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The oriented chromatic number of edge-amalgamation of cycle graph

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Abstract

An oriented $k-$coloring of an oriented graph $\overrightarrow{G}$ is a partition of $V(\overrightarrow{G})$ into k color classes such that no two adjacent vertices belong to the same color class, and all the arcs linking the two color classes have the same direction. The oriented chromatic number of an oriented graph $\overrightarrow{G}$ is the minimum order of an oriented graph $\overrightarrow{H}$ to which $\overrightarrow{G}$ admits a homomorphism to $\overrightarrow{H}$. The oriented chromatic number of an undirected graph G is the maximum oriented chromatic number of all possible orientations of the graph G. In this paper, we show that every edge amalgamation of cycle graphs, which also known as a book graph, has oriented chromatic number less than or equal to six.

Keywords: edge amalgamation of cycle, homomorphism, oriented chromatic number
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1. Introduction

Let $\{G_i, x_iy_i\}$ be a collection of graphs that is every graph G_i has a fixed x_iy_i edge. The edge amalgamation $\{G_i, x_iy_i\}$ is a graph obtain by taking all the $G_i's$ and identifying their terminal edges. We denote the edge amalgamation of cycle graph with $edgeamal\{C_k\}_{i=1}^{n}$. An oriented graph $\overrightarrow{G}$ is a digraph withour loop and no opposite arcs. An oriented $k-$coloring of an oriented

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graph $\overrightarrow{G}$ is a partition of $V(\overrightarrow{G})$ into k color classes such that no two adjacent vertices belong to the same color class, and all the arcs linking two color classes have the same direction.

Let $\overrightarrow{G}$ and $\overrightarrow{H}$ be two oriented graphs, a homomorphism of $\overrightarrow{G}$ to $\overrightarrow{H}$ is a mapping $f : V(\overrightarrow{G}) \to V(\overrightarrow{H})$ that preserve the arcs: if $\overrightarrow{xy} \in E(\overrightarrow{G})$ then $\overrightarrow{f(x)f(y)} \in E(\overrightarrow{H})$. The oriented chromatic number of an oriented graph $\overrightarrow{G}$ is the minimum order of an oriented graph $\overrightarrow{H}$ to which $\overrightarrow{G}$ admits a homomorphism to $\overrightarrow{H}$. The oriented chromatic number of an undirected graph G, $\chi_0(G)$, is the maximum oriented chromatic number of all possible orientations of the graph G. Sopena in [3], [4], and [5] has worked on homomorphism and colorings of oriented graph, and find the chromatic number of oriented graph. We can find other examples in [1] and [2]. The main purpose of this paper is to show that every edge amalgamation of cycle graphs, which also known as a book graph, has oriented chromatic number less than or equal to 6.

2. Main Result

To proof our theorem , we need the following lemma.

Lemma 2.1. *If P is a path, then $\chi_0(P) \leq 3$.*

Proof. Let $P = v_0v_1v_2...v_{n-1}$ and $\overrightarrow{P}$ be any orientation of P. We show that $\overrightarrow{P}$ admits a homomorphism to $\overrightarrow{C_3}$ with arcs are 01, 12, and 20. Define a mapping $f(v_i) : \overrightarrow{P} \to \overrightarrow{C_3}$ as follows:

$$f(v_i) \quad = \begin{cases} 0 & \text{if } i = 0 \\ (f(v_{i-1}) + 1) \pmod 3 & \text{if } v_{i-1}v_i \text{ is an arc} \\ (f(v_{i-1}) - 1) \pmod 3 & \text{if } v_iv_{i-1} \text{ is an arc} \end{cases}$$

Notice that if v_iv_j is an arc in $\overrightarrow{P}$, then $f(v_i)f(v_j) = 01$ or 12 or 20 is an arc in $\overrightarrow{C_3}$. Thus we can conclude that f is a homomorphism from $\overrightarrow{P}$ to $\overrightarrow{C_3}$. Since $\overrightarrow{P}$ admits a homomorphism to an oriented graph of order three, then $\overrightarrow{P}$ can be colored with three colors, so $\chi_0(\overrightarrow{P}) \leq 3$. Since for any orientation of $\overrightarrow{P}$ satisfied $\chi_0(\overrightarrow{P}) \leq 3$, it is implies that $\chi_0(P) \leq 3$. □

Theorem 2.1. *If* $k \in \mathbb{N}$ *and* $k \geq 3$*, then* $\chi_0(edgeamal\{C_k\}_{i=1}^n) \leq 6$*, and this bound is tight.*

Proof. (1) If $k = 3$.

Let $\overrightarrow{G}$ be any orientation of $edgeamal\{C_3\}_{i=1}^n$. Graph $\overrightarrow{G}$ is shown in Figure 1.

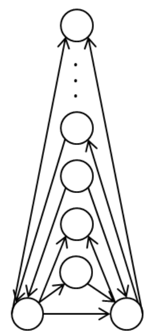


Figure 1: $\overrightarrow{G} = edgeamal\{C_3\}_{i=1}^n$

In this case, the maximum number of different orientation of cycles in $\overrightarrow{G}$ is four, since we can use the previous orientations to the next cycle.

Since the maximum number of different orientation of cycles in $\overrightarrow{G}$ is four, it is implies that $\chi_0(edgeamal\{C_3\}_{i=1}^n) \leq 6$.

(2) If $k \geq 4$.

Let G be $edgeamal\{C_k\}_{i=1}^n$. Graph G is shown in Figure 2.

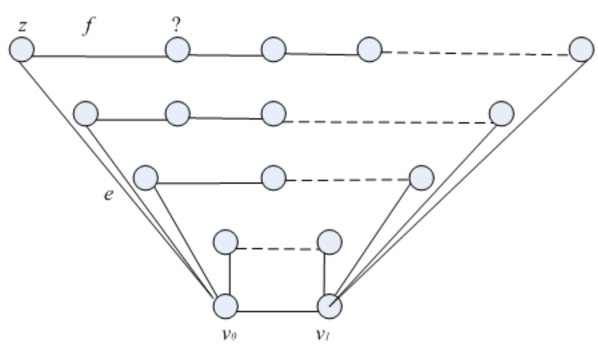


Figure 2: $G = edgeamal\{C_k\}_{i=1}^n$

We show that G can be colored with vertices of this following graph $\overrightarrow{H}$.

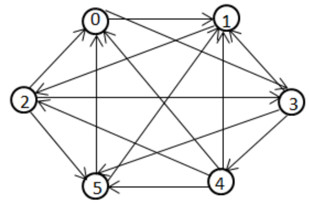
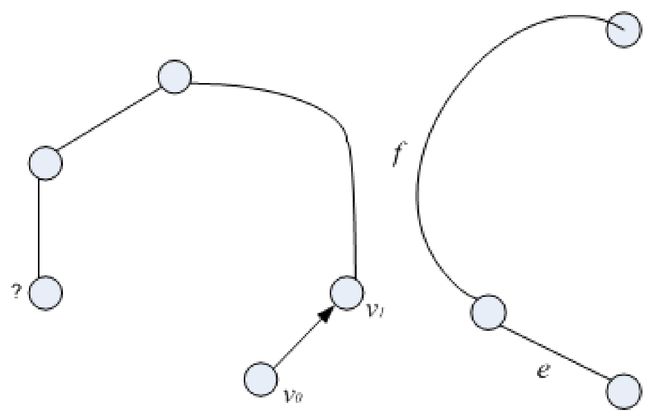


Figure 3: Graph $\overrightarrow{H}$

Look at the outer cycle in graph $\overrightarrow{G}$, if we eliminate edge $e = v_0z$, edge $f = zw$, where w is a vertex that adjacent to vertex z and vertex z, it will form path $\overrightarrow{P}$ as you can see in Figure $4(a)$. Note that vertex w has symbol ? in Figure 4 and Figure 5, since we still do not know the color. According to Lemma 2.1, path $\overrightarrow{P}$ can be colored with three colors that are 0, 1, and 2, thus for vertex with label ”?” there are three possible colors, that is 0, or 1, or 2. Now look at path $\overrightarrow{L}$ in Figure $4(b)$ which consist of vertex with label ”?”, edge f, vertex z, edge e, and vertex v_0.



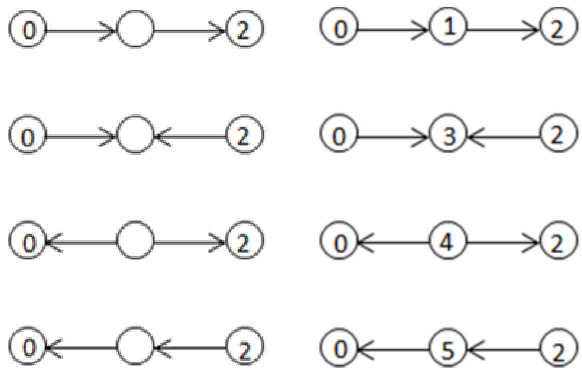
(a) Path $\overrightarrow{P}$ (b) Path $\overrightarrow{L}$

Figure 4

Since vertex with label "?" has 3 possible colors which is 0, or 1, or 2, then we split the possibility into three cases.

Case 1: The color in vertex with label "?" is 2.

All possible orientation for $\overrightarrow{L}$ are shown in Figure $5(a)$. According to coloring in graph $\overrightarrow{H}$, that four orientation can be colored as in Figure $5(b)$.

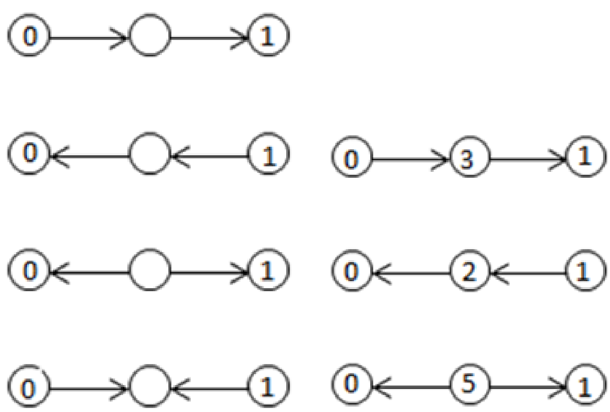


(a) $\overrightarrow{L}$ (b) Color of $\overrightarrow{L}$

Figure 5

Case 2: The color in vertex with label "?" is 1.

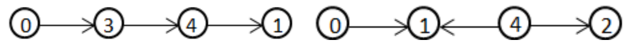
All possible orientation for $\overrightarrow{L}$ is shown in Figure $6(a)$. According to coloring in graph $\overrightarrow{H}$, the first three orientation can be colored as in Figure $6(b)$.



(a) $\overrightarrow{L}$ (b) Colored of $\overrightarrow{L}$

Figure 6: Two possible orientation of $\overrightarrow{L}$

For the fourth orientation of $\overrightarrow{L}$, since there is no vertex in graph $\overrightarrow{H}$ that can be colored the blank vertex, so we change the color 1. This can be done because we colored vertices of $\overrightarrow{L}$ with following the color from Lemma 2.1. There are two possible orientation of $\overrightarrow{L}$ that is shown in Figure $7(a)$ and Figure $7(b)$.

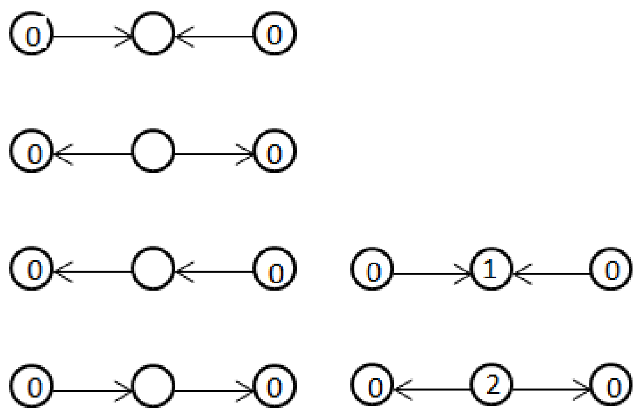


(a) $\overrightarrow{L}$ (b) $\overrightarrow{L}$

Figure 7: Possible orientation and color

Case 3: The color in blank vertex is 0.

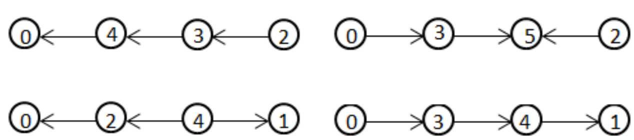
All possible orientation for $\overrightarrow{L}$ is shown in Figure $8(a)$. According to coloring in graph $\overrightarrow{H}$, the first two orientation can be colored as in Figure $8(b)$.



(a) $\overrightarrow{L}$ (b) $\overrightarrow{L}$

Figure 8: Two possible orientations and colors

For the third and fourth orientation of $\overrightarrow{L}$, since there is no vertex in graph $\overrightarrow{H}$ that can be colored the blank vertex, so we change the color 0. This can be done because we colored vertices of $\overrightarrow{L}$ with following the color from Lemma 2.1. For each orientations, the graph have two possible orientations. The result is shown in Figure 9.



(a) $\overrightarrow{L}$ (b) $\overrightarrow{L}$

Figure 9: Two possible orientations and colors

Thus all possible orientation of path $\overrightarrow{L}$ can be colored with vertices of $\overrightarrow{H}$, it is implies that graph $\overrightarrow{G}$ can be colored with vertices of $\overrightarrow{H}$. Since the order of $\overrightarrow{H}$ is six, then it implies that $\chi_0(\overrightarrow{G}) \leq 6$. Therefore we can conclude that $\chi_0(edgeamal\{C_k\}_{i=1}^n) \leq 6$ since for any orientation of cycle in $\overrightarrow{G}$ we have $\chi_0(\overrightarrow{G}) \leq 6$. □

The following graph $edgeamal\{C_3\}_{i=1}^4$ is an example where $\chi_0(edgeamal\{C_k\}_{i=1}^n) = 6$.

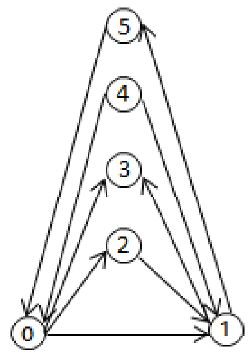


Figure 10: Colored $Edgeamal\{C_3\}_{i=1}^4$

□

3. Conclusion

The largest possible oriented chromatic number of edge-amalgamation of cycle graphs is six and this bound is tight. For the future research, we can try to find the oriented chromatic number of other family of graphs.

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