i = 0, 1, \ldots, t - 1.\]
Next we show $lmd_{F}$ and present the representation with respect to $a(m,n)$. If all vertices of $V \cup W$ are adjacent such that $r(x,y) = (m,n)$, then there are three possibilities to choose vertex of $W$.

Next we show $lmd(F_{(m,n)}) \geq \lceil \frac{n+2}{4} \rceil$. Assume $W$ is a local metric set of a generalized fan graph $F_{(m,n)}$ with $|W| < \lceil \frac{n+2}{4} \rceil$. There are three possibilities to choose vertex of $W$.

1. If all vertices of $W$ in $V(K_{m}) = \{v_{i} | 1 \leq i \leq m\} \subset V(F_{(m,n)})$ then there are vertices $x,y \in V(K_{m})$ are adjacent such that $r(x,W) = r(y,W) = (1,1,\ldots,1)$.

2. If some vertices of $W$ in $V(K_{m}) = \{v_{i} | 1 \leq i \leq m\} \subset V(F_{(m,n)})$ and other vertices in $V(P_{n}) = \{u_{j} | 1 \leq j \leq n\}$, then there are vertices $x,y \in V(K_{m})$ are adjacent such that $r(x,W) = r(y,W) = (2,2,\ldots,1,1)$.

3. If all vertices of $V(P_{n}) = \{u_{j} | 1 \leq j \leq n\} \subset V(F_{(m,n)})$, then there are vertices $x,y \in V(K_{m})$ are adjacent such that $r(x,W) = r(y,W) = (2,2,\ldots,1,1)$.

| $j$ | $r(u_{j}|W)$ |
|-----|-------------|
| 2, 2, 2, 2 | $j = \begin{cases} (n-4); & \text{for } n = a \\ (n-5); & \text{for } n = b \\ (n-4,n-6); & \text{for } n = c \\ (n-5,n-7); & \text{for } n = d \end{cases}$ |
| 2, 2, 2, 2, 2, 2 | $j = \begin{cases} (1,5,9,\ldots,n-5,n); & \text{for } n = a \\ (1,5,9,\ldots,n-6,n); & \text{for } n = b \\ (1,5,9,\ldots,n-7,n); & \text{for } n = c \\ (1,5,9,\ldots,n-4,n); & \text{for } n = d \end{cases}$ |
| 2, 2, 2, 2, 2, 2, 2 | $j = \begin{cases} (n-4); & \text{for } n = a \\ (n-5); & \text{for } n = b \\ (n-4,n-6); & \text{for } n = c \\ (n-5,n-7); & \text{for } n = d \end{cases}$ |
| 2, 2, 0, 2, 0, 2, 2 | $j = \begin{cases} (n-4); & \text{for } n = b \\ (n-5); & \text{for } n = c \\ (n-6); & \text{for } n = d \end{cases}$ |
| 2, 2, 2, 2, 2, 2, 2 | $j = \begin{cases} (n-4); & \text{for } n = a \\ (n-5); & \text{for } n = b \\ (n-4,n-6); & \text{for } n = c \\ (n-5,n-7); & \text{for } n = d \end{cases}$ |
| 2, 2, 2, 0, 2, 0, 2 | $j = \begin{cases} (n-4); & \text{for } n = a \\ (n-5); & \text{for } n = b \\ (n-4,n-6); & \text{for } n = c \\ (n-5,n-7); & \text{for } n = d \end{cases}$ |
| 2, 2, 2, 2, 2, 2, 2 | $j = \begin{cases} (n-4); & \text{for } n = b \\ (n-5); & \text{for } n = c \\ (n-6); & \text{for } n = d \end{cases}$ |

with $a = \{6,10,14,18,\ldots\}$, $b = \{7,11,15,19,\ldots\}$, $c = \{8,12,16,20,\ldots\}$, and $d = \{9,13,17,21,\ldots\}$. All vertices $v_{i}$ and some vertices $u_{j}$ with $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$ have the same representation with respect to $W$ but not adjacent each other, so $W$ is the local metric set. Then $lmd(F_{(m,n)}) \leq \lceil \frac{n+2}{4} \rceil$. 

Next we show $lmd(F_{(m,n)}) \geq \lceil \frac{n+2}{4} \rceil$. Assume $W$ is a local metric set of a generalized fan graph $F_{(m,n)}$ with $|W| < \lceil \frac{n+2}{4} \rceil$. There are three possibilities to choose vertex of $W$.
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\[ V(K_m) \text{ are adjacent such that} \\
\quad r(x|W) = r(y|W) = (2, 2, \ldots, 1, 1) \]

From all possibilities to choose vertex of \( W \) there are at least two adjacent vertices with the same representations, so \( W \) is not local metric set. This contradicts with the fact that \( W \) is a local metric set of \( (W_n^t) \). Hence \( lmd(F_{(m,n)}) \geq \lfloor \frac{2n+2}{4} \rfloor \). These complete the proof of the theorem.

3. Conclusion

According to the discussion above it can be concluded that the local metric dimension of a \( t \)-fold wheel graph, \( P_n \odot K_m \) graph, and a generalized fan graph are as stated in Theorem 2.2, Theorem 2.3, and Theorem 2.4 respectively.

Problem 1. Determine the total metric dimension of \( P_n \odot^k K_m \).

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References


