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Z_{2nm}-supermagic labeling of $C_n\square C_m$

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Abstract

A Γ-supermagic labeling of a graph $G = (V, E)$ with $|E| = k$ is a bijection from E to an Abelian group Γ of order k such that the sum of labels of all incident edges of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$. We present a Z_{2nm}-supermagic labeling of Cartesian product of two cycles, $C_n\square C_m$ for n odd. This along with an earlier result by Ivančo proves that a Z_{2nm}-supermagic labeling of $C_n\square C_m$ exists for every $n, m \geq 3$.

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1. Introduction

The Cartesian product of cycles $C_{n_1}, C_{n_2}, \cdots, C_{n_s}$, denoted $C_{n_1}\square C_{n_2}\square\cdots\square C_{n_s}$ can be viewed as the Cayley graph of Abelian group $Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_s}$ generated by group elements $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$. It is an intriguing question whether we can label the elements (that is, edges, vertices, or both) of such a graph with elements of the group (or another Abelian group of an appropriate order) so that the sum of the labels of the elements incident or adjacent to every edge or vertex is the same group element μ, called a *magic constant*. We provide exact definitions of the above notions in Section **??**.

It seems natural to label just edges and sum the edge labels incident with each vertex. This can be viewed as a setup of a network in which we can choose any vertex as the source (or sink) to obtain the same flow.

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Another natural approach is to only label vertices and look at the sum of labels of vertices adjacent to each vertex. The latter notion has been studied in several papers already. Froncek [3] studied Z_{nm}-labeling of $C_n \square C_m$, and Cichacz [1] proved more results for some other Abelian groups. Cichacz and Froncek [2] investigated circulant graphs in this context.

This paper is the first attempt to study the edge labeling version. We present a method for a Z_{2nm}-supermagic labeling of Cartesian product of two cycles, $C_n \square C_m$ for n odd, and prove that such labeling exists for every magic constant of type $\mu = 4\nu$, where ν is any element on Z_{2nm}.

Preliminaries

First we define the Cartesian product of graphs. Because we focus on products of two cycles in this paper, we restrict our definition to that case. A more general definition for s graphs can be obtained recursively.

Definition 1.1. The *Cartesian product* $G = G_1 \square G_2$ of graphs G_1 and G_2 with disjoint vertex and edge sets V_1, V_2, and E_1, E_2 respectively, is the graph with vertex set $V = V_1 \times V_2$ where any two vertices $u = (u_1, u_2), v = (v_1, v_2) \in G$ are adjacent in G if and only if either $u_1 = v_1$ and $u_2v_2 \in E_2$ or $u_2 = v_2$ and $u_1v_1 \in E_1$.

In particular, the Cartesian product $C_n \square C_m$ of two cycles C_n and C_m is a 4-regular graph, consisting of m copies of C_n (which we will call *horizontal*) and n copies of C_m (which we call *vertical*). The example of $C_3 \square C_4$ in Figure 1 illustrates the terminology. One should notice that the product is commutative, that is, $C_n \square C_m \equiv C_n \square C_m$.

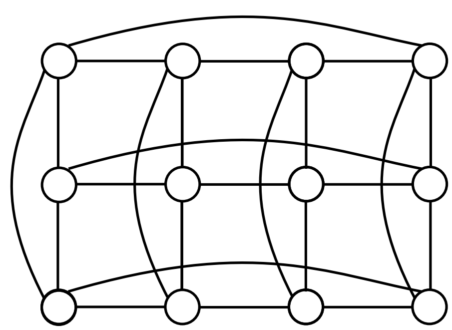

Figure 1. The Cartesian product $C_3 \square C_4$

Now we define the labelings we are investigating.

Definition 1.2. A Γ-*supermagic labeling* of a graph $G = (V, E)$ with $|E| = k$ is a bijection f from E to an Abelian group Γ of order k such that the sum of labels of all incident edges of every vertex $x \in V$, called the *weight* of x and denoted $w(x)$, is equal to the same element $\mu \in \Gamma$, called the *magic constant*. That is,

$$w(x) = \sum_{y:xy \in E} f(xy) = \mu$$

for every vertex $x \in V$.

In fact, our definition is a generalization of a previously studied notion of *supermagic labeling*, where the labels are just consecutive positive integers. This type of labeling is also often called *vertex-magic edge labeling*.

Definition 1.3. A *supermagic labeling* of a graph $G = (V, E)$ with $|E| = k$ is a bijection g from E to the set $\{1, 2, \ldots, k\}$ such that the sum of labels of all incident edges of every vertex $x \in V$, called the *weight* of x and denoted $w(x)$, is equal to the same integer c, called the *magic constant*. That is,

$$w(x) = \sum_{y:xy \in E} g(xy) = c$$

for every vertex $x \in V$.

As we also briefly discuss analogous results obtained earlier for distance magic labeling, we define it here as well.

Based on the notion of distance magic graphs, Froncek [3] introduced the concept of Γ-distance magic labeling.

Definition 1.4. A Γ*-distance magic labeling* of a graph $G = (V, E)$ with $|V| = p$ is a bijection f from V to and Abelian group Γ of order p such that the sum of labels of all adjacent vertices of every vertex $x \in V$, called the *weight* of x and denoted $w(x)$, is equal to the same element $\mu \in \Gamma$, called the *magic constant*. That is,

$$w(x) = \sum_{y:xy \in E} f(y) = \mu$$

for every vertex $x \in V$.

As in the case of edge labelings, even here results on labeling with positive integers preceded those on labeling with group elements.

Definition 1.5. A *distance magic labeling* of a graph $G = (V, E)$ with $|V| = p$ is a bijection g from V to the set $\{1, 2, \ldots, p\}$ such that the sum of labels of all adjacent vertices of every vertex $x \in V$, called the *weight* of x and denoted $w(x)$, is equal to the same integer c, called the *magic constant*. That is,

$$w(x) = \sum_{y:xy \in E} g(y) = c$$

for every vertex $x \in V$.

We first list results on distance magic and Γ-distance magic labelings of Cartesian cycle products, as they have been a motivation of our research.

Rao, Singh, and Parameswaran in [5] proved the following.

Theorem 1.6 (Rao, Singh, Parameswaran 2004)**.** *The graph $C_n \Box C_m$ has a distance magic labeling if and only if $n = m \geq 6$ and $n, m \equiv 2 \pmod 4$.*

Froncek [3] proved a complete result on Γ-distance magic labeling of Cartesian product of two cycles with cyclic groups.

Theorem 1.7 (Froncek 2013)**.** *The Cartesian product $C_n \square C_m$ has a Z_{nm}-distance magic labeling if and only if $n, m \geq 3$ and nm is even.*

Cichacz and Froncek [2] proved the following.

Theorem 1.8 (Cichacz and Froncek 2016)**.** *Let G be an r-regular graph on n vertices, where r is odd. Then there does not exist an Abelian group Γ of order n having exactly one involution (an element that is its own inverse) admitting a Γ-distance magic labeling of G.*

Cichacz [1] proved a more general result for other Abelian groups.

Theorem 1.9 (Cichacz 2013)**.** *Let n, m, t, s be positive integers, $n, m \geq 3$ and $l = \mathrm{lcm}(n, m)$. Let $\Gamma = Z_{lt} \times A$, where A is an Abelian group of order s and $nm = lts$. Then the Cartesian product $C_n \square C_m$ has a Γ-distance magic labeling.*

Results analogous to Theorem 1.6 for supermagic edge labeling were proved by Ivančo [4].

Theorem 1.10 (Ivančo 2000)**.** *$C_n \square C_n$ has a supermagic labeling for any $n \geq 3$.*

Theorem 1.11 (Ivančo 2000)**.** *Let $n, m \geq 2$ be integers. Then $C_{2n} \square C_{2m}$ has a supermagic labeling.*

Ivančo also conjectured that the Cartesian product $C_n \square C_m$ allows a supermagic labeling for any $n, m \geq 3$.

In the following sections, we prove that the Z_{2nm}-supermagic labeling equivalent of the conjecture is true.

2. Results

First we present a construction for a product of two odd cycles. We denote the horizontal m-cycles by $B^0, B^1, \ldots, B^{n-1}$ and the vertical n-cycles by $C^0, C^1, \ldots, C^{m-1}$. Then a vertex x_{ij} belongs to B^i and C^j.

We start by labeling the edges of B^0 (going from left to right and skipping every other edge) by consecutive even elements, $0, 2, 4, \ldots, 2m-2$. Because m is odd, all edges receive labels. Then we continue labeling B^1 with the next m even elements, $2m, 2m+2, \ldots, 4m-2$. Again, as the number of cycles is even, we label edges of all n horizontal m-cycles while using all even elements of Z_{2nm}. In general, edges in B^{2s} are labeled $2s, 2s+2, 2s+4, \ldots, 2s+2m-2$, where the superscript is taken mod n.

The layout of the labels is shown in Figure 2.

	$x_{i0}x_{i1}$	$x_{i1}x_{i2}$	$x_{i2}x_{i3}$	$x_{i3}x_{i4}$	$\cdots$	$x_{i(m-1)}x_{i0}$
B^0	0	$m+1$	2	$m+3$	$\cdots$	$m-1$
B^1	$2m$	$3m+1$	$2m+2$	$3m+3$	$\cdots$	$3m-1$
B^2	$4m$	$5m+1$	$4m+2$	$5m+3$	$\cdots$	$5m-1$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
B^{n-2}	$-4m$	$-3m+1$	$-4m+2$	$-3m+3$	$\cdots$	$-3m-1$
B^{n-1}	$-2m$	$-m+1$	$-2m+2$	$-m+3$	$\cdots$	$-m-1$

Figure 2. Labeled horizontal cycles, n, m odd

We will call the sum of labels of the horizontal edges incident with a vertex x_{ij} the *horizontal partial weight of* x_{ij} and denote it by $w_h(x_{ij})$. Similarly, the sum of labels of the vertical edges incident with x_{ij} will be called the *vertical partial weight of* x_{ij} and denoted by $w_v(x_{ij})$. More precisely,

$$w_h(x_{ij}) = f(x_{i(j-1)}x_{ij}) + f(x_{ij}x_{i(j+1)})$$

and

$$w_v(x_{ij}) = f(x_{(i-1)j}x_{ij}) + f(x_{ij}x_{(i+1)j}).$$

The partial weights $w_h(x_{ij})$ of vertices are listed in Figure 3. Notice that the partial weights in each column form a coset of Z_{2nm} induced by the subgroup $\langle 4m \rangle$. Therefore, we need to label the vertical cycles so that each cycle has partial weights w_v forming an appropriate coset as well. We will call the cosets of type $\langle 4m \rangle + 2t$ *even* and those of type $\langle 2m \rangle + 2t + 1$ *odd*.

Namely, for a column with partial weights in a coset $\langle 4m \rangle + 2t$, we want the vertical cycle with partial weights w_v forming the coset $\langle 4m \rangle - 2t$, listed in the opposite order. Then we have $w_h(x_{ij}) = 4mi + 2t$ and $w_v(x_{ij}) = 4m(n-i) - 2t$ which yields $w(x_{ij}) = 4mi + 2t + 4nm - 4mi - 2t = 4nm = 0$.

	x_{i0}	x_{i1}	x_{i2}	x_{i3}	$\cdots$	$x_{i(m-1)}$
B^0	$m-1$	$m+1$	$m+3$	$m+5$	$\cdots$	$3m-3$
B^1	$5m-1$	$5m+1$	$5m+3$	$5m+5$	$\cdots$	$7m-3$
B^2	$9m-1$	$9m+1$	$9m+3$	$9m+5$	$\cdots$	$11m-3$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
B^{n-2}	$-7m-1$	$-7m+1$	$-7m+3$	$-7m+5$	$\cdots$	$-5m-3$
B^{n-1}	$-3m-1$	$-3m+1$	$-3m+3$	$-3m+5$	$\cdots$	$-m-3$

Figure 3. Partial weights in horizontal cycles, n, m odd

We achieve this goal by labeling edges of each C^j consecutively with elements of a coset induced by the subgroup $\langle 4m \rangle$. However, because n, m are both odd, we have $\langle 4m \rangle = \langle 2m \rangle$. We also set $m' = (m-1)/2$ to simplify notation in the labeling presented below in Figure 4.

There are two cases, depending on whether m' is odd or even. They both have the same partial weights, because they only differ by the placement of element nm in their labels. The labels in each column (that is, a coset) either all contain that element, or none do. Moreover, for any two consecutive columns, exactly one of them has nm added in each term. Thus, the sum of two neighboring edges making up the partial weight in every other cycle contains $2nm$, which in Z_{2nm} is indeed equal to zero. We need to do that to ensure that the labels form *odd* cosets, as we have used all even ones for the horizontal cycles.

The partial weights for vertical cycles are presented in Figure 5. Adding the partial weights in Figures 3 and 5, we can see that the total weights all equal zero and thus the labeling is supermagic. We provide a more rigorous proof below.

Theorem 2.1. *For n, m both odd, $C_n \square C_m$ can be labeled with group elements from Z_{2nm} to form a Z_{2nm}-supermagic labeling with magic constant 0.*

Proof. Let again $m = 2m' + 1$. We define the horizontal labels as

$$f(x_{ij}x_{i(j+1)}) = \begin{cases} 2mi + j & \text{for } j \text{ even} \\ 2mi + m + j & \text{for } j \text{ odd.} \end{cases}$$

Therefore, for j even we get

$$\begin{aligned} w_h(x_{ij}) &= f(x_{i(j-1)}x_{ij}) + f(x_{ij}x_{i(j+1)}) \\ &= (2mi + m + j - 1) + (2mi + j) \\ &= (4i + 1)m + 2j - 1 \end{aligned}$$

and for j odd,

$$\begin{aligned} w_h(x_{ij}) &= f(x_{i(j-1)}x_{ij}) + f(x_{ij}x_{i(j+1)}) \\ &= (2mi + j - 1) + (2mi + m + j) \\ &= (4i + 1)m + 2j - 1. \end{aligned}$$

So, in both cases the partial weight is

$$w_h(x_{ij}) = (4i + 1)m + 2j - 1. \tag{1}$$

For the vertical labels, we need to distinguish two cases. When m' is even, we define

$$f(x_{ij}x_{(i+1)j}) = \begin{cases} -m(2i + 1) - m' - j & \text{for } j \text{ even} \\ -m(2i + 1) - nm - m' - j & \text{for } j \text{ odd.} \end{cases}$$

m' **even**	C_0	C_1	C_2	$\cdots$	C_{m-1}
$x_{0j}x_{1j}$	$-m-m'$	$-m-nm-m'-1$	$-m-m'-2$	$\cdots$	$-m-m'-(m-1)$
$x_{1j}x_{2j}$	$-3m-m'$	$-3m-nm-m'-1$	$-3m-m'-2$	$\cdots$	$-3m-m'-(m-1)$
$x_{2j}x_{3j}$	$-5m-m'$	$-5m-nm-m'-1$	$-5m-m'-2$	$\cdots$	$-5m-m'-(m-1)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$x_{(n-2)j}x_{(n-1)j}$	$3m-m'$	$3m-nm-m'-1$	$3m-m'-2$	$\cdots$	$3m-m'-(m-1)$
$x_{(n-1)j}x_{0j}$	$m-m'$	$m-nm-m'-1$	$m-m'-2$	$\cdots$	$m-m'-(m-1)$

m' **odd**	C_0	C_1	C_2	$\cdots$	C_{m-1}
$x_{0j}x_{1j}$	$-m-nm-m'$	$-m-m'-1$	$-m-nm-m'-2$	$\cdots$	$-m-nm-m'-(m-1)$
$x_{1j}x_{2j}$	$-3m-nm-m'$	$-3m-m'-1$	$-3m-nm-m'-2$	$\cdots$	$-3m-nm-m'-(m-1)$
$x_{2j}x_{3j}$	$-5m-nm-m'$	$-5m-m'-1$	$-5m-nm-m'-2$	$\cdots$	$-5m-nm-m'-(m-1)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$x_{(n-2)j}x_{(n-1)j}$	$3m-nm-m'$	$3m-m'-1$	$3m-nm-m'-2$	$\cdots$	$3m-nm-m'-(m-1)$
$x_{(n-1)j}x_{0j}$	$m-nm-m'$	$m-m'-1$	$m-nm-m'-2$	$\cdots$	$m-nm-m'-(m-1)$

Figure 4. Labeled vertical cycles, n, m odd

$$\begin{array}{lrrrrcr}
 & C_0 & C_1 & C_2 & C_3 & \cdots & C_{m-1} \\
 & & & & & & \\
x_{0j} & -m+1 & -m-1 & -m-3 & -m-5 & \cdots & -3m+3 \\
x_{1j} & -5m+1 & -5m-1 & -5m-3 & -5m-5 & \cdots & -7m+3 \\
x_{2j} & -9m+1 & -9m-1 & -9m-3 & -9m-5 & \cdots & -11m+3 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
x_{(n-2)j} & 7m+1 & 7m-1 & 7m-3 & 7m-5 & \cdots & 5m+3 \\
x_{(n-1)j} & 3m+1 & 3m-1 & 3m-3 & 3m-5 & \cdots & m+3
\end{array}$$

Figure 5. Partial weights in vertical cycles, n, m odd

Then for j even we have

$$\begin{aligned}
w_v(x_{ij}) &= f(x_{(i-1)j}x_{ij}) + f(x_{ij}x_{(i+1)j}) \\
&= (-m(2i-1) - m' - j) + (-m(2i+1) - m' - j) \\
&= -4mi - 2m' - 2j
\end{aligned}$$

and for j odd we have

$$\begin{aligned}
w_v(x_{ij}) &= f(x_{(i-1)j}x_{ij}) + f(x_{ij}x_{(i+1)j}) \\
&= (-m(2i-1) - nm - m' - j) + (-m(2i+1) - nm - m' - j) \\
&= -4mi - 4nm - 2m' - 2j \\
&= -4mi - 2m' - 2j.
\end{aligned}$$

Now substituting back $m = 2m' + 1$, we obtain

$$w_v(x_{ij}) = -4mi - 2m' - 2j = -4mi - (m-1) - 2j = -(4i+1)m - 2j + 1. \qquad (2)$$

It now follows from (1) and (2) that

$$w(x_{ij}) = w_h(x_{ij}) + w_v(x_{ij}) = ((4i+1)m + 2j - 1) + (-(4i+1)m - 2j + 1) = 0,$$

as desired.

When m' is odd, we define

$$f(x_{ij}x_{(i+1)j}) = \begin{cases} -m(2i+1) - nm - m' - j & \text{for } j \text{ even} \\ -m(2i+1) - m' - j & \text{for } j \text{ odd.} \end{cases}$$

Then for j even we have

$$\begin{aligned} w_v(x_{ij}) &= f(x_{(i-1)j}x_{ij}) + f(x_{ij}x_{(i+1)j}) \\ &= (-m(2i-1) - nm - m' - j) + (-m(2i+1) - nm - m' - j) \\ &= -4mi - 4nm - 2m' - 2j \\ &= -4mi - 2m' - 2j \end{aligned}$$

and for j odd we have

$$\begin{aligned} w_v(x_{ij}) &= f(x_{(i-1)j}x_{ij}) + f(x_{ij}x_{(i+1)j}) \\ &= (-m(2i-1) - m' - j) + (-m(2i+1) - m' - j) \\ &= -4mi - 2m' - 2j \end{aligned}$$

as well. We observe that the vertical partial weight is the same as for m' even, so we again obtain

$$w_v(x_{ij}) = -(4i+1)m - 2j + 1. \tag{3}$$

This is the same temporary weight as for m' even in (2), so adding (1) and (3) we have again

$$w(x_{ij}) = w_h(x_{ij}) + w_v(x_{ij}) = 0,$$

which completes the proof. □

The construction for n odd and m even is similar. Using the same notation, we set $n = 2n'+1, m = 2m'$ and label edges of B_0 consecutively with elements $0, 2, \ldots, 2m-2$. Then we continue with B_1 starting at $2m$ and so on, utilizing all even elements of Z_{2nm}. The labeling is presented in Figure 6.

The partial weights in this case are not all different, as they only use elements from the cosets $\langle 2m \rangle + 4t + 2$. In particular, coset $\langle 2m \rangle + 4t + 2$ appears in columns t and $m' + t$. However, while the cosets are in the same position for $t = 0$, for other values of t the values in column $m' + t$ are cyclically shifted up by $n' + 1$ positions (or by n' down, which is indeed the same) compared with column t. The partial weights are also shown in Figure 6.

For vertical cycles, we again use all odd cosets, namely for C_j the coset $\langle 2m \rangle - (2j-1)$. Looking at Figure 7 the labels may be a bit confusing, starting with $C_{m'}$. This is because we made some simplifications in the formula for $f(x_{ij}x_{i(j+1)})$. For instance, since $m = 2m'$, we have

$$f(x_{0m'}x_{1m'}) = (n-1)m - (2(m'+1)-1) = (n-1)m - (m-1) = (n-2)m + 1.$$

A detailed proof follows.

Theorem 2.2. *For n odd and m even, $C_n \square C_m$ can be labeled with group elements from Z_{2nm} to form a Z_{2nm}-supermagic labeling with magic constant 0.*

	$x_{i0}x_{i1}$	$x_{i1}x_{i2}$	$x_{i2}x_{i3}$	$\cdots$	$x_{im'}x_{i(m'+1)}$	$x_{i(m'+1)}x_{i(m'+2)}$	$\cdots$	$x_{i(m-1)}x_{i0}$
B^0	0	2	4	$\cdots$	m	$m+2$	$\cdots$	$2m-2$
B^1	$2m$	$2m+2$	$2m+4$	$\cdots$	$3m$	$3m+2$	$\cdots$	$4m-2$
B^2	$4m$	$4m+2$	$4m+4$	$\cdots$	$5m$	$5m+2$	$\cdots$	$6m-2$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$B^{n'}$	$(n-1)m$	$(n-1)m+2$	$(n-1)m+4$	$\cdots$	nm	$nm+2$	$\cdots$	$(n+1)m-2$
$B^{n'+1}$	$(n+1)m$	$(n+1)m+2$	$(n+1)m+4$	$\cdots$	$(n+2)m$	$(n+2)m+2$	$\cdots$	$(n+3)m-2$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
B^{n-2}	$-4m$	$-4m+2$	$-4m+4$	$\cdots$	$-3m$	$-3m+2$	$\cdots$	$-2m-2$
B^{n-1}	$-2m$	$-2m+2$	$-2m+4$	$\cdots$	$-m$	$-m+2$	$\cdots$	-2

	x_{i0}	x_{i1}	x_{i2}	$\cdots$	$x_{im'}$	$x_{i(m'+1)}$	$\cdots$	$x_{i(m-1)}$
B^0	$2m-2$	2	6	$\cdots$	$2m-2$	$2m+2$	$\cdots$	$4m-6$
B^1	$6m-2$	$4m+2$	$4m+6$	$\cdots$	$6m-2$	$6m+2$	$\cdots$	$8m-6$
B^2	$10m-2$	$8m+2$	$8m+6$	$\cdots$	$10m-2$	$10m+2$	$\cdots$	$12m-6$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$B^{n'}$	-2	$-2m+2$	$-2m+6$	$\cdots$	-2	2	$\cdots$	$2m-6$
$B^{n'+1}$	$4m-2$	$2m+2$	$2m+6$	$\cdots$	$4m-2$	$4m+2$	$\cdots$	$6m-6$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
B^{n-2}	$-6m-2$	$-8m+2$	$-8m+6$	$\cdots$	$-6m-2$	$-6m+2$	$\cdots$	$-4m-6$
B^{n-1}	$-2m-2$	$-4m+2$	$-4m+6$	$\cdots$	$-2m-2$	$-2m+2$	$\cdots$	-6

Figure 6. Labeled horizontal cycles and partial weights, n odd, m even

	C_0	C_1	C_2	$\cdots$	$C_{m'}$	$C_{m'+1}$	$\cdots$	C_{m-1}
$x_{0j}x_{1j}$	$-2m+1$	$(n-1)m-1$	$(n-1)m-3$	$\cdots$	$(n-2)m+1$	$(n-2)m-1$	$\cdots$	$(n-3)m+3$
$x_{1j}x_{2j}$	$-4m+1$	$(n-3)m-1$	$(n-3)m-3$	$\cdots$	$(n-4)m+1$	$(n-4)m-1$	$\cdots$	$(n-5)m+3$
$x_{2j}x_{3j}$	$-6m+1$	$(n-5)m-1$	$(n-5)m-3$	$\cdots$	$(n-6)m+1$	$(n-6)m-1$	$\cdots$	$(n-9)m+3$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$x_{n'j}x_{(n'+1)j}$	$-(n'+1)m+1$	-1	-3	$\cdots$	$-m+1$	$-m-1$	$\cdots$	$-2m+3$
$x_{(n'+1)j}x_{(n'+2)j}$	$-(n'+3)m+1$	$-2m-1$	$-2m-3$	$\cdots$	$-3m+1$	$-3m-1$	$\cdots$	$-4m+3$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$x_{(n-2)j}x_{(n-1)j}$	$2m+1$	$(n+3)m-1$	$(n+3)m-3$	$\cdots$	$(n+2)m+1$	$(n+2)m-1$	$\cdots$	$(n+1)m+3$
$x_{(n-1)j}x_{0j}$	1	$(n+1)m-1$	$(n+1)m-3$	$\cdots$	$nm+1$	$nm-1$	$\cdots$	$(n-1)m+3$

	C_0	C_1	C_2	$\cdots$	$C_{m'}$	$C_{m'+1}$	$\cdots$	C_{m-1}
x_{0j}	$-2m+2$	-2	-6	$\cdots$	$-2m+2$	$-2m-2$	$\cdots$	$-4m+6$
x_{1j}	$-6m+2$	$-4m-2$	$-4m-6$	$\cdots$	$-6m+2$	$-6m-2$	$\cdots$	$-8m+6$
x_{2j}	$-10m+2$	$-8m-2$	$-8m-6$	$\cdots$	$-10m+2$	$-10m-2$	$\cdots$	$-12m+6$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$x_{n'j}$	2	$2m-2$	$2m-6$	$\cdots$	2	-2	$\cdots$	$-2m+6$
$x_{(n'+1)j}$	$-4m+2$	$-2m-2$	$-2m-6$	$\cdots$	$-4m+2$	$-4m-2$	$\cdots$	$-6m+6$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_{n-2}	$6m+2$	$8m-2$	$8m-6$	$\cdots$	$6m+2$	$6m-2$	$\cdots$	$4m+6$
x_{n-1}	$2m+2$	$4m-2$	$4m-6$	$\cdots$	$2m+2$	$2m-2$	$\cdots$	6

Figure 7. Labeled vertical cycles and partial weights, n odd, m even

Proof. Let $n = 2n' + 1$ and $m = 2m'$. The horizontal labels are defined as

$$f(x_{ij}x_{i(j+1)}) = 2mi + 2j.$$

Hence, for $j \neq 0$ we have the horizontal partial weights

$$\begin{aligned} w_h(x_{ij}) &= f(x_{i(j-1)}x_{ij}) + f(x_{ij}x_{i(j+1)}) \\ &= (2mi + 2(j-1)) + (2mi + 2j) \\ &= 4mi + 4j - 2 \end{aligned} \tag{4}$$

and for $j = 0$ we have

$$\begin{aligned} w_h(x_{i0}) &= f(x_{i(m-1)}x_{i0}) + f(x_{i0}x_{i1}) \\ &= (2mi + 2(m-1)) + 2mi \\ &= 4mi + 2m - 2 \end{aligned} \tag{5}$$

The vertical edges are labeled

$$f(x_{ij}x_{(i+1)j}) = m(n-1) - 2mi - (2j-1) = m(n-2i-1) - 2j + 1$$

for $j \neq 0$ and

$$f(x_{i0}x_{(i+1)0}) = -2m(i+1) + 1$$

otherwise. Therefore, the partial weights in the vertical cycles are

$$\begin{aligned} w_v(x_{ij}) &= f(x_{(i-1)j}x_{ij}) + f(x_{ij}x_{(i+1)j}) \\ &= (m(n-2i+1) - 2j + 1) + (m(n-2i-1) - 2j + 1) \\ &= 2nm - 4mi - 4j + 2 \\ &= -4mi - 4j + 2 \end{aligned} \tag{6}$$

and for $j = 0$ we have

$$\begin{aligned} w_h(x_{i0}) &= f(x_{(i-1)0}x_{i0}) + f(x_{i0}x_{(i+1)0}) \\ &= (-2mi + 1) + (-2m(i+1) + 1) \\ &= -4mi - 2m + 2. \end{aligned} \tag{7}$$

Adding (4) and (6), we get

$$\begin{aligned} w(x_{ij}) &= w_h(x_{ij}) + w_v(x_{ij}) \\ &= 4mi + 4j - 2 + (-4mi - 4j + 2) \\ &= 0 \end{aligned}$$

and adding (5) and (7), we get

$$\begin{aligned} w(x_{i0}) &= w_h(x_{i0}) + w_v(x_{i0}) \\ &= 4mi + 2m - 2 + (-4mi - 4m + 2) \\ &= 0, \end{aligned}$$

which completes the proof. □

Now we are ready to state our main result.

Theorem 2.3. *The Cartesian product $C_n \square C_m$ admits a Z_{2nm}-supermagic labeling for all $n, m \geq 3$.*

Proof. For n odd, the proof follows from Theorems 2.1 and 2.2. For both n, m even it follows from Ivančo's result in Theorem 1.11. Obviously, if the weight of every vertex is the same positive integer c, then by performing addition in Z_{2nm} rather than in $\mathbb{Z}$ we obtain a Z_{2nm}-supermagic labeling with $\mu = c \bmod 2nm$. □

Although both labelings in Theorems 2.1 and 2.2 result in magic constant $\mu = 0$, it is easy to observe that when we have any Z_{2nm}-supermagic labeling with $\mu = 0$, then we can also find labelings for any magic constant $\mu = 4\nu$, where ν is an element of Z_{2nm}. We prove a slightly more general result.

Observation 2.4. When a 4-regular graph G of order p admits a Z_{2p}-supermagic labeling with magic constant μ, then there is a Z_{2p}-supermagic labeling of G with $\mu + 4\nu$ for any ν in Z_{2p}. On the other hand, no such labeling with μ odd can exist.

Proof. We start with a Z_{2p}-supermagic labeling f inducing a magic constant μ, and define f_ν as $f_\nu(xy) = f(xy) + \nu$. This is indeed again a bijection from E to Z_{2p} and $w_\nu(x) = w(x) + 4\nu$ for every x in V.

On the other hand, there is no labeling with an odd magic constant. When $\mu = 2\nu + 1$, then

$$\sum_{x \in V} w_\nu(x) = p(2\nu + 1) = 2p\nu + p = p.$$

Since every edge label contributes to the weights of two vertices, we also have

$$\sum_{x \in V} w_\nu(x) = 2 \sum_{xy \in E} f_\nu(xy) = 2p(2p - 1) = 0,$$

which is a contradiction. □

We combine the previous claims into one as follows.

Theorem 2.5. *The Cartesian product $C_n \square C_m$ admits a Z_{2nm}-supermagic labeling with magic constant μ*

(i) if and only if $\mu \equiv 0 \pmod 2$ when n, m are both odd,

(ii) if and only if $\mu \equiv 2 \pmod 4$ *when* n, m *are both even,*

(iii) for every $\mu \equiv 0 \pmod 4$ *when* n *is odd and* m *is even,*

whenever $n, m \geq 3$*. Moreover, no such labeling with* μ *odd exists for* n *odd and* m *even.*

Proof. It follows from Observation 2.4 that $\mu \equiv 0 \pmod 2$ for all three cases.

For n, m both odd, the result follows from Theorem 2.1 and Observation 2.4 and the fact that in this case, Z_{2nm} is of order $2nm \equiv 2 \pmod 4$ and hence $4\nu \neq 0$ generates the subgroup $\langle 2 \rangle$.

For n, m both even, the labeling with positive integers in Theorem 1.11 gives the magic constant $c = 4nm + 2$. To see that, we observe that

$$\sum_{x \in V} w(x) = 2 \sum_{xy \in E} g(xy),$$

and because $|V| = nm$ and $w(x) = c$ for every x in V, we have

$$nmc = 2 \sum_{t=1}^{2nm} t = 2nm(2nm + 1)$$

and hence

$$c = 2(2nm + 1).$$

Reducing c modulo $2nm$, we get $\mu = 2$. It follows from Observation 2.4 that a desired labeling exists for any $\mu \equiv 2 \pmod 4$. To see that we cannot have $\mu \equiv 0 \pmod 4$, we use the fact that $C_n\square C_m$ for n and m even is bipartite with partite sets V_0 and V_1. Because every edge has one end-vertex in V_0, we have

$$\sum_{x \in V_0} w(x) = |V_0|\mu = \sum_{xy \in E} f(xy),$$

and because $|V_0| = 2n'm'$, we have

$$2n'm'\mu = \sum_{a \in Z_{2nm}} a = nm. \tag{8}$$

However, when $\mu \equiv 0 \pmod 4$, say $\mu = 4\nu$, the left-hand side in (8) is

$$2n'm'\nu = 2n'm'4\nu = 2(2n')(2m')\nu = 2nm\nu = 0, \tag{9}$$

because the multiplication is performed in Z_{2nm}. Comparing (8) and (9), we obtain

$$nm = 0,$$

which is impossible in Z_{2nm}. This contradiction shows that $\mu \not\equiv 0 \pmod 4$.

Finally, for n odd and m even, Z_{2nm} is of order $2nm \equiv 0 \pmod 4$ and hence $4\nu \neq 0$ generates the subgroup $\langle 4 \rangle$. The result then follows from Theorem 2.2 and 2.4. $\square$

3. Conclusion

We currently do not know any labeling with $\mu \equiv 2 \pmod 4$ for case (iii) in Theorem 2.5. Hence, we pose an open problem.

Problem. *Does there exist a Z_{2nm}-supermagic labeling of the Cartesian product $C_n \square C_m$ with magic constant $\mu \equiv 2 \pmod 4$ for n odd and m even?*

There are two other obvious directions in which one could investigate Z_{2nm}-supermagic labelings of Cartesian products of cycles. One is edge Γ-labelings of products of s cycles for $s \geq 3$. Another one is labeling of $C_n \square C_m$ with other Abelian groups of order $2nm$. The ultimate goal is to completely characterize all Abelian groups and cycle lengths such that there exists a Γ-supermagic labeling of $C_{n_1} \square C_{n_2} \square \cdots \square C_{n_s}$ where Γ is of order $sn_1 n_2 \ldots n_s$.

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