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Local antimagic vertex coloring of unicyclic graphs

Nuris Hisan Nazula[a], Slamin[b], Dafik[a]

[a]*Mathematics Education Study Program, University of Jember, Indonesia*
[b]*Informatics Study Program, University of Jember, Indonesia*

nuris.h.nazula@gmail.com, slamin@unej.ac.id, d.dafik@unej.ac.id

Abstract

The local antimagic labeling on a graph G with $|V|$ vertices and $|E|$ edges is defined to be an assignment $f : E \to \{1, 2, \cdots, |E|\}$ so that the weights of any two adjacent vertices u and v are distinct, that is, $w(u) \neq w(v)$ where $w(u) = \Sigma_{e \in E(u)} f(e)$ and $E(u)$ is the set of edges incident to u. Therefore, any local antimagic labeling induces a proper vertex coloring of G where the vertex u is assigned the color $w(u)$. The local antimagic chromatic number, denoted by $\chi_{la}(G)$, is the minimum number of colors taken over all colorings induced by local antimagic labelings of G. In this paper, we present the local antimagic chromatic number of unicyclic graphs that is the graphs containing exactly one cycle such as kite and cycle with two neighbour pendants.

Keywords: local antimagic labeling, vertex coloring, unicyclic graphs, kite, cycle with two neighbour pendants
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1. Introduction

Let $G = (V, E)$ be a finite, simple, connected and undirected graph. The *local antimagic labeling* on a graph G with $|V|$ vertices and $|E|$ edges is defined to an assignment $f : E \to \{1, 2, \cdots, m\}$ so that the weights any two adjacent vertices are distinct, that is, $w(u) \neq w(v)$ where $w(u) = \Sigma_{e \in E(u)} f(e)$ and $E(u)$ is the set of edges incident to u. Therefore, any local antimagic labeling induces a proper vertex coloring of G where the vertex u is assigned the color $w(u)$. The local antimagic chromatic number, denoted by $\chi_{la}(G)$, is the minimum number of colors taken

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over all colorings induced by local antimagic labelings of G. This concept was recently introduced by Arumugam *et al.* [1].

In the paper [1], Arumugam *et al.* presented the exact value of the local antimagic chromatic number of some families of graphs as follows.

- Complete graph K_n on $n \geq 3$ vertices, $\chi_{la}(K_n) = n$.
- Star $K_{1,n-1}$ on $n \geq 3$ vertices, $\chi_{la}(K_{1,n-1}) = n$.
- Path P_n on $n \geq 3$ vertices, $\chi_{la}(P_n) = 3$.
- Cycle C_n on $n \geq 3$ vertices, $\chi_{la}(C_n) = 3$.
- Friendship graph F_n for $n \geq 2$, $\chi_{la}(F_n) = 3$.
- Friendship graph F_n for $n \geq 2$ by removing an edge e, $\chi_{la}(F_n - \{e\}) = 3$.
- Complete bipartite graph $K_{m,n}$ for $m, n \geq 2$, $\chi_{la}(K_{m,n}) = 2$ if and only if $m \equiv n$ (mod 2).
- Complete bipartite graph $K_{2,n}$ for $n \geq 2$, $\chi_{la}(K_{2,n}) = 2$ for even $n \geq 2$ and $\chi_{la}(K_{2,n}) = 3$ for odd $n \geq 3$ or $n = 2$.
- Graph L_n for $n \geq 2$ that is obtained by inserting a vertex to each edge vv_i, $1 \leq i \leq n-1$, of the star, $\chi_{la}(L_n) = n + 1$.
- Wheel W_n of order $n+1$ for $n \geq 3$, $\chi_{la}(W_n) = 4$ if $n \equiv 1, 3$ (mod 4), $\chi_{la}(W_n) = 3$ if $n \equiv 2$ (mod 4), and $3 \leq \chi_{la}(W_n) \leq 5$ if $n \equiv 0$ (mod 4).

Furthermore, Arumugam *et al.* [1] showed that for any tree T with l leaves, $\chi_{la}(T) \geq l + 1$ and for the graph $H = G + \bar{K}_2$ where G is a graph of order $n \geq 4$, then

$$\chi_{la}(G) + 1 \leq \chi_{la}(H) \leq \begin{cases} \chi_{la}(G) + 1 & \text{for } n \text{ is even} \\ \chi_{la}(G) + 2 & \text{otherwise.} \end{cases}$$

In this paper, we present the local antimagic chromatic number of unicyclic graphs such kite and cycle with two neighbour pendants. A graph is called *unicyclic* if it is connected and contains exactly one cycle. Therefore, a graph is unicyclic if and only if it is connected and has size equal to its order [4]. A kite, denoted by $Kt_{n,m}$, consists of a cycle of length n with a m-edge path (the tail) attached to one vertex [2].

2. Main Results

We start this section with a new result on the local antimagic chromatic number of the kite graph in the following theorem.

Theorem 2.1. *For the kite $Kt_{n,m}$ with $n \geq 3$ and $m \geq 1$, $\chi_{la}(Kt_{n,m}) = 3$.*

Proof. Let $Kt_{n,m}$ be the kite with $n \geq 3$ and $m \geq 1$. The vertex set of $Kt_{n,m}$ is $V = \{u_i | 1 \leq i \leq n\} \cup \{v_j | 1 \leq j \leq m\}$ and the edge set is $E = \{u_i u_{i+1} | 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{u_1 v_1\} \cup \{v_j v_{j+1} | 1 \leq j \leq m-1\}$.

Label the edges of $Kt_{n,m}$ using a bijection $f : E \to \{1, 2, \ldots, n+m\}$ below.

$$\begin{aligned}
f(u_i u_{i+1}) &= \begin{cases} \frac{m+1}{2} + \frac{i+1}{2} - 1 & \text{for odd } m \text{ and odd } i \\ \frac{m+i}{2} & \text{for even } m \text{ and even } i \\ \frac{m+1}{2} + \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor - \frac{i}{2} + 1 & \text{for odd } m \text{ and even } i \\ \frac{m}{2} + \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n+1}{2} \rfloor - \frac{i+1}{2} + 1 & \text{for even } m \text{ and odd } i \end{cases} \\
f(u_n u_1) &= \begin{cases} \lfloor \frac{n}{2} \rfloor + \frac{m+1}{2} & \text{for odd } m \\ \lfloor \frac{n+1}{2} \rfloor + \frac{m}{2} & \text{for even } m \end{cases} \\
f(u_1 v_1) &= \begin{cases} \frac{m}{2} & \text{for even } m \\ \frac{m+1}{2} + n & \text{for odd } m \end{cases} \\
f(v_j v_{j+1}) &= \begin{cases} \frac{m-1}{2} - \frac{j+1}{2} + 1 & \text{for odd } m \text{ and odd } j \\ \frac{m-j}{2} & \text{for even } m \text{ and even } j \\ \frac{m-1}{2} + \frac{j}{2} + n + 1 & \text{for odd } m \text{ and even } j \\ \frac{m}{2} + \frac{j+1}{2} + n & \text{for even } m \text{ and odd } j \end{cases}
\end{aligned}$$

It is easy to see that f is a local antimagic labeling and the weight of vertices are

$$\begin{aligned}
w(u_i) &= \begin{cases} \frac{3m}{2} + 2\lfloor \frac{n+1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor & \text{for even } m \text{ and } i = 1 \\ n + m + 1 & \text{for even } (m+i) \\ n + m & \text{for odd } (m+i) \\ \frac{3m+3}{2} + \lfloor \frac{n}{2} \rfloor + n & \text{for odd } m \text{ and } i = 1 \end{cases} \\
w(v_i) &= \begin{cases} n + m & \text{for even } (m+j) \\ n + m + 1 & \text{for odd } (m+j) \end{cases}
\end{aligned}$$

Thus, $\chi_{la}(Kt_{n,m}) \leq 3$. To show the lower bound, we can use the local antimagic chromatic number of cycle C_n due to Arumugam *et al.* [1]. Since for $n \geq 3$, $\chi_{la}(C_n) = 3$ and the kite $Kt_{n,m}$ contains a cycle C_n, it easy to see that $\chi_{la}(Kt_{n,m}) \geq 3$. Therefore $\chi_{la}(Kt_{n,m}) = 3$. □

Figure 1 shows an example of the local antimagic vertex coloring of the kite $Kt_{5,6}$ with the local antimagic chromatic number equals to 3.

We note that a n-pan graph, denoted by Pg_n, is the graph obtained by joining a cycle graph C_n to a singleton graph K_1 with a bridge. In other words, the n-pan graph is a special case of the kite graph $Kt_{n,m}$ when $m = 1$. Consequently,

Corollary 2.1. *For the n-pan graph Pg_n with $n \geq 3$, $\chi_{la}(Pg_n) = 3$.* □

In the next theorem, we present the local antimagic chromatic number of another unicyclic graph, that is the cycle with two neighbour pendants, as follows.

Theorem 2.2. *For the cycle with two neighbour pendants Cp_n with $n \geq 3$, $\chi_{la}(Cp_n) = 4$.*

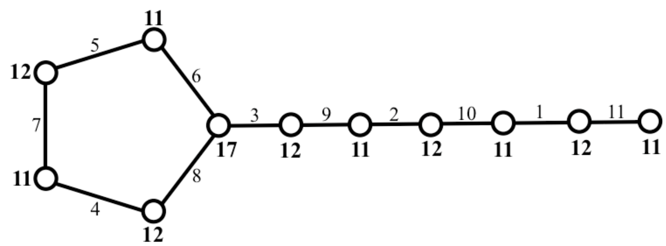

Figure 1. The local antimagic vertex coloring of $Kt_{5,6}$ with $\chi_{la}(Kt_{5,6}) = 3$

Proof. Let Cp_n be the cycle with two neighbour pendants Cp_n with $n \geq 3$. The vertex set of Cp_n is $V = \{u_i | 1 \leq i \leq n\} \cup \{v_1, v_2\}$ and the edge set is $E = \{u_i u_{i+1} | 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{u_1 v_1, u_2 v_2\}$.

Label the edges of Cp_n using a bijection $f : E \to \{1, 2, \ldots, n+2\}$ below.

$$
\begin{aligned}
f(u_i u_{i+1}) &= \begin{cases} \frac{i+1}{2} & \text{for odd } i \\ n+1-\frac{i}{2} & \text{for even } i \end{cases} \\
f(u_n u_1) &= \lceil \frac{n+1}{2} \rceil \\
f(u_i v_i) &= n+i \qquad\qquad \text{for } i = 1, 2
\end{aligned}
$$

It is easy to see that f is a local antimagic labeling and the weight of vertices are

$$
\begin{aligned}
w(u_i) &= \begin{cases} \lfloor \frac{3n+6}{2} & \text{for } i = 1 \\ 2n+3 & \text{for } i = 2 \\ n+2 & \text{for odd } i \geq 3 \\ n+1 & \text{for even } i \geq 4 \end{cases} \\
w(v_i) &= n+i \qquad\qquad \text{for } i = 1, 2
\end{aligned}
$$

Thus $\chi_{la}(Cp_n) \leq 4$. To show the lower bound, we suppose that $f(u_1 v_1) = m_1$ and $f(u_2 v_2) = m_2$. Then $w(v_1) = m_1$, $w(v_2) = m_2$, $w(u_1) > m_1$ and $w(u_2) > m_2$. Clearly, $w(v_1) \neq w(v_2)$. Since u_1 is neighbour of u_2, then $w(u_1) \neq w(u_2)$. This implies that $\chi_{la}(Cp_n) \geq 4$. We conclude that $\chi_{la}(Cp_n) = 4$. □

Figure 2 shows an example of the local antimagic vertex coloring of the cycle with two neighbour pendants Cp_6 with the local antimagic chromatic number equals to 4.

3. Conclusion

Another family of unicyclic graph is a sun. A sun, denoted by Su_n, is a cycle on n vertices C_n with an edge terminating in a vertex of degree 1 attached to each vertex [2]. The local antimagic chromatic number of the sun Su_n has not been discovered. Consequently, we have the following open problems.

Problem 1. *Determine the local antimagic chromatic number of sun.*

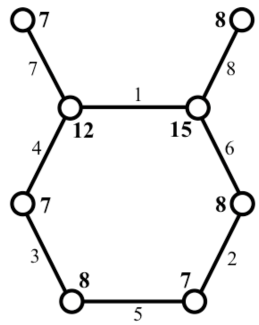

Figure 2. The local antimagic vertex coloring of Cp_6 with $\chi_{la}(Cp_6) = 4$

In general,

Problem 2. *Determine the local antimagic chromatic number of some families of graph.*

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